MAPS WHICH INDUCE THE ZERO MAP ON HOMOTOPY

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1. Introduction. In this paper, all spaces will have the homotopy type of simply connected CW-complexes, and will have base points which are preserved by maps and homotopies. We denote by [X, Y] the set of homotopy classes of maps from X to Y, and by $N[X, Y]$ the subset of those homotopy classes $[f]$ which induce the zero homomorphism on homotopy, that is, $f_\#: \pi_i(X) \to \pi_i(Y)$ is the zero homomorphism for each i. We wish to find conditions which would imply that such a map is necessarily null-homotopic. We express our conditions in terms of Postnikov systems and Moore-Postnikov decompositions of certain fibre spaces.

In **(3)**, D. W. Kahn obtained a condition for $N[X, Y]$ to be zero. This paper represents an attempt to improve this result. We prove below a result (Theorem 2) which implies Kahn's result. This work was done while the author was a Fellow of the Summer Research Institute of the Canadian Mathematical Congress in 1966.

2. Suppose that Y is a simply connected space and ΩY is the loop space of Y. Let $\Omega Y \to PY \to Y$ be the path space fibration over Y. Then a Moore-Postnikov decomposition of this fibre space (see **(4; 5)**) gives a sequence of fibre spaces and commutative diagrams:

$$\begin{array}{ccccccc} & K(\pi_{i-1}(Y), i-2) & & & & & \\ & \swarrow & & & & & \\ \ldots \to & Y_i \to & Y_{i-1} \to & \ldots \to & Y_3 \to & Y_2 = Y \\ & \uparrow \nearrow & & & & & \\ & PY & & & & & \end{array} \tag{2.1}$$

In case $\pi_{i-1}(Y) = 0$, we can clearly identify Y_i and Y_{i-1}. In general, we shall do this and omit the unnecessary terms in this sequence. A Postnikov system for Y provides a sequence of fibre spaces and commutative diagrams:

$$\begin{array}{ccccc} & K(\pi_i(Y), i) & & & \\ & \swarrow & & & \\ \ldots \to & Y^i \to & Y^{i-1} \to & \ldots \to & Y^2 \\ & g_i \nwarrow & \nearrow g_{i-1} & & \\ & & Y & & \end{array} \tag{2.2}$$

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such that the map g_i induces an isomorphism on homotopy groups in dimensions $\leqslant i$.

In diagram (2.1), let us denote the composite map $Y_i \to \ldots \to Y$ by q_i. This is a fibre map and the fibre space $Y_i \to Y$ is induced from the fibre space $PY^{i-1} \to Y^{i-1}$ by the map g_{i-1}. Thus the fibre of q_i is ΩY^{i-1}, and Y_i is $(i-1)$-connected and q_i induces isomorphisms on homotopy in dimensions $\geqslant i$. Let $\delta_i \in H^i(Y_i, \pi_i(Y))$ be the fundamental class of Y_i. Let f be a map from a space X to Y_i. If X is a CW-complex, then $f^*\delta_i$ is the obstruction to lifting the map f to Y_{i+1}.

We recall that the fibration

$$K(\pi_{i-1}(Y), i-2) \xrightarrow{j} Y_i \xrightarrow{p} Y_{i-1}$$

is principal (see **(4)**), that is, there is an action

$$\mu\colon K(\pi_{i-1}(Y), i-2) \times Y_i \to Y_i$$

of the fibre on the total space with the following property: for classes u, $v \in [X, Y_i]$, $p_\#(u) = p_\#(v)$ if and only if there exists a class

$$\omega \in [X, K(\pi_{i-1}(Y), i-2)]$$

with

$$v \simeq \mu(\omega \times u)\Delta\colon Y_i \to Y_i \times Y_i \to K(\pi_{i-1}(Y), i-2) \times Y_i \xrightarrow{\mu} Y_i.$$

Moreover, if l_1 and l_2 are the usual inclusions

$$K(\pi_{i-1}(Y), i-2) \subset K(\pi_{i-1}(Y), i-2) \times Y_i,$$
$$Y_i \subset K(\pi_{i-1}(Y), i-2) \times Y_i,$$

respectively, then $\mu l_1 \simeq j$ and $\mu l_2 \simeq$ identity.

3. In this section, X will have the homotopy type of a finite-dimensional complex. For any space Y, we wish to consider $N[X, Y]$. Suppose that $0 < n(1) < n(2) < \ldots$ are the dimensions in which Y has non-zero homotopy groups, and let $\pi_i = \pi_{n(i)}(Y)$.

