GROWTH PROPERTIES AND SEQUENCES OF ZEROS OF ANALYTIC FUNCTIONS IN SPACES OF DIRICHLET TYPE

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Abstract

For $0 < p < \infty$, we let $\mathcal{D}_p$ denote the space of those functions $f$ that are analytic in the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ and satisfy $\int_{1/2}^{3/2} (1 - |z|)^{p-1} |f'(z)|^p dx dy < \infty$. The spaces $\mathcal{D}_p$ are closely related to Hardy spaces. We have, $\mathcal{D}_p \subset H_p$, if $0 < p \leq 2$, and $H_p \subset \mathcal{D}_p$, if $2 \leq p < \infty$. In this paper we obtain a number of results about the Taylor coefficients of $\mathcal{D}_p$-functions and sharp estimates on the growth of the integral means and the radial growth of these functions as well as information on their zero sets.


Keywords and phrases: Spaces of Dirichlet type, Hardy spaces, Bergman spaces, integral means, radial growth, sequences of zeros.

1. Introduction and main results

We denote by $\Delta$ the unit disc $\{z \in \mathbb{C} : |z| < 1\}$. If $f$ is a function which is analytic in $\Delta$ and $0 < r < 1$, we set

$$M_p(r, f) = \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{it})|^p dt \right)^{1/p}, \quad 0 < p < \infty,$$

$$I_p(r, f) = M_p(r, f), \quad 0 < p < \infty,$$

$$M_\infty(r, f) = \sup_{|z|=r} |f(z)|.$$

For $0 < p \leq \infty$, the Hardy space $H_p$ consists of all analytic functions $f$ in the disc for which $\|f\|_{H_p} \overset{\text{def}}{=} \sup_{0<r<1} M_p(r, f) < \infty$. We refer the reader to [10] and [13] for the theory of Hardy spaces.
If \(0 < p < \infty\) and \(\alpha > -1\), we let \(A_p^\alpha\) denote the (standard) weighted Bergman space, that is, the set of analytic functions \(f\) in \(\Delta\) such that
\[
\int_{\Delta} (1 - |z|)^\alpha |f(z)|^p \, dA(z) < \infty.
\]
Here, \(dA(z) = (1/\pi) \, dx \, dy\) denotes the normalized Lebesgue area measure in \(\Delta\). The standard unweighted Bergman space \(A_0^\alpha\) is simply denoted by \(A^\alpha\). We mention \([11]\) and \([17]\) as general references for the theory of Bergman spaces.

The space \(\mathcal{D}_a^p\) (\(p > 0\), \(\alpha > -1\)) consists of all functions \(f\) which are analytic in \(\Delta\) such that \(f' \in A_p^\alpha\). The space \(\mathcal{H}_0^2\) is the classical Dirichlet space \(\mathcal{D}\). For other values of \(p\) and \(\alpha\) the spaces \(\mathcal{D}_a^p\) have been extensively studied in a number papers such as \([27, 28, 30, 33]\) for \(p = 2\) and \([4, 8, 34, 36]\) for other values of \(p\). If \(p < \alpha + 1\), it is well known that \(\mathcal{D}_a^p = A_{a+p}^p\) with equivalence of norms (see \([12, \text{Theorem 6}]\)). For \(\alpha = p - 2\), the space \(\mathcal{D}_a^p\) is the Besov space \(B^p\) (compare to \([3]\)).

The space \(\mathcal{D}_a^p\) is said to be a Dirichlet space if \(p \geq \alpha + 1\). In this paper we shall be primarily interested in the ‘limit case’ \(p = \alpha + 1\), that is, in the spaces \(\mathcal{D}_{p-1}^p\), \(0 < p < \infty\), which are closely related to Hardy spaces. Indeed, a classical result of Littlewood and Paley \([19]\) (see also \([20]\)) asserts that
\[
(1) \quad H^p \subset \mathcal{D}_{p-1}^p, \quad 2 \leq p < \infty.
\]
On the other hand, we have
\[
(2) \quad \mathcal{D}_{p-1}^p \subset H^p, \quad 0 < p \leq 2,
\]
(see \([34, \text{Lemma 1.4}]\)). Notice that, in particular, we have \(\mathcal{D}_1^2 = H^2\). However, we remark that if \(p \neq 2\) then
\[
(3) \quad H^p \neq \mathcal{D}_{p-1}^p.
\]
This can be seen using the characterization of power series with Hadamard gaps which belong to the spaces \(\mathcal{D}_{p-1}^p\).

**Proposition A.** If \(f\) is an analytic function in \(\Delta\) which is given by a power series with Hadamard gaps, \(f(z) = \sum_{k=1}^{\infty} a_k z^{n_k} (z \in \Delta)\) with \(n_{k+1} \geq \lambda n_k\) for all \(k\) (\(\lambda > 1\)), then, for every \(p \in (0, \infty)\), \(f \in \mathcal{D}_{p-1}^p\) if and only if \(\sum_{k=1}^{\infty} |a_k|^p < \infty\).

Since for Hadamard gap series as above we have, for \(0 < p < \infty\), \(f \in H^p\) if and only of \(\sum_{k=1}^{\infty} |a_k|^2 < \infty\), we immediately deduce that \(\mathcal{D}_{p-1}^p \neq H^p\) if \(p \neq 2\). We remark that Proposition A follows from \([7, \text{Proposition 2.1}]\). In Section 2 we shall see that Proposition A can also be deduced from the following theorem which gives a condition on the Taylor coefficients of a function \(f\), analytic in \(\Delta\), which implies that \(f \in \mathcal{D}_{p-1}^p\).
THEOREM 1.1. Let $f$ be an analytic function in $\Delta$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ ($z \in \Delta$).

(i) If $0 < p < \infty$ and

$$\sum_{n=0}^{\infty} \left( \sum_{k \in I(n)} |a_k| \right)^p < \infty,$$  

then $f \in \mathcal{D}_{p-1}$.

(ii) If $0 < p \leq 2$ and

$$\sum_{n=1}^{\infty} \left( \sum_{k \in I(n)} |a_k|^2 \right)^{p/2} < \infty,$$  

then $f \in \mathcal{D}_{p-1}$.

Here and throughout the paper, for $n = 0, 1, \ldots$, $I(n)$ is the set of the integers $k$ such that $2^n \leq k < 2^{n+1}$.

If $0 < p \leq 2$, then (4) implies (5). Hence, for $p \in (0, 2]$, (ii) is stronger than (i). We remark also that if $0 < p \leq 2$, then the condition $\sum_{n=0}^{\infty} |a_n|^p < \infty$ implies (5). Consequently, (ii) improves [34, Lemma 1.5].

In Theorem 1.2 we give a condition on the Taylor coefficients of an analytic function $f$ which is necessary for its membership in $\mathcal{D}_{p-1}$ if $2 \leq p < \infty$.

