TWISTED CROSSED PRODUCTS BY COACTIONS

JOHN PHILLIPS and IAIN RAEBURN

Abstract

We consider coactions of a locally compact group G on a C^*-algebra A, and the associated crossed product C^*-algebra $A \times G$. Given a normal subgroup N of G, we seek to decompose $A \times G$ as an iterated crossed product $(A \times G/N) \times N$, and introduce notions of twisted coaction and twisted crossed product which make this possible. We then prove a duality theorem for these twisted crossed products, and discuss how our results might be used, especially when N is abelian.

1991 *Mathematics subject classification* (*Amer. Math. Soc.*): 46 L 55, 22 D 25, 22 D 35.

0. Introduction

It has been known for many years that the group C^*-algebra of a semi-direct product $N \rtimes H$ can be decomposed as a crossed product $C^*(N) \times H$; this observation has both motivated the study of crossed products, and influenced the development of their representation theory, through the various extensions of the Mackey machine (for example, [24, 7]). Applications have required generalisations of this decomposition: the version for non-split group extensions involves twisted crossed products rather than ordinary ones, and for inductive arguments it is necessary to decompose crossed products as well as group algebras. In particular, the various decompositions of a crossed product $A \times G$ as an iterated twisted crossed product $(A \times N) \times G/N$ (for example, [7, 6, 17]) have been important tools in several recent projects (for example, [7, 16, 1]). Here we want to discuss an analogous decomposition for crossed products by

coactions of nonabelian groups, and the appropriate family of twisted crossed products.

Our decomposition theorem is modelled on that of Green [7, Proposition 1] for crossed products by actions. Recall that an action $\beta : G \to \operatorname{Aut} A$ of a locally compact group G is given on a closed normal subgroup N by a twisting map if there is a strictly continuous homomorphism $\tau : N \to UM(A)$ satisfying $\beta|_N = \operatorname{Ad} \tau$ and

$$(0.1) \qquad \beta_s(\tau_n) = \tau_{sns^{-1}} \quad \text{for} \quad s \in G, \ n \in N.$$

A representation $\rho = \pi \times U$ of the crossed product $A \times_\beta G$ is said to preserve the twist τ if $\pi \circ \tau = U|_N$, and then the twisted crossed product $A \times_{\beta, N, \tau} G$ is the quotient of $A \times_\beta G$ by the ideal

$$I_\tau = \bigcap \{ \ker \rho : \rho \text{ is a representation of } A \times_\beta G \text{ preserving } \tau \} .$$

For any action $\alpha : G \to \operatorname{Aut} A$, the canonical embedding i of N in $M(A \times_\alpha N)$ is a twisting map for the natural action β of G on $A \times_\alpha N$, and Green's decomposition asserts that $A \times_\alpha G$ is isomorphic to the twisted crossed product $(A \times_\alpha N) \times_{\beta, N, \tau} G$. The twisted crossed product has properties like those of ordinary crossed products by actions of G/N, so we can profitably think of this as saying $A \times G \cong (A \times N) \times G/N$ — indeed, there is an alternative approach which makes this precise [17].

A coaction δ of G on a C^*-algebra A restricts to a coaction $\delta| = \delta|_{G/N}$ of the quotient G/N by an amenable normal subgroup N; if G were abelian, δ would be given by an action of $\hat{G}$, which would restrict to an action of $N^\perp = (G/N)\hat{}$. We say δ is given by a twist on G/N if there is a corepresentation of G/N in A, which implements $\delta|_{G/N}$ and satisfies an extra consistency condition analogous to (0.1). Formally, this corepresentation is a unitary $W \in UM(A \otimes C^*_r(G/N))$, but slicing it gives a homomorphism j of $C_0(G/N)$ into $M(A)$, and the consistency condition says that $\delta(j(f)) = j(f) \otimes 1$ for $f \in C_0(G/N)$. It was shown in [12] that every representation of $A \times_\delta G$ has the form $\pi \times \mu$ for some pair of representations $\pi : A \to B(\mathcal{H})$ satisfying an appropriate covariance condition (see Section 1), and we say $\pi \times \mu$ preserves the twist W if $\pi \circ j = \mu|_{C_0(G/N)}$. We can now let

$$I_W = \bigcap \{ \ker \pi \times \mu : \pi \times \mu \text{ preserves } W \} ,$$

and define the *twisted crossed product* $A \times_{\delta, G/N, W} G$ to be the quotient $(A \times_\delta G)/I_W$. Thus almost by definition it is a C^*-algebra whose representations are

given by the covariant representations of (A, G, δ) which preserve W, and it can be characterised by this property. The details of this construction are given in Section 2, following a short preliminary section in which we set up notation and review some key material from [12].

