90.34 On triples of integers having the same sum and the same product

Introduction

We consider the problem of finding pairs of different non-ordered n-tuples of integers \((a_1, a_2, a_3, \ldots, a_n)\) and \((b_1, b_2, \ldots, b_n)\) such that:

1) the sums of their elements are equal: \(\sum a_k = \sum b_k = S\),
2) and the products of their elements are equal as well: \(\prod a_k = \prod b_k = P\).

Obviously there is no solution for \(n = 2\), because in that case there is only a single pair corresponding to a given \(S\) and a given \(P\), that is the one made up of the roots of the quadratic equation:

\[x^2 - Sx + P = 0.\]

Therefore the simplest problem of this type concerns triples.

Given a triple \((a, b, c)\) of complex numbers, we can find all the triples \((d, e, f)\) of complex numbers with both \(a + b + c = S = d + e + f\) and \(abc = P = def\) by solving the cubic equations:

\[x^3 - Sx^2 + mx - P = 0,\]

where \(m\) is an arbitrary complex number. For each \(m, d, e\) and \(f\) will be the roots of this equation. But the problem starts to be more interesting and difficult if we wish to have \((a, b, c)\) and \((d, e, f)\) triples made up of rational or Gaussian integers only. We cannot even start from a given triple made up of integers because nothing warrants our finding another triple.

The object of this article is to present a relatively simple method by which to find infinitely many pairs of triples \((a, b, c)\) and \((d, e, f)\) with \(a, b, c, d, e\) and \(f\) all integers, either rational or Gaussian, which satisfy conditions 1 and 2 above.

To this end we need the following definitions:

Sum-equivalence

The triples \((a, b, c)\) and \((d, e, f)\) are said to be sum-equivalent if

\[a + b + c = d + e + f.\]

Cube equivalence

The triples \((a, b, c)\) and \((d, e, f)\) are said to be cube-equivalent if

\[a^3 + b^3 + c^3 = d^3 + e^3 + f^3.\]

Product-equivalence

The triples \((a, b, c)\) and \((d, e, f)\) are said to be product-equivalent if

\[abc = def.\]

With this understanding, we may sum up the aims of this note as follows
• to show that infinitely many pairs of triples of rational or Gaussian integers exist which are both sum and cube equivalent (SC-equivalent).

• to deduce from it that infinitely many pairs of triples of rational or Gaussian integers exist which are both sum and product equivalent (SP-equivalent);

• to study some of the properties of these pairs of triples.

The starting point

Our starting point is a study of the sums of three cubes, and the following, easy to prove, formula:

\[(x + y + z)^3 - (x^3 + y^3 + z^3) = 3(x + y)(y + z)(z + x).\] (1)

Let us suppose that we have two triples \((x, y, z)\) and \((u, v, w)\) such that:

\[x + y + z = u + v + w = S_1\]

and

\[x^3 + y^3 + z^3 = u^3 + v^3 + w^3 = S_3.\]

For the above formula, it is obvious that, if we let

\[(a, b, c) = (x + y, y + z, z + x)\]

and

\[(d, e, f) = (u + v, v + w, w + u),\]

we have

\[a + b + c = d + e + f = 2S_1\]

and

\[abc = def = \frac{1}{3}(S_1^3 - S_3).\]

This situation occurs for instance with \(x = y = z = 1\) and \(u = v = 4,\ w = -5,\) since

\[x + y + z = u + v + w = 3\]

and

\[x^3 + y^3 + z^3 = u^3 + v^3 + w^3 = 3.\]

Correspondingly we have \((a, b, c) = (2, 2, 2)\) and \((d, e, f) = (8, -1, -1).\)

We check that they are SP-equivalent:

\[a + b + c = d + e + f = 6\] and \(abc = def = 8.\)

Interestingly, we have the same equalities with Gaussian integers:

\[-1 + (2 + i) + (2 - i) = 3\] and \((-1)^3 + (2 + i)^3 + (2 - i)^3 = 3.\]

And if we define similarly \((g, h, j) = (1 + i, 4, 1 - i),\) we check that:
\[ \begin{align*}
a + b + c &= d + e + f = g + h + j = 6 \\
abc &= def = ghj = 8.
\end{align*} \]

A richer example is provided by the following results:

\[ \begin{align*}
2^3 + 3^3 + 6^3 &= 1^3 + 5^3 + 5^3 = 13 \cdot 17^3 + (-19)^3 = \\
7^3 + (2 + 3i)^3 + (2 - 3i)^3 &= (-7)^3 + (9 + 4i)^3 + (9 - 4i)^3 = 251
\end{align*} \]

while

\[ \begin{align*}
2 + 3 + 6 &= 1 + 5 + 5 = 13 + 17 + (-19) = 7 + (2 + 3i) + (2 - 3i) = \\
(-7) + (9 + 4i) + (9 - 4i) &= 11.
\end{align*} \]

Therefore the five triples \((5, 9, 8), (6, 10, 6), (30, -2, -6), (9 + 3i, 4, 9 - 3i), (2 + 4i, 18, 2 - 4i)\) are SP-equivalent: their common sum is 22, and their common product is 360, as the reader may check.