THEOREM 1.2. Let $f$ be an analytic function in $\Delta$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ ($z \in \Delta$). If $2 \leq p < \infty$ and $f \in \mathcal{D}_{p-1}$, then

$$\sum_{n=1}^{\infty} \left( \sum_{k \in I(n)} |a_k|^2 \right)^{p/2} < \infty.$$  

If $0 < p < 2$ then (3) can be seen in some other ways. Rudin proved in [29] that there exists a Blaschke product $B$ which does not belong to $\mathcal{D}^1_0$ (see also [24]). Vinogradov [34] extended this result showing that for every $p \in (0, 2)$ there exist Blaschke products $B$ which do not belong to $\mathcal{D}^p_{p-1}$. This clearly gives that $\mathcal{D}^p_{p-1} \neq H^p$ if $0 < p < 2$, a fact which can be also deduced from the results of [9] and [14]. In contrast with what happens for $0 < p < 2$, it is not easy to give examples of functions $f \in \mathcal{D}_{p-1} \setminus H^p$ for a certain $p \in (2, \infty)$ that are not given by power series by Hadamard gaps. Since $H^p \subset \mathcal{D}_{p-1}$ if $p \geq 2$, any Blaschke product belongs to $\mathcal{D}_{p-1}$. Also, for a number of classes $\mathcal{F}$ of analytic functions in $\Delta$ we have $\mathcal{F} \cap \mathcal{D}_{p-1} = \mathcal{F} \cap H^p$ ($0 < p < \infty$). For example, it is very easy to prove the following lemma.

LEMMA 1.3. (i) If $\alpha > 0$, $0 < p < \infty$, and $f(z) = 1/(1 - z)^\alpha$, ($z \in \Delta$), then $f \in H^p$ if and only if $f \in \mathcal{D}_{p-1}$ if and only if $\alpha p < 1$. 

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(ii) If $\alpha, \beta > 0$, $p \in (0, \infty)$, and

$$f(z) = \frac{1}{(1-z)^\alpha (\log(2/(1-z)))^\beta}, \quad (z \in \Delta),$$

then $f \in H^p$ if and only if $f \in G^p_{p-1}$ if and only if $\alpha p < 1$ and $\beta > 0$ or $\alpha p = 1$ and $\beta p > 1$.

A much deeper result is stated in [6, Theorem 1] which asserts that, if $\mathcal{U}$ denotes the class of all univalent (holomorphic and one-to-one) functions in $\Delta$, then $\mathcal{U} \cap H^p = \mathcal{U} \cap G^p_{p-1}$ for all $p > 0$ (see also [25] for the case $p = 1$).

In spite of these facts we shall prove that, for every $p \in (2, \infty)$, there are a lot of differences between the space $H^p$ and the space $G^p_{p-1}$. In Section 3, we shall be mainly concerned in obtaining sharp estimates on the growth of the integral means of $G^p_{p-1}$-functions. If $0 < p \leq 2$ and $f \in G^p_{p-1}$, then $f \in H^p$ and hence, the integral means $M_p(r, f)$ are bounded. This is no longer true for $p > 2$. Our main results in Section 3 are stated in the following two theorems.

**Theorem 1.4.** If $2 < p < \infty$ and $f \in G^p_{p-1}$, then

(i) $M_p(r, f) = O \left( \left( \log \frac{1}{1-r} \right)^{1/2 - 1/p} \right)$, as $r \to 1$.

(ii) $M_2(r, f) = O \left( \left( \log \frac{1}{1-r} \right)^{1/2 - 1/p} \right)$, as $r \to 1$.

**Theorem 1.5.** If $2 < p < \infty$ and $0 < \beta < 1/2 - 1/p$, then there exists a function $f \in G^p_{p-1}$ such that

$$\exp \left( \frac{1}{2\pi} \int_{-\pi}^\pi \log |f(re^{it})| \, dt \right) \neq o \left( \left( \log \frac{1}{1-r} \right)^{\beta} \right), \quad \text{as } r \to 1^-. $$

Since

$$\exp \left( \frac{1}{2\pi} \int_{-\pi}^\pi \log |f(re^{it})| \, dt \right) \leq M_2(r, f),$$

Theorem 1.5 shows that part (ii) of Theorem 1.4 is sharp in a very strong sense.
REMARK. Using Theorem 1.4 we can obtain an upper bound on the integral means $M_q(r, f)$, $2 < q < p$, of a function $f \in \mathcal{D}_{p-1}^p$. Indeed, if $q \in (2, p)$, then $q = p\lambda + 2(1 - \lambda)$, where $\lambda = (q - 2)/(p - 2) \in (0, 1)$. Consequently, using Theorem 1.4 and Hölder’s inequality with exponents $1/\lambda$ and $1/(1 - \lambda)$ we see that, if $f \in \mathcal{D}_{p-1}$ and $2 < q < p$, then

\[
M_q(r, f) = \left( \left( \log \frac{1}{1-r} \right)^\eta \right), \quad \text{as } r \to 1,
\]

where $\eta = \eta(p, q) = p\lambda/q + (p - 2)(1 - \lambda)/pq$ and $\lambda = (q - 2)/(p - 2)$.

In Section 4 we study properties of the sequences of zeros of non trivial $\mathcal{D}_{p-1}$-functions. If $0 < p \leq 2$ then $\mathcal{D}_{p-1}^p \subset H^p$ and hence, the sequence of zeros of a non-identically zero $\mathcal{D}_{p-1}$-function satisfies the Blaschke condition. This does not remain true for $p > 2$. Our main results about the sequences of zeros of functions $f$ in the space $\mathcal{D}_{p-1}$, $2 < p < \infty$, are stated in Theorem 1.6 and Theorem 1.7.

**THEOREM 1.6.** Suppose that $2 < p < \infty$ and let $f$ be a function which belongs to the space $\mathcal{D}_{p-1}^p$ with $f(0) \neq 0$. Let $\{z_k\}_{k=1}^\infty$ be the sequence zeros of $f$ ordered so that $|z_k| \leq |z_{k+1}|$ for all $k$. Then

\[
\frac{1}{N} \sum_{k=1}^N \frac{1}{|z_k|} = o \left( (\log N)^{1/2-1/p} \right), \quad \text{as } N \to \infty.
\]

From now on, if $f$ is a non-identically zero analytic function of zeros and $\{z_k\}_{k=1}^\infty$ is the sequence zeros of $f$ ordered so that $|z_k| \leq |z_{k+1}|$ for all $k$, we shall say that $\{z_k\}_{k=1}^\infty$ is the sequence of ordered zeros of $f$. Theorem 1.7 asserts that Theorem 1.6 is best possible.

**THEOREM 1.7.** If $2 < p < \infty$ and $0 < \beta < 1/2 - 1/p$, then there exists a function $f \in \mathcal{D}_{p-1}^p$ with $f(0) \neq 0$ such that if $\{z_k\}_{k=1}^\infty$ is the sequence of ordered zeros of $f$, then

\[
\prod_{k=1}^N \frac{1}{|z_k|} \neq o \left( (\log N)^\beta \right), \quad \text{as } N \to \infty.
\]

As a consequence of Theorem 1.6 and Theorem 1.7, we obtain the following result.