Suppose now that $\delta : A \to M(A \otimes C_r^*(G))$ is a coaction of G on A and N is a closed normal amenable subgroup of G. We shall show that there is a natural coaction γ of G on the crossed product $A \times_{\delta_|} G/N$, and that the natural embedding j of $C_0(G/N)$ in $M(A \times_{\delta_|} G/N)$ defines a twist W for γ on G/N. Theorem 3.1 asserts that $A \times_\delta G$ is isomorphic to the twisted crossed product $(A \times_{\delta_|} G/N) \times_{\gamma,G/N,W} G$. Our proof of this is based on the universal properties of the various crossed products, and amounts to showing they have the same representation theory.

For our twisted crossed products to be useful, it is important that they have properties like those of ordinary crossed products: $A \times_{\gamma,G/N,W} G$ should resemble $A \times_\epsilon N$. As evidence that this is the case, we show that there is a duality theorem like that of Katayama [10] for untwisted crossed products: every twisted crossed product $A \times_{\gamma,G/N,W} G$ carries a natural dual action $\hat\delta$ of N such that $(A \times_{G/N} G) \times_{\hat\delta} N$ is Morita equivalent to A. (Katayama's theorem is slightly stronger than the case $N = G$ — his gives an isomorphism $(A \times_\delta G) \times_{\hat\delta} G \cong A \otimes K(L^2(G))$ — but the Morita equivalence should suffice for most applications.) Our Theorem 4.1 follows quite easily from Mansfield's imprimitivity theorem for crossed products by coactions [13].

Our interest in this subject arose from the possibility of reducing questions about crossed products by coactions of a solvable group G to twisted crossed products in which the normal subgroup is an abelian subquotient Q of G, and which should behave like crossed products by actions of $\hat Q$. In the final section we discuss briefly how this procedure might give useful information about the K-theory of crossed products by coactions, and consider some other questions this analysis raises. In particular, we look at the relationship between our algebras $A \times_{\delta,G/N,W} G$ and the twisted crossed products $A \times_{\alpha,u} \hat N$ studied in [4, 17].

NOTATION. We denote by λ or λ_G the left regular representation of a locally compact group G on $L^2(G)$, and by M the representation of $C_0(G)$ as multiplication operators on $L^2(G)$. The reduced group C^*-algebra $C_r^*(G)$ is the image of the full group C^*-algebra $C^*(G)$ under (the integrated form of) λ. The comultiplication $\delta_G : C_r^*(G) \to M(C_r^*(G) \otimes C_r^*(G))$ is the integrated form of the representation $s \to \lambda_s \otimes \lambda_s$, which factors through $C_r^*(G)$ because $\lambda \otimes \lambda$ is equivalent to $\lambda \otimes 1$.

If A acts faithfully on a Hilbert space $\mathcal{H}$, so does its multiplier algebra $M(A)$, and we can identify $M(A)$ with $\{T \in B(\mathcal{H}) : TA + AT \subset A\}$ (c.f. [19, Section 3.12]). A homomorphism φ of A into a multiplier algebra $M(C)$ is *nondegenerate* if there is an approximate identity $\{a_i\}$ for A such that $\varphi(a_i) \to 1$ strictly. Every nondegenerate homomorphism φ extends to a strictly continuous homomorphism of $M(A)$ into $M(C)$, which we still call φ (for example, [12, 1.1]). We shall always use 1 for the identity of an algebra, and i for the identity mapping between algebras. If A and B are C^*-algebras, $A \otimes B$ will denote their spatial or minimal tensor product, and $\sigma : A \otimes B \to B \otimes A$ the flip isomorphism. For the definition and properties of the slice maps $S_f : A \otimes B \to A$ for $f \in B^*$, we refer to [12, Section 1].