**Primitive pairs of SP-equivalent triples**

Given a pair of SP-equivalent triples \((a, b, c)\) and \((d, e, f)\) there are several ways to derive from it other pairs of SP-equivalent triples. For instance the pair of opposite triples \((-a, -b, -c)\) and \((-d, -e, -f)\) are also SP-equivalent. More generally, if \(m\) is any Gaussian or rational integer, the pair of triples \((ma, mb, mc)\) and \((md, me, mf)\) will also be SP-equivalent.

The property of SC-equivalence is easy to use for deriving new interesting triplets. For, if we have:

\[ x + y + z = u + v + w \]

and

\[ x^3 + y^3 + z^3 = u^3 + v^3 + w^3 \]

we have also (for instance):

\[ x + (-v) + z = u + (-y) + w \]

and

\[ x^3 + (-v)^3 + z^3 = u^3 + (-y)^3 + w^3 \]

which allows us to state that the triplets \((x - v, -v + z, z + x)\) and \((u - y, -y + w, w + u)\) are SP-equivalent. By swapping one or two elements between both sides of the initial equalities, we shall get several new pairs of SP-equivalent triples, listed in the table below (where the opposites of those displayed are omitted):

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altogether 10 pairs of SP-equivalent triples. This number of pairs might be lessened in the event of repetitions in the values of $x, y, z$, as in the case \{x, y, z\} = \{1, 1, 1\} and \{u, v, w\} = \{4, 4, -5\} of The starting point above.

As an example we may use (see example above):

$$13^3 + 17^3 + (-19)^3 = 7^3 + (2 + 3i)^3 + (2 - 3i)^3$$

$$13 + 17 + (-19) = 7 + (2 + 3i) + (2 - 3i).$$

The corresponding derived pair of rank 3 is (11 — 3i, —21 — 3i, -6) and (-10, -15 - 3i, 9 - 3i), and we check that:

$$a + b + c = d + e + f = -16 - 6i$$

and

$$abc = def = 1440 - 180i.$$

Formulas for finding infinitely many pairs of SC-equivalent triples

Let us consider the following polynomial in $t$:

$$P(t) = \{(t + r)^3 + (t - r)^3 + (-t + p)^3\} - \{(-p)^3 + (-q)^3 + (t + q)^3\}$$

$$= -3t((p + q)t + p^2 + q^2 - 2r^2).$$

Before any further computation, we notice that we have:

$$\{(t + r) + (t - r) + (-t + p)\} - \{(-p) + (-q) + (t + q)\} = 0$$

which is part of the desired result (sum equivalence). The other feature is that $P(t) = 0$ (cube equivalence).

To get $P(t) = 0$, we can set $t = -(p^2 + q^2 - 2r^2)/(p + q)$. After some elementary rearrangements, we obtain:
\[(p^2 + q^2 - 2r^2 - rp - rq)^3 + (p^2 + q^2 - 2r^2 + rp + rq)^3 + (2r^2 + qp - q^2)^3\]
\[= (p^2 + pq)^3 + (q^2 + pq)^3 + (p^2 - pq - 2r^2)^3\]  \hspace{1cm} (2)

a formula valid for any complex values of \(p, q\) and \(r\).

And we check that:
\[(p^2 + q^2 - 2r^2 - rp - rq) + (p^2 + q^2 - 2r^2 + rp + rq) + (2r^2 + qp - q^2)\]
\[= (p^2 + pq) + (q^2 + pq) + (p^2 - pq - 2r^2) = 2p^2 + pq + q^2 - 2r^2.\]

Therefore corresponding pair of SP-equivalent triples is:
\[(a, b, c) = (2(p^2 + q^2 - 2r^2), p^2 + rp + rq + qp, p^2 - rp - rq + qp)\]

and
\[(d, e, f) = ((p + q)^2, p^2 + q^2 - 2r^2, 2(p^2 - r^2)).\]

And we check that:
\[a + b + c = d + e + f = 4p^2 + 2pq + 2q^2 - 4r^2\]

and
\[abc = def = 2(p^2 + q^2 - 2r^2)(p + q)^2(p^2 - r^2).\]

This is enough to prove that there are infinitely many pairs of SP-equivalent triples. They can be generated from these formula by substituting the variables \(p, q, r\) with integral values, either rational or Gaussian.

But here it must be noticed that we have not treated the general case, since our formula leads always to triples where one element of the first one is the double of one element of the other triple \((a = 2e)\), and that this situation is not reproduced for instance in the pair of SP-equivalent triples \{(11 - 3i, -21 - 3i, -6), (-10, -15 - 3i, 9 - 3i)\} studied above.

Another convenient formula, more general than (2), can be found by starting from the polynomial:
\[Q(t) = (mt + p)^3 + (nt + q)^3 - p^3 - q^3 = (mt + r)^3 - (nt - r)^3.\]

but the resulting formula is very complex; the reader may compute it as an exercise.