**COROLLARY 1.8.** If $2 \leq p < q < \infty$ then there exists a sequence $\{z_k\} \subset \Delta$ that is the sequence of zeros of a $\mathcal{D}_{q-1}$-function but is not the sequence of zeros of any $\mathcal{D}_{p-1}$-function.
Hence the situation in this setting is similar to that in the setting of Bergman spaces (see [18, Theorem 1]).

Next we shall get into the proofs of these and some other results. We shall be using the convention that $C_{p, \alpha, \ldots}$ denotes a positive constant which depends only upon the displayed parameters $p, \alpha, \ldots$ but is not necessarily the same at different occurrences.

2. Taylor coefficients of $\mathcal{D}^p_{p-1}$ functions.

We start by recalling the following useful result due to Mateljevic and Pavlovic [21] (see also [5, Lemma 3] and [22]) which will be basic in the proofs of Theorem 1.1 and Theorem 1.2.

**Lemma B.** Let $\alpha > 0$ and $p > 0$. There exists a constant $K$ that depends only on $p$ and $\alpha$ such that, if $\{a_n\}_{n=1}^{\infty}$ is a sequence of non-negative numbers, $t_n = \sum_{k \in I(n)} a_n$ $(n \geq 0)$, and $f(x) = \sum_{n=0}^{\infty} a_n x^{n-1} (x \in (0, 1))$, then

\[
K^{-1} \sum_{n=0}^{\infty} 2^{-n\alpha} t_n^p \leq \int_0^1 (1 - x)^{\alpha-1} f(x)^p \, dx \leq K \sum_{n=0}^{\infty} 2^{-n\alpha} t_n^p.
\]

**Proof of Theorem 1.1.** Take $p \in (0, \infty)$ and let $f$ be analytic in $\Delta$,

\[(12) \quad f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad z \in \Delta.
\]

Suppose that (4) holds. Using Lemma B and (4) we see that

\[
\int_\Delta |f'(z)|^p (1 - |z|^2)^{p-1} \, dA(z) \leq C_p \int_0^1 (1 - r)^{p-1} \left( \sum_{n=1}^{\infty} n |a_n| r^{n-1} \right)^p \, dr
\]

\[
\leq C_p \sum_{n=0}^{\infty} 2^{-np} \left( \sum_{k \in I(n)} k |a_k| \right)^p 
\leq C_p \sum_{n=0}^{\infty} 2^{-np} 2^{(n+1)p} \left( \sum_{k \in I(n)} |a_k| \right)^p
\leq C_p \sum_{n=0}^{\infty} \left( \sum_{k \in I(n)} |a_k| \right)^p < \infty.
\]

Hence, $f \in \mathcal{D}^p_{p-1}$ and the proof of (i) is finished.
Suppose now that $0 < p \leq 2$, $f$ is as in (12) and satisfies (5). Using the fact that $M_p(r, f') \leq M_2(r, f')$ for all $r \in (0, 1)$, making the change of variable $r^2 = s$ and using Lemma B, we obtain

$$
\int_\Delta |f'(z)|^p (1 - |z|^2)^{p-1} dA(z) = 2 \int_0^1 r (1 - r^2)^{p-1} M_p(r, f')^p
dr
\leq 2 \int_0^1 r (1 - r^2)^{p-1} M_2(r, f')^p
dr
\leq 2 \int_0^1 r (1 - r^2)^{p-1} \left( \sum_{n=1}^\infty n^2 |a_n|^2 r^{2n-2} \right)^{p/2}
dr
\leq C \int_0^1 (1 - s)^{p-1} \left( \sum_{n=1}^\infty n^2 |a_n|^2 s^{n-1} \right)^{p/2}
ds
\leq C_p \sum_{n=0}^\infty 2^{-np} \left( \sum_{k \in I(n)} k^2 |a_k|^2 \right)^{p/2}
\leq C_p \sum_{n=0}^\infty \left( \sum_{k \in I(n)} |a_k|^2 \right)^{p/2} < \infty.
$$

Hence, $f \in D_{p-1}^p$. This finishes the proof of (ii). \qed

Next we see that Proposition A can be deduced from Theorem 1.1 as announced.

PROOF OF PROPOSITION A. Let $f$ be an analytic function in $\Delta$ given by a power series with Hadamard gaps

$$
f(z) = \sum_{j=1}^\infty a_j z^{n_j} \quad \text{with} \quad \frac{n_{j+1}}{n_j} \geq \lambda > 1 \quad \text{for all} \quad j,
$$

and suppose that $\sum_{j=1}^\infty |a_j|^p < \infty$. Using the gap condition, we see that there are at most $C_\lambda = \log \lambda + 1$ of the $n_j$s in the set $I(n)$. Then there exists a constant $C_{\lambda, p} > 0$ such that

$$
\sum_{n=0}^\infty \left( \sum_{j \in I(n)} |a_j| \right)^p \leq C_{\lambda, p} \sum_{j=1}^\infty |a_j|^p < \infty,
$$

and consequently, using Theorem 1.1, we deduce that $f \in D_{p-1}^p$.

To prove the other implication suppose that $f$ is as in (13) and $f \in D_{p-1}^p$ for a certain $p > 0$. It is well known (see [38, Chapter V, Vol. I]) that there exist constants $A(\lambda, p)$ and $B(\lambda, p)$ such that

$$
A(\lambda, p) M_p^p(r, f') \leq M_2^p(r, f') \leq B(\lambda, p) M_2^p(r, f'), \quad 0 < r < 1.
$$
This and Lemma B give
\[ \int_{\Delta} |f'(z)|^p (1 - |z|^2)^{p-1} dA(z) = \int_0^1 r (1 - r^2)^{p-1} M_p^p(r, f') dr \]
\[ \geq A(\lambda, p) \int_0^1 r (1 - r^2)^{p-1} M_2^p(r, f') dr \]
\[ \geq A(\lambda, p) \int_0^1 r (1 - r^2)^{p-1} \left( \sum_{j=1}^{\infty} n_j^2 |a_j|^2 r^{2n_j-2} \right)^{p/2} dr \]
\[ \geq A(\lambda, p) \int_0^1 t (1 - t)^{p-1} \left( \sum_{j=1}^{\infty} n_j^2 |a_j|^2 t^{j-1} \right)^{p/2} dt \]
\[ \geq C_p A(\lambda, p) \sum_{n=0}^{\infty} 2^{-np} \left( \sum_{n_j \in I(n)} n_j^2 |a_j|^2 \right)^{p/2} \]
\[ \geq C_p A(\lambda, p) \sum_{n=0}^{\infty} 2^{-np} 2^{np} \left( \sum_{n_j \in I(n)} |a_j| \right)^p \geq C_{\lambda, p} A(\lambda, p) \sum_{j=0}^{\infty} |a_j|^p. \]