1. Crossed products and covariant representations

Let G be a locally compact group and A a C^*-algebra. A coaction of G on A is, roughly speaking, a homomorphism δ of A into $M(A \otimes C^*(G))$ which is comultiplicative. While there are several ways to make this precise, they do apparently lead to the same crossed products (c.f. [22]), and we shall therefore stick with the conventions of [12]. However, we shall want to exploit the point of view of [22], where the crossed product is characterised as a C^*-algebra whose representation theory is the covariant representation theory of the system (A, G, δ), and we shall show how Theorem 3.7 of [12] allows us to do this.

A *coaction* of G on A is a nondegenerate homomorphism δ of A into $M(A \otimes C_r^*(G))$ such that

(1.1) $(\delta \otimes i) \circ \delta = (i \otimes \delta_G) \circ \delta$

(1.2) $\delta(a)(1 \otimes \lambda(z)) \in A \otimes C_r^*(G)$ for all $\lambda(z) \in C_r^*(G)$, $a \in A$.

(Here "nondegenerate" replaces condition 2.6(a) of [12], and condition (1.2) says the range of δ lies in the subalgebra $\tilde{M}(A \otimes C_r^*(G))$ of [12].) Let w_G denote the bounded strictly continuous function $s \to \delta_s$ of G into $UM(C^*(G))$, and view it as a multiplier of $C_0(G, C^*(G)) = C_0(G) \otimes C^*(G)$ — note that the operator $W_G \in U(L^2(G \times G))$ of [12] is $M \otimes \lambda(w_G)$. As in [22], a *covariant representation* of (A, G, δ) is a pair (π, μ) of nondegenerate representations $\pi : A \to B(\mathcal{H})$, $\mu : C_0(G) \to B(\mathcal{H})$ satisfying

$$\pi \otimes i(\delta(a)) = \text{Ad}\,(\mu \otimes \lambda\,(w_G))\,(\pi(a) \otimes 1) \quad \text{in} \quad M(K(\mathcal{H}) \otimes C_r^*(G));$$

equivalently, $(\pi, \mu \otimes \lambda(w_G))$ is a covariant representation in the sense of [12, 3.5]. If δ is a coaction of G on A, and $A \subset B(\mathscr{H})$, the *crossed product* $A \times_\delta G$ is defined in [12] as the C^*-subalgebra of $B(\mathscr{H} \otimes L^2(G))$ generated by

$$\left\{ \delta(a)(1 \otimes M_f) : f \in C_0(G), \ a \in A \right\},$$

but it can be characterised as follows.

THEOREM 1.1. ([12]) *Let* $\delta : A \to M(A \otimes C_r^*(G))$ *be a coaction of* G *on a* C^*-*subalgebra* A *of* $B(\mathscr{H})$, *and define* $j_A = \delta$, $j_{C(G)} = 1 \otimes M$. *Then* j_A, $j_{C(G)}$ *are nondegenerate homomorphisms into* $M(A \times_\delta G)$ *satisfying*

(a) $j_A \otimes i(\delta(a)) = \mathrm{Ad}(j_{C(G)} \otimes \lambda_G(w_G))(j_A(a) \otimes 1)$ *for* $a \in A$;

(b) *for every covariant representation* (π, μ) *of* (A, G, δ) *there is a nondegenerate representation* $\pi \times \mu$ *of* $A \times_\delta G$ *such that*
$$(\pi \times \mu) \circ j_A = \pi \ and \ (\pi \times \mu) \circ j_{C(G)} = \mu;$$

(c) *the set* $\{j_A(a) j_{C(G)}(f) : a \in A, f \in C_0(G)\}$ *spans a dense subspace of* $A \times_\delta G$.

PROOF. That δ and $1 \otimes M$ are nondegenerate homomorphisms into $M(A \times_\delta G)$ is proved in [12, 2.5], and the proof of that result also establishes (c). Equation (a) is essentially the coaction identity:

$$\begin{aligned}
j_A \otimes i(\delta(a)) &= \delta \otimes i(\delta(a)) = i \otimes \delta_G(\delta(a)) \\
&= \mathrm{Ad}(1 \otimes W_G)(\delta(a) \otimes 1) \\
&= \mathrm{Ad}(1 \otimes M \otimes \lambda_G(w_G))(\delta(a) \otimes 1) \\
&= \mathrm{Ad}(j_{C(G)} \otimes \lambda_G(w_G))(j_A(a) \otimes 1).
\end{aligned}$$

If (π, μ) is covariant, the proof of [12, 3.7] shows there is a nondegenerate representation $\pi \times \mu$ of $A \times_\delta G$ such that

$$\pi \times \mu\big(\delta(a)(1 \otimes M(f))\big) = \pi(a)\mu(f),$$

which implies (b).