The following numerical example will stress this point: by letting \(r = 0, p = 2, q = 1\), we obtain:
\[1^3 + 5^3 + 5^3 = 2^3 + 3^3 + 6^3 = 251; \text{ with } 1 + 5 + 5 = 2 + 3 + 5 = 11.\]

But we have seen that 251 can be decomposed into other sums of cubes, which lead to more pairs of SC-equivalent triples.

The question is now: how can we retrieve all the triples whose sum of the cubes is \(S_3 = 251\) and sum is \(S_1 = 11\)?
A procedure to find all triples $SC$-equivalent to a given triple

More generally, let us suppose we are given three integers $x, y, z$ and we define:

$$x + y + z = S_1 \text{ and } x^3 + y^3 + z^3 = S_3.$$ 

Our aim is now to find all the triples $(u, v, w)$ made up of integers such that:

$$u + v + w = S_1 \text{ and } u^3 + v^3 + w^3 = S_3.$$ 

By eliminating $w$ between these two relations, we obtain after all simplifications:

$$-3(u + v)(S_1 - u)(S_1 - v) = S_3 - S_1^3.$$ 

We notice that $S_3 - S_1^3 = -3(x + y)(y + z)(z + x)$ (formula 1), and obtain accordingly:

$$uv + S_1^2 - S_1(u + v) = \frac{(x + y)(y + z)(z + x)}{(u + v)}.$$ 

Therefore, from here we proceed as follows:

(1): tabulate all the integers $D$ dividing $(x + y)(y + z)(z + x)$; it must be noted here that, in order not to miss any solution, one must look for $D$ not only among the positive rational integers, but also among the negative ones, and the Gaussian integers as well.

The search for the divisors $D$ is helped by the factorised form of $(x + y)(y + z)(z + x)$, which explains the interest one has in replacing $S_3 - S_1^3$ in the previous step.

(2): for any integral divisor $D$ of $(x + y)(y + z)(z + x)$, set $u + v = D$ in the equation above; this allows an easy computation of $uv = P$.

Now computing $u$ and $v$ is equivalent to solving the quadratic equation $X^2 - DX + P = 0$ which will yield integral solutions $u, v$ only if its discriminant $D^2 - 4P$ is a rational square or, more generally, the square of a Gaussian integer. And $W = S_1 - u - v = S_1 - D$.

In practice, the number of quadratic equations grows very rapidly as the absolute values of $x, y$ and $z$ increase, and quickly the use of a computer becomes of great help. For instance, it is by using a commercial spreadsheet that we were able to compute in a practical fashion all the above decompositions of 251 into a sum of three cubes, with $S_1 = 11$ for all of them.

In principle, another method could be to solve the equation in $u, v, w$:

$$u^3 + v^3 + w^3 = S_3$$

for a given $S_3$, and then look for pairs of solutions with the same $S_1$. But, as the reader might be convinced by reading [1, 2, 3], solving this Diophantine equation requires theoretical knowledge and computational means well beyond the scope of this simple note. The referee has pointed out [4] which considers similar equations and gives upper and lower bounds on the number of solutions in integers of size at most $B$. 

[1, 2, 3, 4]
References
   (can be downloaded from the Internet site: http://euler.free.fr/docs/HLR93.pdf)

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90.35 More on the family of subsets of a finite set

I enjoyed Thomas Koshy's note [1] which ends with the challenge to find a formula for the binomial sums $s_j(n) = \sum_{h=0}^{\lfloor n/j \rfloor} \binom{n}{mh+j}$, $0 < j < m-1$, $m > 1$. The solution provides a pleasant example of a standard procedure for dealing with such sums of 'arithmetic' subseries of a given—finite or infinite—series which we formalise as a lemma.

Lemma: Let $f(x) = \sum_{h=0}^{\infty} a_h x^h$ and let $\omega = \exp(2\pi i / m)$. Then, if $\sigma_j = \sum_{h=0}^{\infty} a_{mh+j}$, we have $\sigma_j = \frac{1}{m} \sum_{k=0}^{m-1} \omega^{-kj} f(\omega^k)$.

The verification of this is immediate:

$$\sum_{k=0}^{m-1} \omega^{-kj} f(\omega^k) = \sum_{k=0}^{m-1} \omega^{-kj} \sum_{h=0}^{\infty} a_h \omega^{hk} = \sum_{h=0}^{\infty} a_h \sum_{k=0}^{m-1} \omega^{k(h-j)}$$

$$= m \sum_{h=0}^{\infty} a_{mh+j} = m \sigma_j.$$

since $\sum_{k=0}^{m-1} \omega^{k(h-j)} = \begin{cases} m & \text{is a multiple of } m, \\ 0 & \text{is not a multiple of } m. \end{cases}$

Applying the Lemma to $f(x) = (1 + x)^n = \sum_{h=0}^{\infty} \binom{n}{h} x^h$ (so $\sigma_j = \sum_{h=0}^{\infty} \binom{n}{mh+j} = s_j(n)$) gives us: