On Direct and Inverse Interpolation by Divided Differences.

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1. *Introduction.*

The general interpolation series, originated by Newton, has been studied mainly for its algebraic interest, only the special case of equidistant data being developed on the practical side. This is justified by the simplicity of this case, and by the numerous problems for which it suffices, but it may lead to undue simplification of data and to restrictions on experimental and computative methods. Thus tables of functions which are not in common use, or which are carried to many places, must often be limited to relatively few entries, and these might conceivably be not in arithmetical progression, with advantage both of easier tabulation and of more accurate interpolation.* Data from experiment or statistics, again, are often fitted to a parabolic curve of arbitrarily chosen degree, and on rather inadequate grounds. The formation of a difference-table not only avoids the suppression of the original data, but supplies at a glance a useful analysis of them—indicating their consistency and regularity, showing with what accuracy a parabolic curve can represent them, and supplying its expression with minimum labour. For direct interpolation to new points Lagrange's formula, the usual alternative, fails in this respect and, when applied to unfamiliar data, is very apt to mislead. It is wasteful of labour and more liable to error, and cannot easily be extended to include fresh terms.

As the only discussion of divided differences known to me (included in T. N. Thiele's *Interpolationsrechnung*) † is incomplete

* *Cf.* the rules of Tchebichev, Gauss, etc., on choice of points for interpolation and integration. Also Professor Steggall's suggestions for economy of entries in ordinary tables (*Napier Tercentenary Memorial Volume* (1915), p. 319).

† Teubner (1909). Some early suggestions were contained in Gauss' Lectures, published by Encke (*Berliner Astron. Jahrb.* (1830); Abhandlungen I.); these are reproduced in the *Encycl. des Sc. Maths.* (T. 1, Vol. 4 pp. 130–7) See also *Encyc. Brit.*, *Interpolation.*

and further development is needed, I propose to give a short discussion of the necessary theory and to illustrate the working by examples. My results were arrived at independently of Thiele's work, but they are in generally close agreement, and I have adopted some of his ideas. It will be seen that the theory as such has value both algebraically and in relation to the differential calculus, while the remarkable flexibility of the series throws much light on the use of the ordinary difference-table, and gives divided differences the preference on occasion even when the data are equidistant.

2. *Parabolic Interpolation.*

Assuming that the function which is defined by the given values f_r at the point x_r $(r=0, 1, 2 \ldots n)$ can be represented with sufficient accuracy by a power-series in x (a simple transformation such as taking logarithms or reciprocals of x_r sometimes suffices to ensure this), we may write

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \ldots$$
$$+ a_n(x - x_0)(x - x_1)\ldots(x - x_{n-1}) + \ldots,$$

where $a_0, a_1 \ldots a_n$ are determinable in $f_0, f_1 \ldots f_n$ on condition that the remainder of the series be negligible. A fresh point x_{n+1} would enable us to add another term $a_{n+1}(x - x_0)(x - x_1)\ldots(x - x_n)$ without affecting the previously calculated coefficients because it vanishes at all the points on which they depend. Hence we may write

$$f(x) = \phi(x) + (x - x_0)(x - x_1)\ldots(x - x_n)\,\psi(x),$$

where $\phi(x)$ is the interpolation-series of the n^{th} order, and the remainder-term measures the correction which would be made by using an unlimited number of additional data. The nature of the coefficients of $\phi(x)$, the "divided differences," is shown clearly by building up the series in the following way.*

* This form of the theory is due to Ampère (*Ann. de Gergonne* 16 (1826), p. 329), whose work was extended by Cauchy (*C.R.* 11 (1840), p. 775....; *Œuvres* (1), 5). A similar notation was used at the same time by Legendre—*Traité des Fonctions Elliptiques*, Vol. 2 (1826), p. 36. Ampère uses the term "interpolation-functions," but "divided-differences" seems more suitable for practical applications and is used by Oppermann (*J. Inst. Act.* 15 (1869), p. 145), Merrifield (*Brit. Assn. Report*, 1880) and Thiele. For references to other work on the properties of "interpolation functions" see E. Pascal—*Calcolo delle differenze finite*, or his *Repertorium der höheren Mathematik.*

By the Mean Value Theorem

$$f(x) = f_0 + (x - x_0)(x_0 x),$$

where $(x_0 x)$ denotes the function of x and x_0 commonly expressed by $f'\{x_0 + \theta(x - x_0)\}$, $0 < \theta < 1$.

Again,

$$(x_0 x) = (x_0 x_1) + (x - x_1)(x_0 x_1 x)$$
$$(x_0 x_1 x) = (x_0 x_1 x_2) + (x - x_2)(x_0 x_1 x_2 x),$$

and so on. From these equations we get

$$f(x) = f_0 + (x - x_0)(x_0 x_1) + (x - x_0)(x - x_1)(x_0 x_1 x_2) + \dots$$
$$+ (x - x_0)(x - x)\dots(x - x_n)(x_0 x_1 \dots x_n x),$$

the Newton series with remainder term. It is easily proved that $(x_0 x_1 \dots x_n x)$ may be written as $f^{(n+1)}(\bar{x}) / (n+1)!$, where $\bar{x}$ is some point in the range.