COROLLARY 1.2. *Let* δ *be a coaction of* G *on* A. *Suppose* B *is a* C^*-*algebra and* $k_A : A \to M(B)$, $k_{C(G)} : C_0(G) \to M(B)$ *are nondegenerate homomorphisms satisfying* (*the analogues of*) (a), (b) *and* (c) *of Theorem* 1.1. *Then there is an isomorphism* φ *of* $A \times_\delta G$ *onto* B *such that* $\varphi \circ j_A = k_A$ *and* $\varphi \circ j_{C(G)} = k_{C(G)}$.

PROOF. Put B on Hilbert space; then $(k_A, k_{C(G)})$ is a covariant representation of (A, G, δ), which by (b) induces a representation $\varphi = k_A \times k_{C(G)}$ of $A \times_\delta G$. Condition (c) implies that φ maps $A \times_\delta G$ onto B and satisfies

$$\varphi\left(j_A(a)j_{C(G)}(f)\right) = k_A(a)k_{C(G)}(f).$$

Reversing the roles of $A \times_\delta G$ and B gives an inverse $j_A \times j_{C(G)}$ for φ.

We shall frequently be restricting a coaction δ of G to a quotient G/N. Roughly, $\delta|_{G/N}$ (or just $\delta|$) is the composition of δ with the canonical map $i \otimes q : A \otimes C_r^*(G) \to A \otimes C_r^*(G/N)$, but in order for this map to be well-defined on the reduced group C^*-algebras, we need to assume N is amenable. If so, $\delta|$ is a coaction of G/N on A [13, Lemma 4].

2. Twisted coactions and twisted crossed products

Throughout this section, $\delta : A \to M(A \otimes C_r^*(G))$ will be a coaction and N a closed normal *amenable* subgroup of G.

DEFINITION 2.1. A *twist* for δ relative to G/N is a unitary $W \in UM(A \otimes C_r^*(G/N))$ such that

(a) $(W \otimes 1) i \otimes \sigma(W \otimes 1) = i \otimes \delta_{G/N}(W)$;

(b) $\delta|(a) = W(a \otimes 1)W^*$ for $a \in A$;

(c) $\delta \otimes i(W) = i \otimes \sigma(W \otimes 1_{C_r^*(G)})$.

We shall also refer to the pair (δ, W) as a *twisted coaction* of $(G, G/N)$ on A.

REMARK 2.2. If we represent A on a Hilbert space $\mathscr{H}$, condition (a) says that W is a corepresentation of G/N on $\mathscr{H}$, and hence there is a nondegenerate representation j of $C_0(G/N)$ on $\mathscr{H}$ such that $W = j \otimes \lambda_{G/N}(w_{G/N})$ and $j(f) = S_f(W)$ for $f \in A(G/N)$ [12, 3.1] and [15, Appendix]. The formula $j(f) = S_f(W)$ implies that j takes values in $M(A)$, and, since we might as well use the universal representation of A, it follows from [18, 1.3] that j is also nondegenerate as a homomorphism into $M(A)$. Now (b) says that (i, j) is a covariant representation of $(A, G/N, \delta|)$, and (c) that $\delta(j(f)) = j(f) \otimes 1$ — or, in other words, that $j(f)$ is a fixed point for the coaction δ.

REMARK 2.3. In various special cases, these are familiar objects. When $N = \{e\}$, the twisted coactions of (G, G) are precisely the unitary coactions of [12, 2.3(3)] and [20, Section 6], and when $N = G$, the identity is a twist for every coaction of G. When G is abelian, the coaction δ is determined by an action $\alpha : \hat{G} \to \text{Aut } A$: for $a \in A$, $\delta(a)$ is the function $\gamma \to \alpha_\gamma(a)$ in the subalgebra $C_b(\hat{G}, A)$ of $M(A \otimes C_0(\hat{G})) \cong M(A \otimes C_r^*(G))$ (c.f. [12, 2.2(4)]). If α is given on $(G/N)\hat{} = N^\perp$ by a Green twisting map $\tau : (G/N)^\wedge \to UM(A)$, then the strictly continuous function $\gamma \to \tau_\gamma$ determines a multiplier W of $A \otimes C^*(G/N)$, which is a twist for δ relative to G/N: condition (a) says τ is a homomorphism, (b) that $\alpha|_{(G/N)\hat{}} = \text{Ad } \tau$, and (c) reduces to (0.1).