Then we have a Moore-Postnikov decomposition of the fibre space $\Omega Y \to PY \to Y$ as follows:

$$\begin{array}{ccccc} & & K(\pi_{i-1}, n(i-1)-1) & & \\ & \swarrow^{j'_{n(i)}} & & & \\ \ldots \to Y_{n(i)} & \xrightarrow{p'_{n(i)}} & Y_{n(i-1)} \to \ldots \to Y_{n(1)} = Y. & & \\ \nwarrow & & \nearrow & & \\ & PY & & & \end{array} \tag{3.1}$$

We have an analogous result for the fibration $\Omega X \to PX \to X$. Combining various terms we have the diagram

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$$\ldots \to X_{n(i)} \xrightarrow{p_{n(i)}} X_{n(i-1)} \to \ldots \to X_{n(1)} \xrightarrow{p_{n(1)}} X.$$

We note, of course, that $p_{n(i)}$ is the composition

$$X_{n(i)} \to X_{n(i)-1} \to \ldots \to X_{n(i-1)+1} \to X_{n(i-1)}.$$

Let $q_{n(i)} = p_{n(1)} \ldots p_{n(i)} \colon X_{n(i)} \to X$.

Suppose we have a map $f\colon X \to Y$. Let $f_1 = fp_{n(1)}\colon X_{n(1)} \to Y_{n(1)}$. Let $\delta_{n(i)} \in H^{n(i)}(Y_{n(i)}, \pi_i)$ be the fundamental class. Suppose for some $i > 1$ we have defined $f_j\colon X_{n(j)} \to Y_{n(j)}$ for all $j < i$ such that $p'_{n(j)} f_j \simeq f_{j-1} p_{n(j)}$ for such j. Now consider the function $f_{i-1} p_{n(i)}\colon X_{n(i)} \to Y_{n(i-1)}$. Since the obstruction to lifting $f_{i-1} p_{n(i)}$ to $Y_{n(i)}$ lies in $H^{n(i-1)}(X_{n(i)}, \pi_{i-1}) = 0$, we see that there exists a map $f_i\colon X_{n(i)} \to Y_{n(i)}$ such that $p'_{n(i)} f_i \simeq f_{i-1} p_{n(i)}$. Thus we have the elementary

PROPOSITION 1. *A map $f\colon X \to Y$ induces maps $f_i\colon X_{n(i)} \to Y_{n(i)}$ for each i such that $f_1 = fp_{n(1)}$ and $p'_{n(i)} f_i \simeq f_{i-1} p_{n(i)}$.*

Let $F_{n(i-1)}$ be the fibre of $p_{n(i)}\colon X_{n(i)} \to X_{n(i-1)}$. The homotopy sequence of this fibration shows that $F_{n(i-1)}$ is $(n(i-1)-2)$-connected and

$$\pi_j(F_{n(i-1)}) = 0 \quad \text{if } j < n(i-1) - 1 \text{ or } j \geqslant n(i) - 1,$$

$$\partial\colon \pi_{j+1}(X_{n(i-1)}) \cong \pi_j(F_{n(i-1)}) \quad \text{if } n(i-1) - 1 \leqslant j < n(i) - 1,$$

where ∂ is the homotopy boundary. Let

$$\delta'_{n(i-1)} \in H^{n(i-1)}(X_{n(i-1)}, \quad \pi_{n(i-1)}(X)),$$

$$\alpha'_{n(i-1)-1} \in H^{n(i-1)-1}(F_{n(i-1)}, \quad \pi_{n(i-1)}(X)),$$

$$\alpha_{n(i-1)-1} \in H^{n(i-1)-1}(K(\pi_{i-1}, \quad n(i-1) - 1); \pi_{i-1})$$

be the fundamental classes. Then $\tau(\alpha'_{n(i-1)-1}) = \delta'_{n(i-1)}$ and

$$\tau(\alpha_{n(i-1)-1}) = \delta_{n(i-1)},$$

where τ stands for the transgression. Suppose $f\colon X \to Y$ is a map. Then we have induced maps $f_i\colon X_{n(i)} \to Y_{n(i)}$. We observe that

$$(f_i | F_{n(i-1)})_{\#}\colon \pi_{n(i-1)-1}(F_{n(i-1)}) \to \pi_{i-1}$$

and

$$f_{i-1\#}\colon \pi_{n(i-1)}(X_{n(i-1)}) \to \pi_{n(i-1)}(Y_{n(i-1)})$$

are identifiable with

$$f_{\#}\colon \pi_{n(i-1)}(X) \to \pi_{n(i-1)}(Y) = \pi_{i-1}.$$

Each of the above homomorphisms induces a coefficient homomorphism which we shall denote by f^*_c (see **(2)**). Then we have