Now the order in which the data are taken will not affect any quantity, $(x_0 x_1 \dots x_r)$ say, which appears in the process, because this is determined uniquely by the values of $f_0 f_1 \dots f_r$. Hence every divided difference is a symmetrical function of the elements on which it depends, and may be calculated in more than one way. From the first equation above we find

$$(x_0 x_1) = (f_1 - f_0) / (x_1 - x_0),$$

and similarly

$$(x_r x_s) = (f_s - f_r) / (x_s - x_r).$$

From the second equation

$$(x_0 x_1 x_2) = \{(x_0 x_2) - (x_0 x_1)\} / (x_2 - x_1);$$

taking a different order we should obtain the same quantity as being

$$\{(x_1 x_2) - (x_0 x_1)\} / (x_2 - x_0),$$

or

$$\{(x_1 x_2) - (x_0 x_2)\} / (x_1 - x_0).$$

Again

$$(x_0 x_1) = f_0 / (x_0 - x_1) + f_1 / (x_1 - x_0),$$

whence

$$(x_0 x_1 x_2) = \frac{f_0}{(x_0 - x_1)(x_0 - x_2)} + \frac{f_1}{(x_1 - x_0)(x_1 - x_2)} + \frac{f_2}{(x_2 - x_0)(x_2 - x_1)},$$

a form which shows the symmetry of the divided difference. The corresponding expression for $(x_0 x_1 \dots x_n)$ can be established by

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induction, or directly by tracing the coefficient of f_r through the calculation.*

We can now form a difference-table in which each column is obtained from the preceding one by subtraction and division. The notation may be simplified by using only subscripts, *e.g.* (0 1 2), (0 1 2 x).

$$\begin{array}{cccccc} x_0 & f_0 & & & & \\ & & (0\,1) & & & \\ x_1 & f_1 & & (0\,1\,2) & & \\ & & (1\,2) & & (0\,1\,2\,3) & \\ x_2 & f_2 & & (1\,2\,3) & \vdots & \vdots \quad . \\ & & (2\,3) & \vdots & & \\ x_3 & f_3 & \vdots & & & \\ \vdots & \vdots & & & & \end{array}$$

Each difference is the apex of a tabular triangle whose base is in the f-column and includes the elements on which it depends, and is unaffected by any change of order among these elements.† Hence we can choose any order of the data which does not introduce new combinations, such as (1 2 4), and write down ready for calculation the corresponding series

$$f(x) = f_r + (x - x_r)(r\,s) + (x - x_r)(x - x_s)(r\,s\,t) + \dots$$

The coefficients lie on a line through the table which passes always from one column to one or other of the nearest points in the next column. If a new table were formed with the data in this order, this set would form the top diagonal line, the rest of the table being more or less altered. If now we suppose $x_0, x_1 \dots$ to be the order

* This expression of the divided difference shows the relation between the Newton and Lagrange formulæ. Comparison of coefficients of f_r yields a Newton series for the Lagrange term which might serve for a synthetic calculation, viz., for any order of the points beginning at x_r we have

$$\frac{(x - x_0)(x - x_1)\dots(x - x_n)}{(x_r - x_0)(x_r - x_1)\dots\dots(x_r - x_n)} = 1 + \frac{x - x_r}{x_r - x_s}\left(1 + \frac{x - x_s}{x_s - x_t}(1 + \dots)\right).$$

† An alternative form of difference-table was suggested by E. McClintock (*Amer. J. Mth.* 2 (1879), p. 307) in which every difference bears reference to the initial point x_0; the first column contains (01), (02), (03)..., the second (012), (013)... The calculation is claimed to be simpler, but the table is available only for the one order of the data, that of tabulation, or at most for $x_r\ x_0\ x_1\ x_2 \dots$.

of ascending algebraical magnitude we can choose an order, beginning with any one, x_r, and preceeding in ascending *numerical* magnitude of $x - x_s$, which will be suitable for a point x in the neighbourhood of x_r. In this form of table the differences of higher order contain on the whole larger divisors, and the choice of order of the x's for the calculation ensures that each new multiplier is less than the corresponding divisor. Considered algebraically, any other route which ends at a given point, and starts within the base of the corresponding triangle, would yield identically the same result, but in practice the "central-difference" type of formula is least affected by the inaccuracy of the differences and by the remainder-term. There is perfect freedom of choice within the same scheme of calculation to suit any peculiarities of the data, and in some cases more than one route may be combined with advantage, thus obtaining a diminution in certain multipliers or coefficients of the series.