EXAMPLE 2.4. A coaction ϵ of N on A induces a twisted coaction $(\delta, 1 \otimes 1)$ of $(G, G/N)$ on A. To see this, we note that the integrated form of the representation $n \to \lambda_G(n)$ of N gives a nondegenerate injection C of $C_r^*(N) \cong C^*(N)$ in $M(C_r^*(G))$. (For a quick proof that C is injective, observe that $C(z) = 0$ implies $0 = \langle C(z), f \rangle = \langle z, f|_N \rangle$ for $f \in B(G)$, and hence in particular $\langle z, g \rangle = 0$ for all $g \in A(N) = A(G)|_N$ [8].) Now we can define $\delta = (i \otimes C) \circ \epsilon$, and the identity $(C \otimes C) \circ \delta_N = \delta_G \circ C$ almost immediately implies that δ is a coaction. The unitary $W = 1 \otimes 1$ trivially satisfies conditions (a) and (c) of Definition 2.1, and, once we observe that the composition $q \circ C : C^*(N) \to M(C_r^*(G)) \to M(C_r^*(G/N))$ is given on $L^1(N)$ by

$$q \circ C(z) = \left(\int z(n)dn \right) 1_{C_r^*(G/N)} = \langle 1, \lambda(z) \rangle 1_{C_r^*(G/N)},$$

we have

$$\delta|(a) = (i \otimes q) \circ (i \otimes C)(\epsilon(a)) = S_1(\epsilon(a)) \otimes 1_{C_r^*(G/N)} = a \otimes 1,$$

so (b) is trivially true too.

DEFINITION 2.5. Let (δ, W) be a twisted coaction of $(G, G/N)$ on A. A covariant representation (π, μ) of (A, G, δ) *preserves the twist W* if

$$\mu \otimes \lambda_{G/N} (w_{G/N}) = \pi \otimes i(W).$$

REMARK 2.6. Let $j : C_0(G/N) \to M(A)$ be the homomorphism determined by the twist W, as in Remark 2.2. Then (π, μ) preserves W if and only if $\pi \circ j = \mu|_{C_0(G/N)}$. (Note that both here and in the definition, we can freely apply μ to functions on G/N because μ is nondegenerate and $C_0(G/N)$ sits naturally inside $M(C_0(G)) = C_b(G)$.)

REMARK 2.7. One justification for the consistency condition (c) in Definition 2.1 is to make it possible for a reasonable number of covariant representations (π, μ) to preserve W. For suppose (δ, W) satisfy (a) and (b) of Definition 2.1, and (π, μ) is a covariant representation of (A, G, δ) satisfying $\mu \otimes \lambda(w_{G/N}) = \pi \otimes i(W)$. Then

$$
\begin{aligned}
\pi \otimes i \otimes i(\delta \otimes i(W)) & \\
&= \mathrm{Ad}\big(\mu \otimes \lambda_G(w_G) \otimes 1\big)\big(i \otimes \sigma(\pi \otimes i(W) \otimes 1)\big) \\
&= \mathrm{Ad}\big(\mu \otimes \lambda_G(w_G) \otimes 1\big)\big(i \otimes \sigma(\mu \otimes \lambda_{G/N}(w_{G/N}) \otimes 1)\big) \\
&= i \otimes \sigma\big(\mu \otimes \lambda_{G/N}(w_{G/N}) \otimes 1\big) \qquad (C_0(G) \text{ is commutative!}) \\
&= \pi \otimes i \otimes i(i \otimes \sigma(W \otimes 1)).
\end{aligned}
$$

Thus if there are to be enough representations (π, μ) preserving W to separate points of A, then condition (c) must be satisfied.

