GROTHENDIECK'S PROPERTY IN $L^p(\mu, X)$

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Abstract. We prove that, for non purely atomic measures, $L^p(\mu, X)$ is a Grothendieck space if and only if X is reflexive.

1. Introduction. Let (Ω, Σ, μ) be a finite measure space and X a Banach space. We denote by $L^p(\mu, X)$ $(1 \leq p < \infty)$ the Banach space of all X-valued Lebesgue-Bochner p-integrable functions over Ω and by $L^\infty(\mu, X)$ the Banach space of all measurable and essentially bounded functions from Ω to X. The question of when a property passes from the Banach space X to X-valued function spaces has been extensively studied (see [**15**] for a survey on these topics). In this paper, we deal with Grothendieck's property. X is said to be a *Grothendieck space*, whenever weak*-convergence and weak-convergence of sequences coincide in the dual space X^* [**6**], [**13**, Ch. 5]. Grothendieck's property for $C(K, X)$ has been analyzed in [**2**] and [**11**]. It is also known [**1**] that $\ell_p(X)$, $(1 < p < \infty)$ is Grothendieck if and only if X is Grothendieck. We prove that, for non purely atomic measures and $1 < p < \infty$, $L^p(\mu, X)$ is Grothendieck if and only if X is reflexive. The notations and terminology used and not defined in the paper can be found in [**5**] or [**7**].

2. Results. We begin by describing when a Banach space contains a quotient isomorphic to c_0. Let us mention that this result has been obtained in [**10**, Theorem IV.3] for separable Banach spaces.

LEMMA. *X has a quotient isomorphic to c_0 if and only if X^* contains a weak*-null sequence equivalent to the unit basis of ℓ_1.*

Proof. First of all, note that there is a bijection between linear continuous maps T from X into c_0 and weak*-null sequences in X^*. We have $T(x) = (\langle x_n^*, x\rangle)$ for all $x \in X$ and $T^*(\alpha) = \sum_n \alpha_n x_n^*$, for all $\alpha = (\alpha_n) \in \ell_1$.

$(\Rightarrow)$ Assume that $T: X \to c_0$ is a quotient map; then T^* is an isomorphism into, and hence (x_n^*) is equivalent to the unit basis of ℓ_1.
$(\Leftarrow)$ Take $T(x) = (\langle x_n^*, x\rangle)$. Since (x_n^*) is equivalent to the unit basis of ℓ_1, we have that T^* is an isomorphism into. Therefore, the range of T is dense and closed [**4**, p. 168–169], and we deduce that T is onto. □

REMARK. There is a dichotomy for a linear continuous map T from a Banach space X into c_0: either (a) there is an infinite subset $M \subset \mathbb{N}$ such that ST is onto, where S is the canonical projection from $c_0(\mathbb{N})$ onto $c_0(M)$ or (b) T^* is weakly precompact, i.e., T^* sends bounded subsets into weakly conditionally compact subsets. To see this, note that, by the previous Lemma, $T(x) = (\langle x_n^*, x\rangle)$ for some weak*-null sequence (x_n^*) and $T^*(\alpha) = \sum_n \alpha_n x_n^*$. Thus, if (a) does not hold, then, again by our Lemma, $\{x_n^*: n \in \mathbb{N}\}$ is a weakly conditionally compact subset of X^* and therefore its closed absolutely convex hull A also is ([**14**, Addendum]). Finally, note that T^* maps the closed unit ball of ℓ_1 into a subset of A. Condition (b) can be also replaced by the weaker condition (b′) T is unconditionally convergent, because, if T is not unconditionally convergent, there is an operator $S: c_0 \to X$ such that ST is an isomorphism into [**5**, p.54]; thus T^*S^* is a quotient map. Hence, assuming (b), T^*S^* is weakly precompact map onto ℓ_1 and we obtain a contradiction.

Our first theorem can be considered as a dual version of Emmanuele's theorem [**8**] about complemented copies of c_0 in $L^p(\mu, X)$.