The formulae used with the equidistant table are particular cases of the general Newton series, and can be written down from the order of the data.* Thus from a table formed with the points $\ldots -2 \;\; -1 \;\; 0 \;\; 1 \;\; 2 \ldots$ we obtain the Gauss formula

$$f(x) = f_0 + x\,\delta f_{\frac{1}{2}} + x\,(x-1)\,\delta^2 f_0 / 2\,! + \ldots$$

by using the order 0, 1, -1, 2, -2 ... Combining this with the corresponding formula for the order 0, -1, 1, -2, 2 ... gives Stirling's central-difference formula

$$f(x) = f_0 + x\,\mu\,\delta f_0 + x^2\,\delta^2 f_0 / 2\,! + \ldots$$

In practice the equidistant table is formed without using the divisors, which are equal along a column, hence these appear explicitly in the formulae. Thus $(x_0\,x_1 \ldots x_n)$ becomes $\Delta^n f_0 / n\,!$, while the remainder-term may be written

$$x\,(x-1)\,(x-2)\ldots(x-n)\,\Delta^{n+1} f(x) / (n+1)\,!$$

3. *Differentiation and Repeated Data.*

We have assumed that the given points and the new point x are all distinct. If we suppose x_1 to approach x_0, $(x_0\,x_0)$ or $(0\,0)$

* Compare D. C. Fraser's diagram of formulæ for the equidistant table—*J. Inst. Act.* 43 (1909), pp. 235, 442. Also W. F. Sheppard—*J. I. A.* 50 (1916), p. 85; R. Todhunter—do., p. 133.

becomes the symbol for $\underset{x \to x_0}{\text{Lt.}} \{f(x) - f(x_0)\} / (x - x_0)$, *i.e.* $f'(x_0)$.

Given the value of this, no difficulty arises in completing the table, the formulae for calculation of differences such as (0 0 1 2), for instance, all yielding definite values. Now consider a new point x close to x_0, and let $f(x)$ be calculated from the set of equations

$$\left.\begin{aligned} f(x) &= f_0 + (x - x_0)(0\,x) \\ (0\,x) &= (0\,1) + (x - x_1)(0\,1\,x) \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ (0\,1\,2 \ldots \overline{r-2}\,x) &= (0\,1\,2 \ldots \overline{r-1}) + (x - x_{r-1})(0\,1\,2 \ldots \overline{r-1}\,x) \end{aligned}\right\} \quad \ldots\ldots(A)$$

beginning with the last. What actually happens is that we assume a value for the unknown quantity $(0\,1\,2 \ldots \overline{r-1}\,x)$, either supposing it equal to $(0\,1\,2 \ldots r)$ or making some allowance for the neglected remainder-term. Thence we calculate successively $(0\,1\,2 \ldots \overline{r-2}\,x)$, $(0\,1\,2 \ldots \overline{r-3}\,x) \ldots$, $(0\,1\,x)$, $(0\,x)$, and finally $f(x)$ itself. At no stage will any difficulty be introduced if we suppose x equal to x_0, except that of course we stop at $(0\,x)$, which becomes $(0\,0)$. The value of $f'(x_0)$ is thus obtained from the table by the same process as a value of $f(x)$. More generally, let x have any value near to, but distinct from, x_0, and suppose $f(x)$ already calculated as described. Its value and the quantities $(0\,x)$, $(0\,1\,x) \ldots (0\,1\,2 \ldots \overline{r-1}\,x)$ can be written down as an additional diagonal line of the table, and we can calculate $f'(x)$ now as we did $f'(x_0)$. The result of this double process is of course a lesser accuracy in $f'(x)$ than in $f(x)$ or in $f'(x_0)$. The new set of differences obtained is shown in the equations

$$\left.\begin{aligned} (x\,x) &= (0\,x) + (x - x_0)(0\,x\,x) \\ (0\,x\,x) &= (0\,1\,x) + (x - x_1)(0\,1\,x\,x) \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ (012 \ldots \overline{r-3}\,x\,x) &= (012 \ldots \overline{r-2}\,x) + (x - x_{r-2})(012 \ldots \overline{r-2}\,x\,x) \end{aligned}\right\} \quad \ldots\ldots(B)$$

the value of $(0\,1\,2 \ldots \overline{r-2}\,x\,x)$ being again assumed. Similarly, we may calculate $(x\,x\,x)$ and any other. The following table illustrates the inclusion of a derivation and the growth by accretion of new points.

x	$f(x)$				
		$(x\,x)$			
x	$f(x)$		$(x\,x\,x)$		
		$(x\,x)$		$(0\,x\,x\,x)$	
x	$f(x)$		$(0\,x\,x)$		$(0\,1\,x\,x\,x)$
		$(0\,x)$		$(0\,1\,x\,x)$	
x_0	f_0		$(0\,1\,x)$		$(0\,1\,2\,x\,x)$
		$(0\,1)$		$(0\,1\,2\,x)$	
x_1	f_1		$(0\,1\,2)$		$(0\,1\,2\,2\,x)$
		$(1\,2)$		$(0\,1\,2\,2)$	
x_2	f_2		$(1\,2\,2)$		$(0\,1\,2\,2\,3)$
		$(2\,2)$		$(1\,2\,2\,3)$	
x_2	f_2		$(2\,2\,3)$		$(1\,2\,2\,3\,4)$
		$(2\,3)$		$(2\,2\,3\,4)$	
x_3	f_3		$(2\,3\,4)$		
		$(3\,4)$			
x_4	f_4				

It remains to point out the meaning of $(x\,x\,x)$ and similar symbols.