DEFINITION 2.8. Let (δ, W) be a twisted coaction of $(G, G/N)$ on A, and let

$$
I_W = \bigcap \left\{ \ker \pi \times \mu \;\middle|\; \begin{array}{l} (\pi, \mu) \text{ is a covariant representation of} \\ \qquad (A, G, \delta) \text{ which preserves } W \end{array} \right\}.
$$

The *twisted crossed product* is the quotient $A \times_\delta G/I_W$; we denote it by $A \times_{\delta, G/N, W} G$, or just $A \times_{G/N} G$ if no confusion seems likely.

Of course, the idea is that $A \times_{G/N} G$ should be a C^*-algebra whose representations are given by the covariant representations which preserve the twist, and we shall now make this precise. The next lemma provides the basic ingredient, and will also be useful later. Recall that we denote the canonical embeddings of A and $C_0(G)$ in $M(A \times_\delta G)$ by j_A and $j_{C(G)}$, and let $q : A \times_\delta G \to A \times_{\delta, G/N} G$ be the quotient map.

LEMMA 2.9. *Let (δ, W) be a twisted coaction of $(G, G/N)$ on A. Then for $f \in A(G/N)$ we have $q(j_{C(G)}(f)) = q(j_A(S_f(W)))$.*

PROOF. Suppose (π, μ) is covariant and preserves W, and $c \in A \times_\delta G$. Then we have

$$
\begin{aligned}
\pi \times \mu \big(j_{C(G)}(f)c\big) &= \mu(f)\pi \times \mu(c) \\
&= \mu\big(S_f(w_{G/N})\big) \pi \times \mu(c) \\
&= S_f\big(\mu \otimes \lambda_{G/N}(w_{G/N})\big) \pi \times \mu(c) \\
&= S_f\big(\pi \otimes i(W)\big) \pi \times \mu(c) \\
&= \pi \times \mu\big(j_A(S_f(W))\big) \pi \times \mu(c),
\end{aligned}
$$

and hence
$$\left(j_{C(G)}(f) - j_A(S_f(W))\right) c \in I_W = \ker q$$

for all $c \in C$. But according to the definition of the extension of q to $M(A \times_\delta G)$, this says precisely that $q\left(j_{C(G)}(f)\right) = q\left(j_A(S_f(W))\right)$.

COROLLARY 2.10. *If* (π, μ) *is a covariant representation of* (A, G, δ) *with* $I_W \subset \ker \pi \times \mu$, *then* (π, μ) *preserves* W.

PROOF. There is a representation ρ of $(A \times_\delta G)/I_W$ such that $\pi \times \mu = \rho \circ q$, and then $\pi = \rho \circ q \circ j_A$, $\mu = \rho \circ q \circ j_{C(G)}$.

PROPOSITION 2.11. *Let* (δ, W) *be a twisted coaction of* $(G, G/N)$ *on* A, *and let* $k_A = q \circ j_A : A \to M(A \times_{G/N} G)$, $k_{C(G)} = q \circ j_{C(G)} : C_0(G) \to M(A \times_{G/N} G)$. *Then*

(a) $k_A \otimes i(\delta(a)) = \mathrm{Ad}\, k_{C(G)} \otimes i\, (w_G) (k_A(a) \otimes 1)$ *for all* $a \in A$;
(b) $k_A \otimes i(W) = k_{C(G)} \otimes \lambda_{G/N}(w_{G/N})$;
(c) *for every covariant representation* (π, μ) *of* (A, G, δ) *which preserves* W, *there is a nondegenerate representation* $\pi \times_{G/N} \mu$ *of* $A \times_{\delta, G/N, w} G$ *such that* $(\pi \times_{G/N} \mu) \circ k_A = \pi$ *and* $(\pi \times_{G/N} \mu) \circ k_{C(G)} = \mu$;
(d) *the set* $\{k_A(a)k_{C(G)}(f) : a \in A, \ f \in C_0(G)\}$ *spans a dense subspace of* $A \times_{\delta, G/N, w} G$.