PROPOSITION 2. *$f^*_c \delta'_{n(i-1)} = f^*_{i-1} \delta_{n(i-1)}$ for each i.*

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Proof. For simplicity, let us denote $\pi_{n(i-1)}(X)$, $p_{n(i)}$, $p'_{n(i)}$, $f_i| F_{n(i-1)}$ by π'_{i-1}, p_i, p'_i, k_i, respectively. Let δ denote the coboundary in the cohomology sequence of a pair. Then we have the following diagram of commutative squares:

$$\begin{array}{ccccc}
H^{n(i-1)-1}(F_{n(i-1)};\pi'_{i-1}) & \xrightarrow{\delta} & H^{n(i-1)}(X_{n(i)}, F_{n(i-1)};\pi'_{i-1}) & \xrightarrow{p_i^{*-1}} & H^{n(i-1)}(X_{n(i-1)};\pi'_{i-1}) \\
\downarrow f^*_c & & \downarrow f^*_c & & \downarrow f^*_c \\
H^{n(i-1)-1}(F_{n(i-1)};\pi_{i-1}) & \xrightarrow{\delta} & H^{n(i-1)}(X_{n(i)}, F_{n(i-1)};\pi_{i-1}) & \xrightarrow{p_i^{*-1}} & H^{n(i-1)}(X_{n(i-1)};\pi_{i-1}) \\
\uparrow k^*_i & & \uparrow f^*_i & & \uparrow f^*_{i-1} \\
H^{n(i-1)-1}(K(\pi_{i-1}, n(i-1)-1);\pi_{i-1}) & \xrightarrow{\delta} & H^{n(i-1)}(Y_{n(i)}, K(\pi_{i-1}, n(i-1)-1);\pi_{i-1}) & \xrightarrow{p_i'^{*-1}} & H^{n(i-1)}(Y_{n(i-1)};\pi_{i-1})
\end{array}$$

We have

$$\begin{aligned} f^*_c\delta'_{n(i-1)} &= f^*_c\tau(\alpha'_{n(i-1)-1}) = f^*_c p^*{}_i^{-1}\delta(\alpha'_{n(i-1)-1}) \\ &= p^*{}_i^{-1}f^*_c\delta(\alpha'_{n(i-1)-1}) = p^*{}_i^{-1}\delta f^*_c(\alpha'_{n(i-1)-1}). \end{aligned}$$

Now $\alpha'_{n(i-1)-1}$ can be represented by a map

$$\bar{\alpha}'\colon F_{n(i-1)} \to K(\pi'_{i-1}, n(i-1)-1).$$

The homomorphism $f_{\#}\colon \pi_{n(i-1)}(X) \to \pi_{i-1}$ induces the coefficient homomorphism

$$f^*_c\colon H^{n(i-1)}(X_{n(i-1)};\pi_{n(i-1)}(X)) \to H^{n(i-1)}(X_{n(i-1)};\pi_{i-1}).$$

Let $\bar{f}_c\colon K(\pi_{n(i-1)}(X), n(i-1)-1) \to K(\pi_{i-1}, n(i-1)-1)$ be a map which induces $f_{\#}$ on the $(n(i-1)-1)$st homotopy group. Then $f^*_c(\alpha_{n(i-1)-1})$ can be represented by the map $\bar{f}_c\bar{\alpha}'$. Hence as an element of $\operatorname{Hom}(\pi'_{i-1}, \pi_{i-1})$, it is the map

$$\pi'_{i-1} \xrightarrow{id} \pi'_{i-1} \xrightarrow{f_{\#}} \pi_{i-1}.$$

Also, the element $f^*_i(\alpha_{n(i-1)-1})$ can be represented by the map

$$F_{n(i-1)} \xrightarrow{f_i} K(\pi_{i-1}, n(i-1)-1) \xrightarrow{id} K(\pi_{i-1}, n(i-1)-1).$$

Hence it corresponds to the homomorphism

$$\pi_{n(i-1)-1}(F_{n(i-1)}) \xrightarrow[f_{i\#}]{} \pi_{i-1} \xrightarrow[id]{} \pi_{i-1}.$$

Thus we have

$$f^*{}_c(\alpha'_{n(i-1)-1}) = f^*{}_i(\alpha_{n(i-1)-1}).$$

Hence

$$\begin{aligned} f^*{}_c\delta'_n(_{i-1}) &= p^*{}_i{}^{-1}\delta f^*{}_c(\alpha'_{n(i-1)-1}) = p^*{}_i{}^{-1}\delta f^*{}_i(\alpha_{n(i-1)-1}) \\ &= p^*{}_i{}^{-1} f^*{}_i\delta(\alpha_{n(i-1)-1}) = f^*{}_{i-1}\, p^*{}_i{}^{-1}\delta(\alpha_{n(i-1)-1}) \\ &= f^*{}_{i-1}\,\tau(\alpha_{n(i-1)-1}) = f^*{}_{i-1}\,\delta_{n(i-1)}. \end{aligned}$$

Remark 1. Thus we see that if $[f] \in N[X, Y]$, we have that $f^*{}_i\delta_{n(i)} = 0$ for each $i \geqslant 1$.

Remark 2. We now state a simple result which we shall be using repeatedly. Consider the principal fibration $F \to E \to B$. Let μ: $F \times E \to E$ be the action of the fibre. Suppose f: $X \to E$ is a map and ω: $X \to F$ is a map such that $\omega \simeq 0$. Then it is easily seen that $f \simeq \mu(\omega \times f)\Delta$: $X \to X \times X \to F \times E \to E$; for example, see **(4)**.

THEOREM 1. *$[f] \in N[X, Y]$ if and only if each f_i lifts to a map h_i: $X_{n(i-1)} \to Y_{n(i)}$ such that $h_i\, p_{n(i)} \simeq f_i$, $p'_{n(i)}\, h_i \simeq f_{i-1}$.*

Proof. Suppose there exist such maps h_i. Now

$$f_{j\#}: \pi_k(X_{n(j)}) \to \pi_k(Y_{n(j)})$$

coincides with $f_\#$: $\pi_k(X) \to \pi_k(Y)$ for $k \geqslant n(j)$. Since $Y_{n(j)}$ is $(n(j) - 1)$ connected, the result follows easily.

Conversely, suppose $[f] \in N[X, Y]$. The obstruction to lifting f_{i-1} to $Y_{n(i)}$ is $f^*{}_{i-1}\,\delta_{n(i-1)}$. By Remark 1 following Proposition 2, it follows that this is zero. Thus we can find maps h_i: $X_{n(i-1)} \to Y_{n(i)}$ such that $p'_{n(i)}\, h_i \simeq f_{i-1}$. It remains to show that $f_i \simeq h_i\, p_{n(i)}$. Consider the maps f_i, $h_i\, p_{n(i)}$: $X_{n(i)} \to Y_{n(i)}$. Now $p'_{n(i)} f_i \simeq f_{i-1}\, p_{n(i)} \simeq p_{n(i)}\, h_i\, p_{n(i)}$. Since $p'_{n(i)}$: $Y_{n(i)} \to Y_{n(i-1)}$ is a principal fibration, there exists a map ω: $X_{n(i)} \to K(\pi_{i-1}, n(i-1) - 1)$ such that $f_i \simeq \mu(\omega \times h_i\, p_{n(i)})\Delta$, where Δ is the diagonal map of $X_{n(i)}$. Now $\omega \simeq 0$ since $X_{n(i)}$ is $(n(i) - 1)$-connected. Hence $\mu(\omega \times h_i\, p_{n(i)})\Delta \simeq h_i\, p_{n(i)}$. This completes the proof.

Now let $q_{n(i)} = p_{n(1)} \ldots p_{n(i)}$: $X_{n(i)} \to X$. This is a fibration which is induced from the path space fibration $\Omega X^{n(i)-1} \to PX^{n(i)-1} \to X^{n(i)-1}$ by the map $g_{n(i)-1}$: $X \to X^{n(i)-1}$. Thus we have a diagram

$$\begin{array}{ccc} X_{n(i)} & & PX^{n(i)-1} \\ \Big\downarrow q_{n(i)} & & \Big\downarrow \\ X & \xrightarrow[g_{n(i)-1}]{} & X^{n(i)-1} \end{array}$$

We can convert the map $g_{n(i)-1}$ into a fibration. It is easily seen that when we do this, the fibre is precisely $X_{n(i)}$; for example, see **(5)**. Thus we can consider

$$X_{n(i)} \xrightarrow{q_{n(i)}} X \xrightarrow{g_{n(i)-1}} X^{n(i)-1}$$

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as a fibration. We then obtain an exact sequence:

$$0 \to H^{n(i)}(X^{n(i)-1}, \pi_i) \xrightarrow{g^*_{n(i)-1}} H^{n(i)}(X, \pi_i) \xrightarrow{q^*_{n(i)}} H^{n(i)}(X_{n(i)}, \pi_i) \to .$$

Hence $H^{n(i)}(X^{n(i)-1}, \pi_i) = 0$ if and only if

$$q^*_{n(i)}: H^{n(i)}(X, \pi_i) \to H^{n(i)}(X_{n(i)}, \pi_i)$$

is a monomorphism. Thus we obtain a result which is equivalent to Kahn's result.