The last inequality is obvious if \( p \geq 1 \) and, in the case \( 0 < p < 1 \), follows again using the fact that there are at most \( C_\lambda = \log_2 2 + 1 \) of the \( n_j \)'s in the set \( I(n) \). Thus, we have \( \sum_{j=0}^{\infty} |a_j|^p < \infty \). This finishes the proof. \( \square \)

PROOF OF THEOREM 1.2. Suppose that \( 2 \leq p < \infty \) and \( f \in D^p_{p-1} \),
\[ f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad z \in \Delta. \]

Using Lemma B, bearing in mind that \( k = 2^n \) if \( k \in I(n) \), making a change of variable, and using that since \( p \geq 2 \), \( M_2(r, f') \leq M_p(r, f') \), we obtain
\[ \sum_{n=1}^{\infty} \left( \sum_{k \in I(n)} |a_k|^2 \right)^{p/2} \leq \sum_{n=1}^{\infty} 2^{-np} \left( \sum_{k \in I(n)} k^2 |a_k|^2 \right)^{p/2} \]
\[ \leq C_p \int_0^1 (1 - t)^{p-1} \left( \sum_{n=1}^{\infty} n^2 |a_n|^2 t^{n-1} \right)^{p/2} dt \]
\[ \leq C_p \int_0^1 (1 - r^2)^{p-1} \left( \sum_{n=1}^{\infty} n^2 |a_n|^2 r^{2n-2} \right)^{p/2} dt \]
\[ \leq C_p \int_0^1 (1 - r)^{p-1} M_p(r, f')^p < \infty. \]
3. Growth properties of $\mathcal{D}^p_{p-1}$-functions

In this section we are mainly interested in obtaining sharp estimates on the growth of functions $f$ in the spaces $\mathcal{D}^p_{p-1}$ ($2 < p < \infty$).

3.1. Integral means estimates Let us start with estimates on the growth of the maximum modulus $M_\infty(r, f)$. We can prove the following result.

**Theorem 3.1.** Let $f$ be an analytic function in $\Delta$. If $f \in \mathcal{D}^p_{p-1}$, $0 < p < \infty$, then

$$M_\infty(r, f) = o\left(\frac{1}{(1 - r)^{1/p}}\right), \quad \text{as } r \to 1^-.$$

**Proof.** Let $f \in \mathcal{D}^p_{p-1}$ and $z \in \Delta$. Let $D(z)$ denote the open disc

$$\left\{ w \in \mathbb{C} : |z - w| < \frac{1 - |z|^2}{2} \right\}.$$

Clearly, $D(z) \subset \Delta$. Since the function $z \to |f'(z)|^p$ is subharmonic in $\Delta$, we have

$$|f'(z)|^p \leq \frac{C}{|D(z)|} \int_{D(z)} |f'(\omega)|^p dA(\omega) \leq \frac{C}{(1 - |z|^2)^2} \int_{D(z)} |f'(\omega)|^p dA(\omega).$$

It is clear that $(1 - |z|^2) \sim (1 - |\omega|^2)$, $\omega \in D(z)$, $z \in \Delta$. Using this and (15) we obtain

$$|f'(z)|^p \leq \frac{C_p}{(1 - |z|^2)^2} \int_{D(z)} \left[ \frac{1 - |\omega|^2}{1 - |z|^2} \right]^{p-1} |f'(\omega)|^p dA(\omega)$$

$$= \frac{C_p}{(1 - |z|^2)^{p+1}} \int_{D(z)} (1 - |\omega|)^{p-1} |f'(\omega)|^p dA(\omega).$$

On the other hand, since $f \in \mathcal{D}^p_{p-1}$, it follows that

$$\int_{D(z)} (1 - |\omega|)^{p-1} |f'(\omega)|^p dA(\omega) = o(1), \quad \text{as } |z| \to 1^-,$$

which, with (16), implies

$$M_\infty(r, f') = o\left(\frac{1}{(1 - r)^{1+1/p}}\right), \quad \text{as } r \to 1^-,$$

and (14) follows by integration. □
REMARK. We observe that for any $p \in (0, \infty)$, the exponent $1/p$ in (14) is the best possible. Moreover, if we take

$$f_{p, \beta}(z) = (1 - z)^{-1/p} \left( \log \frac{2}{1 - z} \right)^{-\beta}, \quad z \in \Delta,$$

with $\beta > \frac{1}{p}$ then, as we noticed in Lemma 1.3, $f_{p, \beta} \in \mathcal{D}_p^{p-1}$ and it is easy to see that

$$M_\infty(r, f) \approx (1 - r)^{-1/p} \left( \log \frac{1}{1 - r} \right)^{-\beta}, \quad 0 < r < 1.$$ 

So condition (14) in Theorem 3.1 cannot be substituted by the condition

$$M_\infty(r, f) = o \left( \frac{1}{(1 - r)^{1/p}(\log(1/(1 - r)))^{1/p + \varepsilon}} \right), \quad \text{as } r \to 1^-, $$

for any $\varepsilon > 0$.

Now we turn to the proofs of Theorem 1.4 and Theorem 1.5.

PROOF OF THEOREM 1.4. Suppose that $2 < p < \infty$ and $f \in \mathcal{D}_p^{p-1}$. Then

$$\lim_{r \to 1^-} \int_r^1 (1 - s)^{p-1} M_\rho(s, f') \, ds = 0. \quad (18)$$

Since $M_\rho(s, f')$ is an increasing function of $s$

$$\int_r^1 (1 - s)^{p-1} M_\rho(s, f') \, ds \geq M_\rho(r, f') \int_r^1 (1 - s)^{p-1} \, ds \geq C_\rho M_\rho(r, f')(1 - r)^p,$$

which, together with (18), yields

$$M_\rho(r, f') = o \left( (1 - r)^{-1} \right), \quad \text{as } r \to 1^- \quad (19)$$

which, using Minkowski’s integral inequality, implies (7).

Using (19) and the fact that for any fixed $r$ with $0 < r < 1$ the integral means $M_\rho(r, f')$ increase with $p$, we deduce that

$$I_2(r, f') = o \left( (1 - r)^{-2} \right), \quad \text{as } r \to 1^- \quad \text{and then using the well-known inequality (see [26, pages 125–126])}$$

$$\frac{d^2}{dr^2} \left( I_2(r, f) \right) \leq 4I_2(r, f'), \quad 0 < r < 1,$$
we obtain
\[ \frac{d^2}{dr^2} \left( I_2(r, f) \right) = o \left( (1 - r)^{-2} \right) \quad \text{as } r \to 1^- , \]
which, integrating twice, gives
\[ M_2(r, f) = o \left( \log(1/(1 - r))^{1/2} \right) , \quad \text{as } r \to 1 . \]

This is worse than (8). To obtain this we use Theorem 1.2.