We have seen that $(0\,0)$ is the limit of $(0\,1)$ as $x \to x_0$, and we may regard $(0\,1)$ as an intermediate stage between $(0\,0)$ and $(1\,1)$. This we must use to determine $\underset{x_1 \to x_0}{\text{Lt.}} \{(11)-(00)\} / (x_1 - x_0)$, which is $f''(x_0)$; for

$$f''(x_0) = \text{Lt.}\ \{(11)-(01)\} / (x_1 - x_0) + \text{Lt.}\ \{(01)-(00)\} / (x_1 - x_0)$$
$$= \text{Lt.}\ \{(011)+(001)\} = 2\,(000).$$

There are thus two intermediate stages (001) and (011) between (000) and (111), and in general $(x\,x\,x)$ is $f''(x)/2$.

Similarly the difference of n^{th} order $(x\,x\,x \ldots x)$ is $f^{(n)}(x)/n!$, while any other difference with repeated data can be expressed similarly, *e.g.*

$$(0\,1\,x\,x\,x) \text{ is } d^2/dx^2\,(0\,1\,x)/2!,\quad (0\,1\,1\,2) \text{ is } \left| d/dx\,(0\,x\,2) \right|_{x=x_1}$$

We thus arrive at the differential calculus as the special case of interpolation with repeated data, and are enabled to use alternatively Newton's or Taylor's series for the expression of the function concerned. Further, we can use any combination of the two, such as

$$f(x) = f_0 + (x-x_0)(01) + (x-x_0)(x-x_1)(011)$$
$$+ (x-x_0)(x-x_1)^2(0112) + \ldots,$$

which is partly a Taylor's expansion about the point x_1, and partly a Newton expansion about the set of points $x_0, x_1, x_2 \ldots$ This is the theoretical aspect of the flexibility which permits us practically to combine differences and differential coefficients in the one table and procedure of computation.

4. *Preparation and Use of the Difference-Table.*

The procedure in a numerical problem is then as follows: tabulate the given values in order of magnitude of the independent variable x, complete the difference-table, choose an order of the data proceeding outwards from the required point x, and trace the corresponding route through the table, *e.g.* with a pencil. For convenience of description we may label this order as $x_0, x_1, \ldots$; if there is any doubt at a glance which of two points comes next, the choice matters little except at the end of the series. The route thus traced will be a wavy line, beginning at f_0 and moving upwards or downwards always in agreement with the passage from x_0 to x_1 to x_2, and so on. If the data are fairly regularly distributed, the line will correspond to a Gauss series in the ordinary table, but in general is likely to make wider oscillations. Considerations such as are familiar to workers with the equidistant table will guide the choice of route, or of a combination of routes, and particularly at the end the decision at which order of differences to stop, and what allowance to make for neglected orders.*

The following data are taken from Saxelby's *Practical Mathematics*, x being the percentage of lead in an alloy with zinc, θ the melting point of the alloy in degrees Centigrade.† The choice of independent variable is often decided by greater simplicity of divisors in one case, or by such conventions as the customary use of time as independent variable; in other cases we may choose to

* A valuable discussion of the ordinary table will be found in two papers by W. F. Sheppard—*Proc. L. Mth. Soc.*, 4 (1906), p. 320; 10 (1911), p. 139.

† A graphical analysis of these data is given in J. Lipka—*Graphical and Mechanical Computation* (Wiley 1918), p. 146. By plotting first differences of θ as a straight line he obtains the formula $\theta = 141{\cdot}4 + 0{\cdot}620\,x + 0{\cdot}0130\,x^2$, while the tables suggest as most suitable $x = a + b\,\theta + c\,\theta^2 + d\,\theta^3$. The accuracy of the data is not sufficient for proper comparison of methods, but the difference in the value of the derivative is considerable.

suit the problem. The relation assumed between the variables differs in the two cases, so that one table may be more convergent than the other. Hence methods both of direct and of inverse-interpolation are needed. For comparison I give in this case both tables. The calculation may be done with tables, sliderule, or otherwise, taking care to retain sufficient accuracy in the successive orders of differences. It is sometimes necessary to prevent the numbers becoming unwieldy by a change of unit.