PROPOSITION 3. *If X has the homotopy type of a finite-dimensional complex and*

$$q^*_{n(i)}: H^{n(i)}(X, \pi_i) \to H^{n(i)}(X_{n(i)}, \pi_i)$$

is a monomorphism for each $i = 1, 2, \ldots,$ *then* $N[X, Y] = 0$.

Remark. This result could also be proved directly by using our results above.

Now we recall that in **(4)**, Thomas defined two sequences of cohomology operations depending on Y. They are described as follows. In a Postnikov decomposition of Y:

$$\begin{array}{ccc} & l_{n(i)} \swarrow K(\pi_i, n(i)) & \\ \to Y^{n(i)} & \to Y^{n(i-1)} & \to \ldots \end{array}$$

let $k_{n(i)}$ be the ith k-invariant of Y, that is,

$$k_{n(i)}: Y^{n(i)} \to K(\pi_{i+1}, n(i+1)+1).$$

Let

$$\Psi_{n(i)} = -k_{n(i)} \circ l_{n(i)}: K(\pi_i, n(i)) \to K(\pi_{i+1}, n(i+1)+1).$$

This gives a sequence of cohomology operations defined for $i \geqslant 1$. Let

$$\Phi_{n(i)-1} = \sigma\Psi_{n(i)}: K(\pi_i, n(i)-1) \to K(\pi_{i+1}, n(i+1)),$$

where σ is the suspension of cohomology operations. Then, in **(4)**, it is shown that $\Phi_{n(i)-1} = j'^*_{n(i+1)}\delta_{n(i+1)}$ and image of $\Phi_{n(i-1)-1} \subset$ kernel of $\Psi_{n(i)}$ for each $i > 1$. Finally, we observe that since $\Psi_{n(i)}$ is the first k-invariant for the space $Y_{n(i)}$, we have $\Psi_{n(i)}(\delta_{n(i)}) = 0$.

We shall also need the following result, which we quote from Thomas **(4)**.

PROPOSITION 4. *Let g be a map from a* CW-*complex X into $Y_{n(i)}$* $(i \geqslant 2)$. *The map $p'_{n(i)}\, g$ lifts to $Y_{n(i+1)}$ if and only if*

$$g^*\delta_{n(i)} \in \text{image } \Phi_{n(i-1)-1} \subset H^{n(i)}(X, \pi_i).$$

We now recall that, in **(2)**, Kahn showed that a map $f: X \to Y$ induces maps $f^{n(i)-1}: X^{n(i)-1} \to Y^{n(i)-1}$ which, when combined with our constructions, give a diagram of homotopy commutative squares:

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$$
\begin{array}{ccc}
X_{n(i)} & \xrightarrow{\ f_i\ } & Y_{(n i)} \\
{\scriptstyle q_{n(i)}}\big\downarrow & & \big\downarrow{\scriptstyle q'_{n(i)}} \\
X & \xrightarrow{\ f\ } & Y \\
{\scriptstyle g_{n(i)-1}}\big\downarrow & & \big\downarrow{\scriptstyle g'_{n(i)-1}} \\
X^{n(i)-1} & \xrightarrow[f^{n(i)-1}]{} & Y^{n(i)-1}
\end{array}
$$