Say that \( f(z) = \sum_{\infty}^{\infty} a_n z^n , \quad (z \in \Delta) \). Suppose, without loss of generality that \( a_0 = 0 \). Using Hölder’s inequality with the exponents \( p/2 \) and \( p/(p - 2) \) and Theorem 1.2, we obtain
\[
M_2(r, f)^2 = \sum_{n=1}^{\infty} |a_n|^2 r^{2n} = \sum_{n=0}^{\infty} \sum_{k \in I(n)} |a_k|^2 r^{2k} \leq \sum_{n=0}^{\infty} r^{2n+1} \left( \sum_{k \in I(n)} |a_k|^2 \right) \leq \left[ \sum_{n=0}^{\infty} \left( \sum_{k \in I(n)} |a_k|^2 \right)^{p/2} \right]^{2/p} \left[ \sum_{n=0}^{\infty} r^{2n+1} p/(p-2) \right]^{1-2/p} \leq C_{f, p} \left( \log \frac{1}{1 - r} \right)^{1-2/p} .
\]

Since
\[
\exp \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |f(re^{i\theta})| \, d\theta \right) \leq M_2(r, f) , \quad 0 < r < 1 ,
\]
we trivially have the following result.

**Corollary 3.2.** If \( 2 < p < \infty \) and \( f \in \mathcal{D}_{p-1}^p \), then
\[
\exp \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |f(re^{i\theta})| \, d\theta \right) = o \left( \left( \log \frac{1}{1 - r} \right)^{1/2-1/p} \right) , \quad \text{as } r \to 1^- .
\]

Theorem 1.5 shows that Corollary 3.2 and the estimate (8) are sharp in a very strong sense. The following lemma, whose proof is simple and is omitted, will be used in the proof of Theorem 1.5.

**Lemma 3.3.** Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) be an analytic function in \( \Delta \). If \( 0 < \beta \leq 1 \) and \( \sum_{k=0}^{N} |a_k|^2 \approx (\log N)^\beta \), as \( N \to \infty \), then \( I_2(r, f) \approx (\log (1 - r)^{-1})^\beta \) as \( r \to 1^- \).

We make use of the technique introduced by Ullrich in [32]. Let us start introducing some notation.

Let \( \omega = [0, 1]^N \) and \( \omega_1, \omega_2, \ldots \) be ‘the coordinate functions’ \( \omega_j : \Omega \to [0, 1] \). Let \( d\omega \) denote the product measure \( \Omega \) derived from the Lebesgue measure on \([0, 1] \). Now
\( \omega_1, \omega_2, \ldots \) are the Steinhaus variables (independent, identically distributed random variables uniformly distributed on \([0, 1]\)). Note that \( \{e^{2\pi i \omega_j}\}_{j=1}^{\infty} \) is an orthonormal set in \( L^2(\Omega) \), hence, if \( \sum_{j=1}^{\infty} |a_j|^2 < \infty \), then \( \sum_{j=1}^{\infty} a_j e^{2\pi i \omega_j} \) is a well defined element of \( L^2(\Omega) \) with \( L^2 \)-norm \( \left( \sum_{j=1}^{\infty} |a_j|^2 \right)^{1/2} \). The following theorem is [32, Theorem 1].

**THEOREM C.** There exists \( C > 0 \) such that for any sequence of complex numbers \( \{a_j\}_{j=1}^{\infty} \) with \( \sum_{j=1}^{\infty} |a_j|^2 < \infty \), we have

\[
\exp \left[ \int_{\Omega} \log \left| \sum_{j=1}^{\infty} a_j e^{2\pi i \omega_j} \right| \, d\omega \right] \geq C \left( \sum_{j=1}^{\infty} |a_j|^2 \right)^{1/2}.
\]

**PROOF OF THEOREM 1.5.** Suppose that \( 2 < p < \infty \) and \( 0 < \beta < 1/2 - 1/p \). Set \( \varepsilon = 1/2 - 1/p - \beta \), hence, \( \varepsilon > 0 \). We define the sequence \( \{b_j\}_{j=1}^{\infty} \) as \( b_j = j^{-1/p-\varepsilon} \), \( j = 1, 2, \ldots \). Now, for every \( \omega \in \Omega \) we define

\[
f_\omega(z) = \sum_{j=1}^{\infty} b_j e^{2\pi i \omega_j} z^j = \sum_{k=1}^{\infty} a_{k,\omega} z^k, \quad z \in \Delta.
\]

Since \( \sum_{j=1}^{\infty} |b_j|^p < \infty \), using Proposition A we deduce that \( f_\omega \in \mathcal{D}_p^{p-1} \) for every \( \omega \in \Omega \).

We will see that for a.e. \( \omega \in \Omega \)

\[
\exp \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |f_\omega(re^{it})| \, dt \right) \neq o \left( \left( \log(1/(1-r)) \right)^{\beta} \right), \quad \text{as } r \to 1^-.
\]

This will finish the proof.

Suppose that (21) is false. Then there exists a measurable set \( E \subset \Omega \) with positive measure and such that for all \( \omega \in E \)

\[
\exp \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |f_\omega(re^{it})| \, dt \right) = o \left( \left( \log(1/(1-r)) \right)^{\beta} \right), \quad \text{as } r \to 1^-.
\]

This is equivalent to saying that

\[
\lim_{r \to 1^-} \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \left[ \frac{|f_\omega(re^{it})|}{(\log(1/(1-r)))^{\beta}} \right] \, dt = -\infty, \quad \omega \in E.
\]

On the other hand,

\[
\left( \sum_{j=1}^{N} |b_j|^2 \right)^{1/2} \sim \left( \sum_{j=1}^{N} \frac{1}{j^{2/p+2\varepsilon}} \right)^{1/2} \sim \left( \int_{1}^{N} \frac{1}{x^{2/p+2\varepsilon}} \, dx \right)^{1/2} \sim N^{1/2 - 1/p - \varepsilon}, \quad \text{as } N \to \infty.
\]
Thus, there exist $C > 0$ and $N_0 > 0$ such that

\[(24) \quad \left( \sum_{k=1}^{N} |a_{k,\omega}|^2 \right)^{1/2} \leq C \left( \log N \right)^{1/2-1/p-\varepsilon}, \quad N \geq N_0. \]

Using (24) and Lemma 3.3, we deduce that

\[M_2(r, f_\omega) = \int_{\Omega} (f_\omega(r e^{it}))^2 \leq C \left[ \log \frac{1}{1-r} \right]^{1/2-1/p-\varepsilon}, \quad 0 < r < 1, \quad \omega \in \Omega, \]

which implies that for $0 < r < 1$ and $\omega \in \Omega$,

\[(25) \quad \exp \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |f_\omega(re^{it})| \, dt \right) \leq C \left[ \log \frac{1}{1-r} \right]^{1/2-1/p-\varepsilon}. \]

From this we deduce as in (23), that there exists $C > 0$ such that

\[(26) \quad \int_{-\pi}^{\pi} \log \left[ \frac{|f_\omega(re^{it})|}{\left( \log(1/(1-r)) \right)^{\theta}} \right] \, dt \leq C, \quad 0 < r < 1, \quad \omega \in \Omega. \]

Bearing in mind that $E$ has positive measure, (26) and (23) imply

\[(27) \quad \lim_{r \to 1^-} \int_{\Omega} \left[ \int_{-\pi}^{\pi} \log \left( \frac{|f_\omega(re^{it})|}{\left( \log(1/(1-r)) \right)^{\theta}} \right) \, dt \right] \, d\omega = -\infty. \]

Let $N = 1, 2, \ldots$, let $\Omega_N = [0, 1]^N$ and $m_N$ be the Lebesgue measure on $\Omega_N$. Observe now that, for any $N$, we have

\[\int_{\Omega_N} \log |f_\omega(re^{it})| \, dm_N(\omega) \]

\[= \int_{0}^{1} \cdots \int_{0}^{1} \left[ \sum_{j=1}^{N} b_j r^{2j} e^{i2\pi \omega_j + 2i t} + \sum_{j=N+1}^{\infty} b_j r^{2j} e^{i2\pi \omega_j + 2i t} \right] \, d\omega_1 \, d\omega_2 \cdots \, d\omega_N \]

\[= \int_{0}^{1} \cdots \int_{0}^{1} \left[ \sum_{j=1}^{N} b_j r^{2j} e^{i2\pi \omega_j} + \sum_{j=N+1}^{\infty} b_j r^{2j} e^{i2\pi \omega_j} \right] \, d\omega_1 \, d\omega_2 \cdots \, d\omega_N, \text{ a.s.} \]

Letting $N$ tend to $\infty$, we deduce that $\int_{\Omega} \log |f_\omega(re^{it})| \, d\omega$ is independent of $t$. Then using (27) and Fubini's Theorem we obtain

\[(28) \quad \lim_{r \to 1^-} \int_{\Omega} \log \left( \frac{|f_\omega(r)|}{\left( \log(1/(1-r)) \right)^{\theta}} \right) \, d\omega = -\infty. \]
However, if we set \( r_N = 1 - 1/2^N, \ N = 1, 2, \ldots \), by Theorem C and the inequality
\[
e^{-1} \leq r_N^{2j} \leq r_N^{2j}, \quad 1 \leq j \leq N,
\]
we deduce that
\[
\exp \left[ \int_{\Omega} \log |f_\omega(r_N)| \, d\omega \right] = \exp \left[ \int_{\Omega} \log \left( \sum_{j=1}^{\infty} b_j e^{2\pi i \omega} r_N^{2j} \right) \right] \geq C \left( \sum_{j=1}^{N} |b_j|^2 \right)^{1/2} \geq C \left( \sum_{j=1}^{N} \frac{1}{j^{2/p+2\varepsilon}} \right)^{1/2} \geq C \frac{1}{N^{1/p+\varepsilon-1/2}} \geq C \left( \log \frac{1}{1-r_N} \right)^{1/2-1/p-\varepsilon} = C \left( \log \frac{1}{1-r_N} \right) \beta,
\]
which implies
\[
\int_{\Omega} \log \left( \frac{|f_\omega(r_N)|}{(\log(1-r_N))^{-1}} \right)^\beta \, d\omega \geq \log C, \quad \text{for all } N,
\]
which contradicts (28). Consequently, (21) is true and the proof is finished. \( \square \)

3.2. Radial growth of \( D_p^{-1} \)-functions

In this section we obtain some estimates on the radial growth of \( D_p^{-1} \)-functions. If \( 0 < p \leq 2 \) and \( f \in D_p^{-1} \), then \( f \in H_p \) and so \( f \) has nontangential limit a.e. \( \mathbb{T} \). Therefore, we have: If \( 0 < p \leq 2 \) and \( f \in D_p^{-1} \), then \( |f(re^{i\theta})| = O(1) \), as \( r \to 1^- \) for a.e. \( e^{i\theta} \in \partial \Delta \).

Zygmund proved in [37] that if \( f \) is an analytic function in \( \Delta \), then
\[
\int_0^\pi |f'(re^{i\theta})| \, d\rho = o \left( \left( \log \frac{1}{1-r} \right)^{1/2} \right), \quad \text{as } r \to 1^-.
\]
for almost every point \( e^{i\theta} \) in the Fatou set of \( f \), \( F_f \), which consists of those \( e^{i\theta} \in \mathbb{T} \) such that \( f \) has finite nontangential limit at \( e^{i\theta} \). Obviously, (29) implies
\[
|f(re^{i\theta})| = o \left( \left( \log \frac{1}{1-r} \right)^{1/2} \right), \quad \text{as } r \to 1^-.
\]

If \( 2 < p < \infty \), there are functions \( f \in D_p^{-1} \) such that \( F_f \) has Lebesgue measure equal to zero. Indeed, an analytic function \( f \) given by a power series with Hadamard gaps whose sequence of Taylor coefficients \( \{a_k\} \) belongs to \( l^p \setminus l^2 \), is a \( D_p^{-1} \)-function by Proposition A and \( F_f \) has null Lebesgue measure (see [38, Chapter V]). In spite of this, we can prove the following result for \( D_p^{-1} \)-functions.
THEOREM 3.4. If $2 < p < \infty$ and $f \in \mathcal{D}_p_{-1}$, then

\begin{equation}
|f(re^{it})| = o \left[ \left( \log \frac{1}{1 - r} \right)^{1 - 1/p} \right], \quad \text{as } r \to 1^- \text{ for a.e. } e^{it} \in \partial \Delta.
\end{equation}

This is better that the a.e. estimate which can be deduced from (17).

PROOF OF THEOREM 3.4. Let $p$ and $f$ be as in the statement of the theorem. Then

\[
\int_{-\pi}^{\pi} \left( \int_{0}^{1} (1 - r)^{p-1} |f'(re^{it})|^p \, dt \right) \, dr < \infty,
\]

and it follows that the set $\mathcal{A}$ of points $e^{it} \in \partial \Delta$ for which

\[
\int_{0}^{1} (1 - r)^{p-1} |f'(re^{it})|^p \, dt < \infty,
\]

has Lebesgue measure equal to $2\pi$.