				θ	x				
				181	36·9				
			1·63			0·612			
		0·0228		197	46·7		−0·00306		
	−0·00037		2·24			0·447		$0{\cdot}0_4 28$	
$-0{\cdot}0_5 7$		0·0077		235	63·7		−0·00060		$-0{\cdot}0_7 6$
	−0·00071		2·48			0·403		$0{\cdot}0_4 22$	
$-0{\cdot}0_4 5$		−0·0187		270	77·8		0·00154		$-0{\cdot}0_5 1$
	0·00271		2·10			0·477		$-0{\cdot}0_4 83$	
		0·0485		283	84·0		−0 00400		
			2·57			0·389			
				292	87·5				

Let it be required to find an alloy which will melt at 214° C. (steam pressure 300 lbs. / sq. in.). In the right-hand table trace by pencil a line starting at 46·7 and passing through 0·447, −0·00306, $0{\cdot}0_4 28$ (*i.e.* 0·000028), $-0{\cdot}0_7 6$. Either table shows that the 292° point does not agree well with the others. The series to be used, written in chain-form,* is

$$f(\theta) = 46{\cdot}7 + (\theta - 197)\,\{0{\cdot}447 + (\theta - 235)\,\{-0{\cdot}00306 + (\theta - 181)\,\{0{\cdot}0_4 28 + (\theta - 270)\,\{-0{\cdot}0_7 6\}\,\}\ldots\}\,,$$

and the calculation is best carried out by the synthetic method, as in equations (A) and (B).† The multipliers are 17, −21, 33, −56,

* W. Veltmann—*Zeitschr. f. Mth. u. Phys.*, 44 (1899), p. 303—suggests a notation analogous to a continued fraction,

$$f(x) = a_0 + \frac{x - x_0}{a_1 +}\;\frac{x - x_1}{a_2 +}\;\ldots\,,$$

the double line denoting multiplication in place of division.

† Horner's algorithm for solution of numerical equations is a special case when the data are all coincident. The procedure for interpolation is found in Legendre—*loc. cit.* in § 2.

a difference $(012\ldots r\,\theta)$ being multiplied by $(\theta - \theta_r)$. For a single interpolation the work is conveniently arranged in columns, and symbols have been added to identify the new differences. The old ones may be copied down as an initial column if preferred, or simply read off the pencil-line.

− 56	$-0{\cdot}0_7\,6$	$(0123\,\theta)$	$-0{\cdot}0_7\,6$	$(012\,\theta\theta)$
33	$0{\cdot}0_4\,31$	$(012\,\theta)$	$0{\cdot}0_4\,29$	$(01\,\theta\theta)$
− 21	− 0·00204	$(01\,\theta)$	− 0·00265	$(0\,\theta\theta)$
17	0·490	$(0\,\theta)$	0·445	$(\theta\theta)$
	55·0	(θ)		

Thus $0{\cdot}0_4\,31$ is the result of $(56 \times 0{\cdot}0_7\,6) + 0{\cdot}0_4\,28$. For the $(\theta\theta)$ column the multiplier − 56 is not needed, and we use the (θ) column as base, *e.g.* $0{\cdot}0_4\,29$ is $(-33 \times 0{\cdot}0_7\,6) + 0{\cdot}0_4 31$. For $\theta = 214°$ we have therefore $x = 55{\cdot}0$, and $dx/d\theta = 0{\cdot}445$. Even when only *$f'(\theta)$ is required it is probably better to proceed in this way than* to attempt direct determination. In particular,

	$-0{\cdot}0_7\,6$	(01230)
− 73	$0{\cdot}0_4\,32$	(0120)
− 38	− 0·00428	(010)
16	0·543	(00)

at one of the given points $f'(\theta)$ is obtained by a single column, *e.g.* for $f'(197)$ the order of data is 197, 181, 235, 270, and the result 0·543.

As an example of the use of a combined formula let us find θ when $x = 60$, by the left-hand table. It will be convenient to combine the two orders 63·7, 46·7, 77·8, 36·9, 84·0 and 63·7, 77·8, 46·7, 84·0, 36·9, giving a formula of Stirling type.

$$f(x) = 235 + (x - 63{\cdot}7)\{2{\cdot}36 + (x - 62{\cdot}25)\,0{\cdot}0077\}$$
$$- (x - 63{\cdot}7)(x - 46{\cdot}7)(x - 77{\cdot}8)\{0{\cdot}00054 + (x - 60{\cdot}45)\,0{\cdot}0_5\,7\}$$

where 62·25 is the mean of 46·7 and 77·8, 0·00054 is a mean of two differences, and so on. In the calculation we first correct each mean difference (of odd order) by the small multiple of the next difference, and then use the multipliers in pairs. Equations (A) become:

$$f(x) = f_0 + (x - x_0)(0\,x).$$

$$\left.\begin{array}{l} (0\,x) = \{\tfrac{1}{2}\,\overline{(01) + (02)} + (x - \tfrac{1}{2}\,\overline{x_1 + x_2})\,(012)\} + (x - x_1)(x - x_2)(012\,x) \\ (012x) = \{\tfrac{1}{2}\,\overline{(0123) + (0124)} + (x - \tfrac{1}{2}\,\overline{x_3 + x_4})\,(01234)\} \\ \qquad\qquad + (x - x_3)(x - x_4)(01234\,x) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\} \quad (A').$$

Similarly, for the derivative,

$$\left.\begin{aligned} f'(x) = (x\,x) = (0\,x) + (x - x_0)\,(0\,x\,x) \\ (0\,x\,x) = \{(0\,1\,2) + 2\,(x - \tfrac{1}{2}\,\overline{x_1 + x_2})\,(012\,x)\} \\ + (x - x_1)\,(x - x_2)\,(012\,x\,x) \end{aligned}\right\}$$

....