This leads to a commutative diagram

$$
\begin{array}{ccccc}
0 \to H^{n(i)}(Y^{n(i)-1}, \pi_i) & \xrightarrow{g'^*_{n(i)-1}} & H^{n(i)}(Y, \pi_i) & \to & \\
\big\downarrow {\scriptstyle f^{n(i)-1*}} & & \big\downarrow {\scriptstyle f^*} & & \\
0 \to H^{n(i)}(X^{n(i)-1}, \pi_i) & \xrightarrow[g^*_{n(i)-1}]{} & H^{n(i)}(X, \pi_i) & \xrightarrow[q^*_{n(i)}]{} &
\end{array}
$$

where each row is exact; for example, see **(1)**. For $i > 1$, we have the following commutative square:

$$
\begin{array}{ccc}
0 \to H^{n(i-1)-1}(X^{n(i)-1}, \pi_{i-1}) & \xrightarrow{g^*_{n(i)-1}} & H^{n(i-1)-1}(X, \pi_{i-1}) \to 0 \\
\big\downarrow {\scriptstyle \Phi_{n(i-1)-1}} & & \big\downarrow {\scriptstyle \Phi_{n(i-1)-1}} \\
0 \to H^{n(i)}(X^{n(i)-1}, \pi_i) & \xrightarrow[g^*_{n(i)-1}]{} & H^{n(i)}(X, \pi_i) \to
\end{array}
$$

Thus

$$\Phi_{n(i-1)-1} H^{n(i-1)-1}(X, \pi_{i-1}) \subset g^*_{n(i)-1}\, H^{n(i)}(X^{n(i)-1}, \pi_i).$$

Then, for each $i > 1$, we put

$$T^{n(i)}(X, Y) = \frac{[\ker \Psi_{n(i)}] \cap g^*_{n(i)-1} H^{n(i)}(X^{n(i)-1}, \pi_i)}{\Phi_{n(i-1)-1} H^{n(i-1)-1}(X, \pi_{i-1})},$$

where $[\ker \Psi_{n(i)}]$ is the least subgroup of $H^{n(i)}(X, \pi_i)$ which contains the kernel of $\Psi_{n(i)}\colon H^{n(i)}(X, \pi_i) \to H^{n(i+1)+1}(X, \pi_{i+1})$.

Put $\qquad T^{n(1)}(X, Y) = [\ker \Psi_{n(1)}] \cap g^*_{n(1)-1} H^{n(1)}(X^{n(1)-1}, \pi_1).$

Clearly, if $H^{n(i)}(X^{n(i)-1}, \pi_i) = 0$, then $T^{n(i)}(X, Y) = 0$. Our result is:

THEOREM 2. *If X has the homotopy type of a finite-dimensional complex and $T^{n(i)}(X, Y) = 0$ for all $i \geqslant 1$, then $N[X, Y] = 0$.*

Proof. Suppose $[f] \in N[X, Y]$. We need to show that f lifts to each $Y_{n(i)}$. The obstruction to lifting f to $Y_{n(2)}$ is $f^*\delta_{n(1)}$. Now

$$p^*_{n(1)} f^* \delta_{n(1)} = f^*{}_1 \delta_{n(1)} = 0.$$

The fibration

$$X_{n(1)} \xrightarrow[p_{n(1)}]{} X \xrightarrow[g_{n(1)-1}]{} X^{n(1)-1}$$

gives an exact sequence

$$0 \rightarrow H^{n(1)}(X^{n(1)-1}, \pi_1) \xrightarrow{g^*_{n(1)-1}} H^{n(1)}(X, \pi_1) \xrightarrow{p^*_{n(1)}} H^{n(1)}(X_{n(1)}, \pi_1) \rightarrow .$$

Hence $f^*\delta_{n(1)} = g^*_{n(1)-1}(a)$ for a unique $a \in H^{n(1)}(X^{n(1)-1}, \pi_1)$. Now

$$\Psi_{n(1)} f^*\delta_{n(1)} = f^*\Psi_{n(1)} \delta_{n(1)} = 0.$$

Thus

$$\begin{aligned} f^*\delta_{n(1)} &\in [\ker \Psi_{n(1)}] \cap g^*_{n(1)-1} H_{n(1)}(X^{n(1)-1}, \pi_1) \\ &= T^{n(1)}(X, Y) \\ &= 0 \quad \text{by hypothesis.} \end{aligned}$$

Thus we can find a map $l_2: X \rightarrow Y_{n(2)}$ such that $p'_{n(2)} l_2 \simeq f$. Now

$$p'_{n(2)} l_2 q_{n(2)} \simeq f q_{n(2)} \simeq f_1 p_{n(2)} \simeq p'_{n(2)} f_2.$$

Hence there exists a map $\omega: X_{n(2)} \rightarrow K(\pi_1, n(1) - 1)$ such that $l_2 q_{n(2)} \simeq \mu(\omega \times f_2)\Delta$, where Δ is the diagonal map $X_{n(2)} \rightarrow X_{n(2)} \times X_{n(2)}$. Since $\omega \simeq 0$, it follows that $l_2 q_{n(2)} \simeq f_2$. Suppose f lifts to a map $l_i: X \rightarrow Y_{n(i)}$ for $i > 2$ with $q'_{n(i)} l_i \simeq f$ and $l_i q_{n(i)} \simeq f_i$. We need to show that f lifts to $Y_{n(i+1)}$. Now put $\mu = l^*_i \delta_{n(i)}$. We have $q^*_{n(i)}(\mu) = q^*_{n(i)} l^*_i \delta_{n(i)} = f^*_i \delta_{n(i)} = 0$. The fibration

$$X_{n(i)} \xrightarrow[q_{n(i)}]{} X \xrightarrow[g_{n(i)-1}]{} X^{n(i)-1}$$

gives an exact sequence

$$0 \rightarrow H^{n(i)}(X^{n(i)-1}, \pi_i) \xrightarrow{g^*_{n(i)-1}} H^{n(i)}(X, \pi_i) \xrightarrow{q^*_{n(i)}} H^{n(i)}(X_{n(i)}, \pi_i) \rightarrow .$$

Hence we have $\mu = g^*_{n(i)-1}(\nu)$ for a unique class $\nu \in H^{n(i)}(X^{n(i)-1}, \pi_i)$. Also,

$$\Psi_{n(i)}(\mu) = \Psi_{n(i)} l^*_i \delta_{n(i)} = l^*_i \Psi_{n(i)} \delta_{n(i)} = 0.$$

Thus $\mu \in [\ker \Psi_{n(i)}] \cap g^*_{n(i)-1} H^{n(i)}(X^{n(i)-1}, \pi_i)$. By hypothesis,

$$T^{n(i)}(X, Y) = 0.$$

Hence $l^*_i \delta_{n(i)} = \mu \in \text{im } \Phi_{n(i-1)-1}$. By Proposition 5, it follows that $p'_{n(i)} l_i$ lifts to $Y_{n(i+1)}$, that is, there exists a map $l_{i+1}: X \rightarrow Y_{n(i+1)}$ with $p'_{n(i)} p'_{n(i+1)} l_{i+1} \simeq p'_{n(i)} l_i$. Thus $q'_{n(i+1)} l_{i+1} \simeq p'_{n(2)} \ldots p'_{n(i)} p'_{n(i+1)} l_{i+1} \simeq q'_{n(i)} l_i \simeq f$. Now

$$\begin{aligned} p'_{n(i)} p'_{n(i+1)} l_{i+1} q_{r(i+1)} &\simeq p'_{n(i)} l_i q_{n(i+1)} \\ &\simeq p'_{n(i)} l_i q_{n(i)} p_{n(i+1)} \\ &\simeq p'_{n(i)} f_i p_{n(i+1)} \\ &\simeq p'_{n(i)} p'_{n(i+1)} f_{i+1}. \end{aligned}$$

This means that there exists a map $\omega_1: X_{n(i+1)} \to K(\pi_{i-1}, n(i-1)-1)$ with $p'_{n(i+1)}\, l_{i+1}\, q_{n(i+1)} \simeq \mu(\omega_1 \times p'_{n(i+1)} f_{i+1})\Delta$, where Δ is the diagonal map $X_{n(i+1)} \to X_{n(i+1)} \times X_{n(i+1)}$. Since $\omega_1 \simeq 0$, we have

$$p'_{n(i+1)}\, l_{i+1}\, q_{n(i+1)} \simeq p'_{n}(_{i+1)} f_{i+1}.$$

Again, this means that there exists a map $\omega_2: X_{n(i+1)} \to K(\pi_i, n(i)-1)$ with $l_{i+1}\, q_{n(i+1)} \simeq \mu(\omega_2 \times f_{i+1})\Delta$, where Δ is the diagonal map $X_{n(i+1)} \to X_{n(i+1)} \times X_{n(i+1)}$. Since $\omega_2 \simeq 0$, it follows that

$$l_{i+1}\, q_{n(i+1)} \simeq f_{i+1}.$$

This completes the induction and the proof.

Now, following **(4)**, we shall define a sequence of non-negative integers $\tau_{n(i)}$ as follows. Suppose that π_i is a cyclic group, and let $f: S^{n(i)} \to Y$ represent a generator. Define $\tau_{n(i)}$ to be the least positive integer such that

$$\tau_{n(i)}\, S_i \in f^*\, H^{n(i)}(Y, \pi_i),$$

where S_i generates the cyclic group $H^{n(i)}(S^{n(i)}, \pi_i)$. If $f^*\, H^{n(i)}(Y, \pi_i) = 0$, or if π_i is not cyclic, put $\tau_{n(i)} = 0$. Denote by $\tau^*_{n(i)}$ the cohomology operation given by multiplying each cohomology class by the integer $\tau_{n(i)}$. We shall consider this operation only in dimension $n(i)$ and with coefficients in π_i. Define, for each $i > 1$,

$$R^{n(i)}(X, Y) = \frac{\ker \tau^*_{n(i)} \cap [\ker \Psi_{n(i)}]}{\ker \tau^*_{n(i)} \cap \operatorname{im} \Phi_{n(i-1)-1}},$$

where

$$\tau^*_{n(i)}: H^{n(i)}(X, \pi_i) \to H^{n(i)}(X, \pi_i),$$

$$\Psi_{n(i)}: H^{n(i)}(X, \pi_i) \to H^{n(i+1)+1}(X, \pi_{i+1}),$$

$$\Phi_{n(i-1)-1}: H^{n(i-1)-1}(X, \pi_{i-1}) \to H^{n(i)}(X, \pi_i).$$

If $i = 1$, put $R^{n(1)}(X, Y) = [\ker \Psi_{n(1)}] \cap g^*_{n(1)-1}\, H^{n(1)}(X^{n(1)-1}, \pi_1)$. Then we have

THEOREM 3. *Let X be a space having the homotopy type of a finite-dimensional* CW-*complex. If $R^{n(i)}(X, Y) = 0$ for all $i \geqslant 1$ and*

$$\tau^*_{n(i)}: H^{n(i)}(X^{n(i)-1}, \pi_i) \to H^{n(i)}(X^{n(i)-1}, \pi_i)$$

is zero for all $i \geqslant 2$, then $N[X, Y] = 0$.

Proof. Since $R^{n(1)}(X, Y) = 0 = T^{n(1)}(X, Y)$, we have, as in the proof of Theorem 2, that there exists a map $l_2: X \to Y_{n(2)}$ with $p'_{n(2)}\, l_2 \simeq f$ and $l_2\, q_{n(2)} \simeq f_2$. Suppose that for some $i > 2$, we have a map $l_i: X \to Y_{n(i)}$ with $q'_{n(i)}\, l_i \simeq f$ and $l_i\, q_{n(i)} \simeq f_i$. We need to show that f lifts to $Y_{n(i+1)}$. Put $\mu = l^*_i\, \delta_{n(i)}$. Then $q^*_{n(i)}(\mu) = q^*_{n(i)}\, l^*_i\, \delta_{n(i)} = f^*_i\, \delta_{n(i)} = 0$. The fibration

$$X_{n(i)} \xrightarrow[q_{n(i)}]{} X \xrightarrow[g_{n(i)-1}]{} X^{n(i)-1}$$

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gives an exact sequence

$$0 \to H^{n(i)}(X^{n(i)-1}, \pi_i) \xrightarrow[g^*_{n(i)-1}]{} H^{n(i)}(X, \pi_i) \xrightarrow[q^*_{n(i)}]{} H^{n(i)}(X_{n(i)}, \pi_i) \to .$$

Hence we have that $\mu = g^*_{n(i)-1}(\nu)$, where $\nu \in H^{n(i)}(X^{n(i)-1}, \pi_i)$. Also

$$\Psi_{n(i)}(\mu) = \Psi_{n(i)} l^*_i \delta_{n(i)} = l^*_i \Psi_{n(i)} \delta_{n(i)} = 0.$$

Thus $\mu \in \ker \Psi_{n(i)}$. Further,

$$\tau^*_{n(i)}(\mu) = \tau^*_{n(i)} g^*_{n(i)-1}(\nu) = g^*_{n(i)-1} \tau^*_{n(i)}(\nu).$$

Since $\nu \in H^{n(i)}(X^{n(i)-1}, \pi_i)$, the hypotheses of the theorem imply that $\tau^*_{n(i)}(\mu) = 0$. Thus $\mu \in [\ker \Psi_{n(i)}] \cap \ker \tau^*_{n(i)}$. By hypothesis, $R^{n(i)}(X, Y) = 0$. Hence $\mu \in \operatorname{im} \Phi_{n(i-1)-1}$. It follows from Proposition 4 that we can find a map $l_{i+1}: X \to Y_{n(i+1)}$ with $p'_{n(i)} p'_{n(i+1)} l_{i+1} \simeq p'_{n(i)} l_i$. The proof is completed by reproducing the last part of the proof of Theorem 2.

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