THEOREM 1. *Let* (Ω, Σ, μ) *be a non purely atomic measure space, let* $1 < p \leq \infty$ *and assume that* X^* *contains a copy of* ℓ_1. *Then* $L^p(\mu, X)$ *contains a quotient isomorphic to* c_0.

Proof. We shall construct a weak*-null sequence in $L^p(\mu, X)^*$ equivalent to the unit basis of ℓ_1; thus by the above lemma, we shall obtain a quotient isomorphic to c_0. There is no loss of generality in considering the case of $[0,1]$ with the Lebesgue measure.

Let (x_n^*) be a sequence in X^* equivalent to the standard basis of ℓ_1; i.e. there are positive constants α and β such that for all finite sequences $a_1, a_2, \ldots, a_n$ of scalars we have

$$\alpha \sum_{i=1}^{n} |a_i| \leq \left\| \sum_{i=1}^{n} a_i x_i^* \right\|_{X^*} \leq \beta \sum_{i=1}^{n} |a_i|.$$

Consider the sequence (r_n) of Rademacher functions on $[0,1]$ and define a sequence of simple functions by $f_n := r_n x_n^* \in L^q(\mu, X^*) (n \in \mathbb{N})$, where $1/q + 1/p = 1$. Since $|r_n(t)| = 1$ for all $t \in [0,1]$ and $n \in \mathbb{N}$, we have

$$\alpha \sum_{i=1}^{n} |a_i| \leq \left\| \sum_{i=1}^{n} a_i r_i(t) x_i^* \right\|_{X^*} \leq \beta \sum_{i=1}^{n} |a_i|, \quad \text{for all } t \in [0,1],$$

whenever $a_1, a_2, \ldots, a_n$ are scalars. Hence, by integration,

$$\alpha \sum_{i=1}^{n} |a_i| \leq \left\| \sum_{i=1}^{n} a_i f_i \right\|_{L^q(\mu, X^*)} \leq \beta \sum_{i=1}^{n} |a_i|.$$

In other words, (f_n) is a $L^q(\mu, X^*)$-sequence equivalent to the unit basis of ℓ_1. Since $L^q(\mu, X^*)$ can be isometrically embedded in $L^p(\mu, X)^*$ [**6**, *p*. 97], it follows that (f_n) is a sequence in $(L^p(\mu, X))^*$ equivalent to the unit basis of ℓ_1.

Let us show that it is also a $\sigma(L^p(\mu, X)^*, L^p(\mu, X))$-null sequence. Take $f \in L^p(\mu, X)$. Since (x_n^*) is bounded and the measure is finite, we have

$$\begin{aligned} \lim_n |\langle f_n, f \rangle| &= \lim_n \left| \int_{[0,1]} r_n(t) \langle x_n^*, f(t) \rangle \, d\mu(t) \right| \\ &\leq \lim_n \|x_n^*\| \left\| \int_{[0,1]} r_n(t) f(t) \, d\mu(t) \right\|_X = 0. \end{aligned}$$

□

If X contains a copy of ℓ_1, then X^* also contains a copy of ℓ_1 [**5**, p. 211] and, by Theorem 1, $L^p(\mu, X)$ contains a quotient isomorphic to c_0 $(1 < p < \infty)$. However, note that this quotient does not come, in general, from a complemented copy of ℓ_1, since $L^p(\mu, X)$ contains a complemented copy of ℓ_1 if and only if X does [**12**].

THEOREM 2. *If* (Ω, Σ, μ) *is not purely atomic and* $1 < p < \infty$, *then* $L^p(\mu, X)$ *is a Grothendieck space if and only if* X *is reflexive.*

Proof. On the one hand, if X is reflexive, then $L^p(\mu, X)$ is reflexive, and hence Grothendieck.