(B′).

Here the original differences of even order are corrected by small multiples of the new differences of next order.

	$-0{\cdot}0_5\,7$			$-0{\cdot}0_5\,7$	$(012\,x\,x)$
$-17{\cdot}8$ }	$-0{\cdot}00054$	$-0{\cdot}00054$	$(012\,x)$		
$13{\cdot}3$ }	$0{\cdot}0101$			$0{\cdot}0118$	$(0\,x\,x)$
$-3{\cdot}7$	$2{\cdot}34$	$2{\cdot}49$	$(0\,x)$	$2{\cdot}45$	$(x\,x)$
	235	$225{\cdot}8$	(x)		

The first column contains the multipliers, the second the corrected differences; the latter can in general be completed only after $f(x)$ has been calculated.

In particular, again, $f'(x_0)$ is given by a single application of equations (A′). Any linear combination, not necessarily a simple mean, of two routes can be used in a similar way, provided it simplifies the arithmetical work without increasing too greatly the complexity of the scheme. Such formulae are not convenient, however, when it is desired to incorporate the new values in the table.

5. *Extension of the Table and Inclusion of Derivatives.*

In order that the sets of differences which appear in the synthetic calculation of $f(x)$ and $f'(x)$ should take their place in the table itself, it is necessary to rearrange this so that the line of differences employed becomes the lowest diagonal line. Since we are not concerned with the rest of the new table we may make this line horizontal and add the new differences as a fresh line, the table then taking the form.

$$\begin{array}{llllllll} x_2 & f_2 & & & & & & \\ x_1 & f_1 & & & & & & \\ x_0 & f_0 & (01) & (012) & (0123) & (01234) & \ldots & (012\ldots r) \\ x & f(x) & (0x) & (01x) & (012x) & (0123x) & \ldots & (012\ldots\overline{r-1}x) \\ x & f(x) & (xx) & (0xx) & (01xx) & (012xx) & \ldots & (012\ldots\overline{r-2}xx) \\ x' & f(x') & (xx') & (xxx') & (0xxx') & (01xxx') & \ldots & (012\ldots\overline{r-3}\,xxx'). \end{array}$$

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Another new point x' has been added, $f(x')$ being calculated from the last line above it. Thus $(0xxx') = (01xx) + (x' - x_1)(01xxx')$. The multiplier $(x' - x_1)$ is found from the ends of the base of the triangle of which $(01xxx')$ is apex, *i.e.* it is the difference of the two elements involved which lie furthest apart in the x-column. Theoretically any number of new points, repeated or not, could be thus added in succession, but in practice this would involve accumulation of error, and, moreover, the principle of central-differences requires the points to lie in the same region.

To illustrate I shall take a fresh example, of a kind not uncommon in various fields, in which one variable x occurs as a succession of small integers. Such cases often arise through absence of terms in a regular sequence, and it is possible to complete the sequence by the methods of finite differences. This is equivalent to the use of Lagrange's formula and open to the same objections, while the divided-difference table with x as independent variable is particularly easy to construct. The sines of the angles 0°, 30°, 45°, 60°, 90° are easily remembered. Using their simple ratios we have a table which will give the sine of any angle between 30° and 60° with considerable accuracy. For other angles the same data can be rearranged, *e.g.* – 45°, – 30°, 0°, 30°, 45°.

x	$\sin \pi x/12$				
0	0				
		0·25000			
2	0·50000		– 0·01429 7		
		0·20711		– 0·00244 95	
3	0·70711		– 0·02409 5		$0{\cdot}0_3\,13\ 542$
		0·15892		– 0·00163 70	
4	0·86603		– 0·03064 3		
		0·06699			
6	1·0				

The remainder-term of the series will be

$$x(x-2)(x-3)(x-4)(x-6)\,(02346x).$$

The value of $(02346x)$ is that of $f^{(5)}(x)/5!$ at some point, $f(x)$ being $\sin \pi x/12$, hence it is $\not> 0{\cdot}00001$. For x between 2 and 4, $x(x-2)\,..\,(x-6)$ is $\not> 4$, hence the error in the sine due to limitation of data is at most of order 0·00004, *i.e.* less than half a unit in the fourth place. The derivative is

$$(\pi/12)\cos \pi x/12, \text{ or } \pi/12 \sin(\pi/2 - \pi x/12),$$

giving a method of determining π. The error of $f'(x)$ is greater than that of $f(x)$, so that the error in π may amount to a unit or two in the third place. Taking for example the angle 36° ($x = 2·4$) we have:—

0
4
3

2	0·50000	0·20711	−0·02409 5	−0·00244 95	$0·0_3 13$ 542
2·4	0·58781	0·21953	−0·02069 6	−0·00212 45	,,
2·4		0·21171	−0·01929 1	−0·00234 12	,,
2·4			−0·02025 9	−0·00241 95	,,
2·4				−0·00236 53	,,

The Taylor series for $\sin \pi x/12$ in powers of $(x - 2·4)$ is therefore

$$\sin \pi x/12 = 0·58781 + 0·21171\,(x - 2·4) - 0·02026\,(x - 2·4)^2 \\ - 0·00237\,(x - 2·4)^3 - 0·000135\,(x - 2·4)^4 + \ldots,$$

the correct value being

$$\sin \pi x/12 = 0·58779 + 0·21180\,(x - 2·4) - 0·02014\,(x - 2·4)^2 \\ - 0·00244\,(x - 2·4)^3 - 0·000115\,(x - 2·4)^4 + \ldots.$$

To obtain as accurate a measurement of π as possible, let us calculate cos 45°:—

0
4
2

3	0·70711	0·20711	−0·024095	−0·0024495	0·00013542
3		0·18506	−0·022052	−0·0020433	,,

whence $\pi = 12 \times 0·18506/0·70711 = 3·1406$. The Stirling formula for $f'(x_0)$ from equations (A') takes in this case a very simple form, viz.

$$f'(3) = \tfrac{1}{2}(0·20711 + 0·15892) + \tfrac{1}{2}(0·0024495 + 0·0016370) = 0·18506.$$

Subtabulation may be carried out systematically in this way, but accumulation of error renders a return to the original table advisable at intervals. Alternatively we may calculate the sub-differences of second or higher order directly and proceed by simple addition.

Greater accuracy may be obtained by using more initial data, and one way of doing this is to include the derivatives,

$$(\pi/12)\cos(\pi x/12).$$

From the points 0°, 30°, 45° alone, using both sines and cosines, a table can be formed giving at least equal accuracy to the above.

x	$\sin \pi x/12$				
0	0				
		0·26180			
0	0		−0·00590 0		
		0·25000		−0·00286 75	
2	0·50000		−0·01163 5		$0{\cdot}0_4 6$ 86
		0·22673		−0·00266 17	
2	0·50000		−0·01962 0		$0{\cdot}0_4 9$ 72
		0·20711		−0·00237 00	
3	0·70711		−0·02199 0		
		0·18512			
3	0·70711				

The calculation differs in no way from that already described, and higher derivatives could similarly be included. The process is simply a modification of the ordinary series calculation of $\sin x$. employing a combination of a Taylor and a Newton series, but when the data are purely empirical this use of both $f(x)$ and $f'(x)$ in the same table is very valuable. (The Lagrange formula was extended to this case by Hermite). It may happen that for some point a we are given $f'(a)$ but not $f(a)$. We can interpolate for $f(a)$ from the other data and correct the result by the value of $f'(a)$. Let $\phi(x)$ denote the series used, so that

$$f(x) = \phi(x) + (x - x_0)(x - x_1) \ldots (x - x_r)(012 \ldots rx)$$

$$\therefore \; f'(x) = \phi'(x) + (x - x_0) \ldots (x - x_r)(012 \ldots rx)\{\Sigma 1/(x - x_s)\}$$

if we assume for the moment that $(012 \ldots rx)$ is constant.
Hence

$$f(a) = \phi(a) + \{f'(a) - \phi'(a)\}/\Sigma 1/(a - x_s),$$

where $\phi(a)$, $\phi'(a)$ are the interpolated values. For example, to

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find sin 45° from the data 0, 0, 2, 2 alone, given the derivative 0·18512 :—

0				
0				
2				
2	0·50000	0·22673	−0·011635	−0·0028675
3	0·70649	0·20649	−0·020237	,,
3		0·18339	−0·023104	,,

The remainder-term is 3 . 3 . 1 . 1 . (00223) $\therefore$ $\Sigma 1/(a-x_s) = 2^2/_3$, giving $f(a) = 0{\cdot}70649 + (3 \times 0{\cdot}00173)/8 = 0{\cdot}70714$. A further approximation could be made by using this value of $f(a)$ to calculate (002233), and hence the term omitted in $f'(x)$ by assuming $(012 \ldots rx)$ constant. In the present case the error in $f(a)$ is already within the limits of the table.