Take and fix $e^{it} \in \mathcal{A}$. Take also $\varepsilon > 0$. Then there exists $r_\varepsilon \in (0, 1)$ such that

\begin{equation}
\int_{r_\varepsilon}^{1} (1 - s)^{p-1} |f'(se^{it})|^p \, ds < \varepsilon.
\end{equation}

Using (32) and Hölder's inequality with exponents $p$ and $p/(p - 1)$, we obtain for $r_\varepsilon < r < 1$,

\begin{equation}
\int_{0}^{r} |f'(se^{it})| \, ds = \int_{0}^{r} |f'(se^{it})| \, ds + \int_{r}^{r_\varepsilon} |f'(se^{it})| \, ds \leq C_{f,\varepsilon} + \int_{r_\varepsilon}^{r} \frac{(1 - s)^{1-1/p}}{(1 - s)^{1-1/p}} |f'(se^{it})| \, ds \leq C_{f,\varepsilon} + \varepsilon \left( \log \frac{1}{1 - r} \right)^{1 - 1/p}.
\end{equation}

Consequently, we have proved that

\[
\limsup_{r \to 1} \left( \log \frac{1}{1 - r} \right)^{1/p - 1} \int_{0}^{r} |f'(se^{it})| \, ds \leq \varepsilon.
\]

Since $\varepsilon > 0$ and $e^{it} \in A$ are arbitrary, we have

\[
\int_{0}^{r} |f'(se^{it})| \, ds = o \left[ \left( \log \frac{1}{1 - r} \right)^{1 - 1/p} \right], \quad \text{as } r \to 1^-,
\]

for all $e^{it} \in A$. This implies that (31) holds for all $e^{it} \in A$, which has Lebesgue measure equal to $2\pi$. This finishes the proof.
We do not know whether or not the exponent $1 - 1/p$ in Theorem 3.4 is sharp but we know that it cannot be substituted by any exponent smaller than $1/2 - 1/p$. Indeed, we can prove the following result.

**Theorem 3.5.** If $2 < p < \infty$, then there exists a function $f \in \mathcal{D}_p^{p-1}$ such that

$$
\lim_{r \to 1^-} \frac{|f(re^{it})|}{\left( \log \frac{1}{1-r} \right)^{1/2-1/p} \left( \log \log \frac{1}{1-r} \right)^{-1}} = \infty, \quad \text{for a.e. } e^{it} \in \partial \Delta.
$$

**Proof.** Take $p > 2$. Define

$$a_k = \frac{1}{k^{1/p} \log 2k}, \quad k = 1, 2, \ldots, \quad \text{and} \quad f(z) = \sum_{k=1}^{\infty} a_k z^{2k}, \quad z \in \Delta.$$

Since $\sum_{k=1}^{\infty} |a_k|^p < \infty$, by Proposition A, we have that $f \in \mathcal{D}_p^{p-1}$.

On the other hand,

$$
\left( \sum_{k=1}^{N} |a_k|^2 \right)^{1/2} = \left( \sum_{k=1}^{N} \frac{1}{k^{2/p} \log^2 2k} \right)^{1/2} \sim \left( \int_{1}^{N} \frac{1}{x^{2/p} \log^2 2x} \, dx \right)^{1/2} \sim \frac{N^{1/2-1/p}}{\log N}, \quad \text{as } N \to \infty,
$$

and then it is easy to see that

$$
M_2(r, f) = I_2(r, f)^{1/2} \sim \left( \frac{\log \frac{1}{1-r}}{\log \log \frac{1}{1-r}} \right)^{1/2-1/p}, \quad \text{as } r \to 1^-.
$$

Now, by the law of the iterated logarithm for lacunary series (see [35]) we have that

$$
\lim_{r \to 1^-} \frac{|f(re^{it})|}{\left[ I_2(r, f) \log \log I_2(r, f) \right]^{1/2}} = 1, \quad \text{for a.e. } e^{it} \in \partial \Delta.
$$

Now we observe that (36) and (35) imply (34). This finishes the proof. \qed

### 4. Zeros of $\mathcal{D}_p^{p-1}$ functions

#### 4.1. Products of the zeros of $\mathcal{D}_p^{p-1}$ functions

We start by recalling the following result due to Horowitz, (see [18, page 65]).
**Lemma D.** Let $f$ be an analytic function in $\Delta$ with $f(0) \neq 0$ and let $\{z_k\}$ be the sequence of ordered zeros of $f$. If $0 < p < \infty$, $0 \leq r < 1$, and $N$ is a positive integer, then

$$|f(0)|^p \prod_{k=1}^{N} \frac{r^p}{|z_k|^p} \leq M_p(r, f)^p. \quad (37)$$

This lemma and the estimates for the integral means of $D_p^p$-functions obtained in Section 3.1 are the basic ingredients in the proofs of Theorem 1.6 and Theorem 1.7. This method was used by Horowitz in [18] for the Bergman spaces and later by the first author of this paper, Nowak, and Waniurski in [15] for the Bloch space $B$ and some other related spaces.

**Proof of Theorem 1.6.** Let $p$, $f$, and $\{z_k\}_{k=1}^{\infty}$ be as in the statement of Theorem 1.6. Using Theorem 1.4, we see that $f$ satisfies (8) and using Lemma D with $p = 2$, we deduce that

$$\prod_{k=1}^{N} \frac{r}{|z_k|} \leq C M_2(r, f) \leq C \left( \log \frac{1}{1-r} \right)^{1/2-1/p}, \quad \text{if } r \text{ is close enough to } 1. \quad (38)$$

Now, taking $r = 1 - 1/N$ with $N$ big enough in (38) and bearing in mind that $(1 - 1/N)^N > 1/2e$, we deduce that

$$\prod_{k=1}^{N} \frac{1}{|z_k|} \leq C (\log N)^{1/2-1/p}. \quad (39)$$

This finishes the proof. \hspace{1cm} \Box

Our next objective is to prove Theorem 1.7 which asserts that Theorem 1.6 is sharp. We start recalling some notation and facts from Nevanlinna theory (see [16, 23] or [31]) which will be needed in our proof.

Let $f$ be a non-constant analytic function in $\Delta$. For any $a \in \mathbb{C}$ and $0 < r < 1$, we denote by $n(r, a, f)$ the number of zeros $f - a$ in the disc $|z| \leq r$, where each zero is counted according to its multiplicity. We define also

$$N(r, a, f) \overset{\text{def}}{=} \int_{0}^{r} \frac{n(t, a, f) - n(0, a, f)}{t} \, dt + n(0, a, f) \log r, \quad 0 < r < 1. \quad (40)$$

For simplicity, we shall write $n(r, f) = n(r, 0, f)$, $N(r, f) = N(r, 0, f)$. The Nevanlinna characteristic function $T(r, f)$ is defined by

$$T(r, f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ |f(0, e^{i\theta})| \, d\theta, \quad 0 < r < 1.$$
The proximity function \( m(r, a, f) \) is given by
\[
m(r, a, f) \overset{\text{def}}{=} \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ \frac{1}{|f(re^{it}) - a|} \, dt, \quad 0 < r < 1.
\]

Now we can state the First Fundamental Theorem of Nevanlinna.

**THEOREM E.** Let \( f \) be a non-constant analytic function in \( \Delta \). Then
\[
m(r, a, f) + N(r, a, f) = T(r, f) + O(1), \quad \text{as } r \to 1^-.
\]

for every \( a \in \mathbb{C} \).

Now we can prove the following result.