On the other hand, assume that $L^p(\mu, X)$ is a Grothendieck space. Since X is isomorphic to a complemented subspace of $L^p(\mu, X)$, X is Grothendieck. Duals of Grothendieck spaces are weakly sequential complete. Thus, via Rosenthal's theorem we obtain that either X^* is reflexive or X^* contains a copy of ℓ_1. Therefore, if X is non-reflexive, by Theorem 1, $L^p(\mu, X)$ has a quotient isomorphic to c_0 and we get a contradiction. We note that this implication also holds for $p = +\infty$. □

In the context of Banach lattices, Theorem 2 can be also rephrased in terms of quotients isomorphic to c_0. The main fact is a proposition that partially answers a question posed by Diestel [**6**] (to give an internal characterization of Grothendieck spaces).

PROPOSITION. *Let X be a Banach lattice. Then X is a Grothendieck space if and only if X contains no quotient isomorphic to c_0.*

Proof. ($\Rightarrow$) Note that Grothendieck's property is inherited by quotients.
($\Leftarrow$) Assume that X is not Grothendieck. Then, we can find a weak*-null sequence $(x_n^*) \subset X^*$ without any weakly null subsequence. Since X has no quotient isomorphic to c_0, X cannot have a complemented copy of ℓ_1 and this implies that X^* cannot have a copy of c_0. By a known result on Banach lattices, we deduce that X^* is weakly sequentially complete. Therefore, appealling to Rosenthal's theorem, we deduce that (x_n^*) has a subsequence equivalent to the unit basis of ℓ_1. This contradicts the initial Lemma. □

COROLLARY 1. *Let X be a Banach lattice and $1 < p < \infty$.*

(1) *If μ is purely atomic, then $L^p(\mu, X)$ contains a quotient isomorphic to c_0 if and only if X contains a quotient isomorphic to c_0.*

(2) *If μ is not purely atomic, then $L^p(\mu, X)$ contains a quotient isomorphic to c_0 if and only if X is not reflexive.*

Proof. (1) Note that if (x_n^*) is a weak* null sequence in $\ell_q(X^*) \equiv (\ell_p(X))^*$ equivalent to the unit basis of ℓ_1, then there must be $k \in \mathbb{N}$ such that $(x_n^*(k)) \subset X^*$ is equivalent to the unit basis of ℓ_1. (2) follows form the proposition above and Theorem 2. □

This corollary is not true for arbitrary Banach spaces. Namely, take a quasireflexive separable Banach space $X$of order $n \geq 1$. On the one hand, since every quotient of X is quasireflexive of order n [**3**], X has no quotient isomorphic to c_0. On the other hand, assume that X is a Grothendieck space. Since X is separable, by Diestel [**6**], the identity in X is weakly compact; thus X is reflexive.

For $p = +\infty$, Theorem 2 is not true, in general, In this case, a concept from local Banach theory appears as a necessary condition for being Grothendieck.

COROLLARY 2. *Let (Ω, Σ, μ) be a non purely atomic measure. If $L^\infty(\mu, X)$ is a Grothendieck space, then X is reflexive and X does not contain all ℓ_n^1 uniformly complemented.*

Proof. As we pointed out in the proof of Theorem 2, X must be reflexive.

On the other hand, suppose that there are operators $J_n : \ell_1^n \to X$, $P_n : X \to \ell_1^n$, such that $P_n J_n$ is the identity in ℓ_1^n and $\|J_n\| \, \|P_n\| \leq \lambda$ for some $\lambda > 0$ and for all $n \in \mathbb{N}$. Then, $(\oplus_n \ell_1^n)_\infty$ is isomorphic to a complemented subspace of $\ell_\infty(X)$ which in turns is clearly

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complemented in $L^\infty(\mu, X)$. Since $(\oplus_n \ell_1^n)_\infty$ contains a complemented copy of ℓ_1 [9], we obtain a contradiction. □

Concerning the condition in Corollary 2, note that there are reflexive Banach spaces which contain all ℓ_1^n uniformly complemented. An example: $(\oplus_n \ell_1^n)_2$.

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