6. *Inverse Interpolation.*

It is not always practically advisable to arrange the data in form for direct solution of a problem, either because the table lacks sufficient convergence or because the reverse table is relatively simple to construct, or already constructed. The process of determining x from a table of differences of $f(x)$ is necessarily a tentative one of the nature of successive approximation, and is equivalent to the solution of an algebraic equation written in factorial form. The diminishing series of coefficients makes this specially suitable for iterative calculation, and as a rule we require only a single root whose location is already roughly known. If $f(x)$ lie between f_0 and f_1 a first approximation to x is given by simple proportion, $x' = x_0 + \{f(a) - f_0\}/(01)$. Writing $\phi(x)$ for the interpolation series in this region and k for the given value of $f(x)$, the equation to be solved is $\phi(x) - k = 0$. The most rapid form of iterative solution is Newton's tangent method, giving the next approximation $x'' = x' - \{\phi(x') - k\}/\phi'(x')$, and so on. The whole calculation can be carried through as described in the last section, using at first only a small accuracy and returning to the original table to increase this when the root has been located within small bounds. In place of calculating ϕ' each time, we may calculate ϕ only and use a suitably chosen and numerically simple approximation to ϕ', making a fresh choice if necessary in the course of the work. The differences

$(0x')$, $(x'x'')$...which appear in the calculation of ϕ indicate the value of ϕ' for this purpose. This is the general iterative solution of form $x = x + \lambda[\phi(x) - k]$, the points converging more slowly, but each calculation being quicker. For example, to find the angle whose sine is 0·6 :—

3					
3					
2					
2	0·50000	0·22673	– 0·019620	– 0·0023700	$0{\cdot}0_4972$
(2·5	0·6088	0·2175	– 0·0184	– 0·00242	,,
2·5		0·2076	– 0·0198	– 0·00247	,,)
2·458	0·60000	0·21835	– 0·018307	– 0·0024227	,,

The first approximation is $2 + 0{\cdot}1/0{\cdot}207 = 2{\cdot}5$, say. Then $\phi(2{\cdot}5) = 0{\cdot}6088$, $\phi'(2{\cdot}5) = 0{\cdot}2076$, $\therefore$ next approximation is $2{\cdot}5 - 0{\cdot}088/0{\cdot}208 = 2{\cdot}458$. Calculation directly from the original data gives $f(2{\cdot}458) = 0{\cdot}60000$, so that this value is as correct as the table allows. The remainder-term is negligible to the fifth place, so that x may be taken to be 2·4580, or 36° 52′·20 with possible error of a unit in the last figure.

Extensions of Newton's method, employing also $\phi''(x)$, may be used to quicken the approximation, but are seldom required in problems where the accuracy of the data is limited. An algebraic equation with exact coefficients may be solved to any degree of approximation in this way, however, and with considerable rapidity, once the roots have been located. At any stage the polynomial form with new origin may be obtained, and the solution can be expressed in Lagrange's continued-fraction form if desired. Transcendental equations may be thus solved also, and the most suitable points for tabulation will not always be in arithmetical progression.

Difficulty arises when there are two roots close together, the approximation becoming slow and the indications uncertain. This case is best treated by using $\phi''(x)$ to separate the roots. If a root be a, we have

$$k = \phi(a) = \phi(x) + (a - x)\,\phi'(x) + (a - x)^2\,\phi''(x)/2 + \dots .$$

The first two terms give Newton's value, the first three, solved as a quadratic, furnish approximations to both the close roots. It may sometimes be sufficient to use three terms of the interpolation series, *i.e.* $k = f_0 + (x - x_0)(01) + (x - x_0)(x - x_1)(012)$. Between two such roots lies a point at which $f'(x) = 0$, and it is often required from the table to find where $f'(x)$ has this or any other given value. This problem can be solved in the same way, though not with the same accuracy. Thus to find the angle whose cosine is 0·8, we have $(xx) = 0{\cdot}8\,\pi/12 = 0{\cdot}20944$. By simple proportion between $f'(2)$ and $f'(3)$—if these were not available we could use some such approximation as $(xx) = (01) + (2x - x_0 - x_1)(012)$—we get 2·4, say. Calculation gives

$$\phi'(2{\cdot}4) = 0{\cdot}2118,\ \phi''(2{\cdot}4) = -0{\cdot}04,\ \therefore\ x = 2{\cdot}4 - (\phi' - k)/\phi'' = 2{\cdot}46.$$

And so on.

7. *Integration.*

The Lagrange formula has the advantage of exhibiting the value of the unknown function at any point as a linear function of the given values. This is often useful in dealing with familiar data and some computers prefer the Lagrange formula on this account even for equidistant data. The same fact makes it specially valuable in numerical integration, since, for any given distribution of ordinates, the result is simply a linear function of their values with fixed coefficients. Unfortunately these are often inconvenient numbers, and in practice substitute formulae are used, which are either arbitrary approximations to the Cotes' formulae or combinations of more than one. When, however, the distribution of ordinates is peculiar to the problem in question, the computation of the coefficients is a heavy task, and the same applies to the direct integration of the Newton series, which is often to be preferred. Alternatively the Taylor series may be formed and integrated, but this entails loss of accuracy when carried out in the only practicable way. A more direct use of the difference-table is made by Thiele, who finds relations connecting the differences of the integrated function with those of the ordinates, and so builds up a new table from some differences of high order assumed constant. The procedure is rather complicated and further investigation is needed.