**PROPOSITION 4.1.** If \( 2 < p < \infty \) and \( f \) is a non-constant \( \mathcal{D}_{p-1} \) function, then
\[
n(r, a, f) = O\left( \frac{1}{1-r} \log \log \frac{1}{1-r} \right), \quad \text{as } r \to 1^-, \text{ for all } a \in \mathbb{C}.
\]

**PROOF.** Using the arithmetic-geometric mean inequality we obtain
\[
T(r, f) \leq \frac{1}{4\pi} \int_{-\pi}^{\pi} \log \left( |f(re^{it})|^2 + 1 \right) \, dt
\]
\[
\leq \frac{1}{2} \log \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( |f(re^{it})|^2 + 1 \right) \, dt \right) \leq \frac{1}{2} \log \left( I_2(r, f) + 1 \right),
\]
which, with part (ii) of Theorem 1.4, gives
\[
T(r, f) = O\left( \log \log \frac{1}{1-r} \right), \quad \text{as } r \to 1^-.
\]
Using Theorem E, we deduce that
\[
N(r, a, f) = O\left( \log \log \frac{1}{1-r} \right), \quad \text{as } r \to 1^-, \text{ for all } a \in \mathbb{C}.
\]

Now, it is well known (see [2, page 22]) that this implies (41). \( \square \)

Now, we can proceed with the proof of Theorem 1.7.

**PROOF OF THEOREM 1.7.** Take \( p \) and \( \beta \) with \( 2 < p < \infty \) and \( 0 < \beta < 1/2 - 1/p \). Take \( f \in \mathcal{D}_{p-1}^\beta \) with \( f(0) \neq 0 \) and
\[
\exp \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |f(re^{it})| \, dt \right) \neq o\left( \left( \log \frac{1}{1-r} \right)^\beta \right), \quad \text{as } r \to 1^-,
\]
such a function exists by Theorem 1.5. Using (44) we see that there exist a sequence \( \{r_j\}_{j=1}^{\infty} \subset (0, 1) \) with \( r_j \uparrow 1 \) and a positive constant \( C \) (independent of \( j \)), such that

\[
\exp \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |f(\alpha e^{it})| \, dt \right) \geq C \left( \log \frac{1}{1-r_j} \right)^{\beta}, \quad j = 1, 2, \ldots.
\]

We shall write \( n(r) \) instead of \( n(r, f) \) for simplicity. Using Jensen's formula (see [1, page 206]) and (45) we deduce that

\[
|f(0)| \prod_{k=1}^{n(r_j)} \frac{r_j}{|z_k|} \geq C \left( \log \frac{1}{1-r_j} \right)^{\beta}, \quad j = 1, 2, \ldots,
\]

which implies that

\[
n(r_j) \to \infty, \quad \text{as } j \to \infty.
\]

On the other hand, Proposition 4.1 implies that there exists \( C > 0 \) such that

\[
n(r) \leq C \frac{1}{1-r} \log \log \frac{1}{1-r}, \quad \text{if } r \text{ is sufficiently close to } 1.
\]

This implies that

\[
\log n(r) \leq C \log \frac{1}{1-r}, \quad \text{if } r \text{ is sufficiently close to } 1,
\]

which, together with (46), shows that there exists \( j_0 \in \mathbb{N} \) such that for every \( j \geq j_0 \)

\[
|f(0)| \prod_{k=1}^{n(r_j)} \frac{r_j}{|z_k|} \geq C \left[ \log n(r_j) \right]^{\beta}.
\]

This finishes the proof. \( \Box \)

### 4.2. A substitute of Blaschke condition

If \( 2 < p < \infty \) the sequence \( \{z_k\} \) of ordered zeros of a non trivial \( \mathcal{D}^p_{p-1} \) function need not satisfy the Blaschke condition. Indeed, the Blaschke condition is equivalent to saying that \( \prod_{n=1}^{N} (1/|z_n|) = O(1) \) and we have seen that this is not always true. Using Theorem 1.6 and arguing exactly as in the proof of [15, Theorem 5] we can prove the following result.

**Theorem 4.2.** Let \( 2 < p < \infty \) and \( f \in \mathcal{D}^p_{p-1} \) with \( f \not\equiv 0 \). Let \( \{z_k\}_{k=1}^{\infty} \) be the sequence of zeros of \( f \). Then

\[
\sum_{|z_k| > 1^{-1/\epsilon}} (1 - |z_k|) \left( \log \log \frac{1}{1 - |z_k|} \right)^{-\alpha} < \infty
\]

for all \( \alpha > 1 \).
Next, we shall prove that the condition $\alpha > 1$ is needed in Theorem 4.2.

**THEOREM 4.3.** Let $2 < p < \infty$. Then there exists a function $f \in \mathcal{D}_p^p$ with $f \neq 0$, whose sequence of zeros $\{z_k\}_{k=1}^\infty$ satisfies

$$\sum_{|z_k| > 1/1/e} (1 - |z_k|) \left( \log \log \frac{1}{1 - |z_k|} \right)^{-1} = \infty. \quad (49)$$

**PROOF.** Set $g(z) = \sum_{k=1}^\infty k^{-(p+2)/4} z^{2^k}, z \in \Delta$. Since $g$ is given by a power series with Hadamard gaps and $\sum_{k=1}^\infty k^{-(p+2)/4} < \infty$, it follows that $g \in \mathcal{D}_p^p$.

We shall follow the argument of the proof of [15, Theorem 6]. Set

$$r_n = 1 - 2^{-n}, \quad n = 1, 2, 3, \ldots. \quad (50)$$

It is easy to see that, for all sufficiently large $n$, $I_2(r_n, g) \geq C n^{1/2 - 1/p}$, which, since $\log(1/(1 - r_n)) = n \log 2$, implies that

$$I_2(r_n, g) \geq C \left( \log \frac{1}{1 - r_n} \right)^{1/2 - 1/p} \quad \text{if } n \text{ is sufficiently large.} \quad (51)$$

Now, since $\log(1/(1 - r_n)) \sim \log(1/(1 - r_{n+1}))$, as $n \to \infty$, and since $I_2(r, g)$ and $(\log(1/(1 - r)))^{1/2 - 1/p}$ are increasing functions of $r$, we deduce

$$I_2(r, g) \geq C \left( \log \frac{1}{1 - r} \right)^{1/2 - 1/p}, \quad (52)$$

if $r$ is sufficiently close to 1.

Using this and arguing as in [15, page 126] we deduce that there exist a complex number $a$ with $g(0) \neq a$, a positive constant $\beta$, and a number $r_0 \in (0, 1)$ such that

$$N(r, a, g) \geq \beta \log \log \frac{1}{1 - r}, \quad r \in (r_0, 1). \quad (53)$$

Take such an $a \in \mathbb{C}$ and set $f(z) = g(z) - a, z \in \Delta$. Then $f \in \mathcal{D}_p^p$ and $f(0) \neq 0$. Also (53) can be written as

$$N(r, f) \geq \beta \log \log \frac{1}{1 - r}, \quad r \in (r_0, 1). \quad (54)$$

Let $\{z_n\}$ be the sequence of zeros of $f$. Using Proposition 4.1 and arguing as in [15, page 127], we obtain (49). \qed
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