PROOF. Conditions (a) and (d) follow immediately from the corresponding properties of $(A \times_\delta G, j_A, j_{C(G)})$. To establish (b), we let $f \in A(G/N)$ and compute (recalling that we identify $C_0(G/N)$ with its image in $M(C_0(G))$):

$$\begin{aligned}
S_f\left(j_{C(G)} \otimes \lambda_{G/N}(w_{G/N})\right) &= j_{C(G)}\left(S_f(w_{G/N})\right)\\
&= j_{C(G)}(f)\\
&= j_A(S_f(W)) \quad \text{by the lemma}\\
&= S_f\left(j_A \otimes i(W)\right).
\end{aligned}$$

Since this holds for every $f \in A(G/N)$, it implies (b). If (π, μ) is covariant and preserves W, then the representation $\pi \times \mu$ of $A \times_\delta G$ vanishes on I_W, and hence factors through a representation $\pi \times_{G/N} \mu$ of $A \times_\delta G/I_W$ such that $(\pi \times_{G/N} \mu) \circ q = \pi \times \mu$. Thus (c) follows from the equations $(\pi \times \mu) \circ j_A = \pi$, $(\pi \times \mu) \circ j_{C(G)} = \mu$.

REMARK 2.12. The standard arguments (cf. Corollary 1.2) show that these properties characterise the twisted crossed product. To be precise: if $(B, l_A, l_{C(G)})$ is a triple consisting of a C^*-algebra B and nondegenerate homomorphisms of A, $C_0(G)$ into $M(B)$ satisfying (the analogues of) (a), (b), (c) and (d), then there is an isomorphism φ of $A \times_{\delta, G/N, w} G$ onto B such that $\varphi \circ k_A = l_A$ and $\varphi \circ k_{C(G)} = l_{C(G)}$.

EXAMPLE 2.13. If $\delta = \operatorname{Ad} W : a \to W(a \otimes 1)W^*$ is a unitary coaction, then $A \cong A \times_{\delta, G, w} G$. By the previous remark, it is enough to show that if $j : C_0(G) \to M(A)$ is defined by $j(f) = S_f(W)$ as in Remark 2.2, then the triple (A, i, j) satisfies the conditions in Proposition 2.11. But (a) is the statement $\delta = \operatorname{Ad} W$, (b) is the equation $W = j \otimes \lambda_G(w_G)$ defining j, and (d) holds because j is nondegenerate; to get (c), note that if (π, μ) preserves W, then $\mu = \pi \circ j$ (see Remark 2.6), and we can define $\pi \times \mu = \pi$.

EXAMPLE 2.14. Suppose $(\delta, 1 \otimes 1)$ is induced from a coaction ϵ of N, as in Section 2.4. Then we claim that $A \times_{\delta, G/N, 1 \otimes 1} G \cong A \times_\epsilon N$. By Corollary 1.2, it is enough to find maps $k_A, k_{C(N)}$ such that $(A \times_{G/N} G, k_A, k_{C(N)})$ satisfies the conditions characterising $A \times_\epsilon N$ in Theorem 1.1. For k_A, we take the usual embedding of A in $M(A \times_{G/N} G)$. To define $k_{C(N)}$, we show that $k_{C(G)}$ factors through the quotient map $C_0(G) \to C_0(N)$. To see this, note that for $g \in A(G/N)$, we have

$$
\begin{aligned}
k_{C(G)}(g) &= S_g\big(k_{C(G)} \otimes \lambda_{G/N}(w_{G/N})\big) \\
&= S_g\big(k_A \otimes i(1_A \otimes 1_{C_r^*(G/N)})\big) \\
&= g(N)1_{A \times_{G/N} G},
\end{aligned}
$$

and by continuity this must also hold for any $g \in C_0(G/N)$. Now a standard approximation argument shows that for $f \in C_0(G)$, $f|_N = 0$ implies $k_{C(G)}(f) = 0$, and hence we can define $k_{C(N)}$ by $k_{C(N)}(f|_N) = k_{C(G)}(f)$. To check (a), we recall that $C : C^*(N) \to M(C_r^*(G))$ is injective, and compute

$$
\begin{aligned}
i \otimes C\big(k_A \otimes i(\epsilon(a))\big) &= k_A \otimes i(\delta(a)) \\
&= \operatorname{Ad} k_{C(G)} \otimes \lambda_G(w_G)\,(k_A(a) \otimes 1) \\
&= \operatorname{Ad} k_{C(N)} \otimes \lambda_G(w_G|_N)\,(k_A(a) \otimes 1) \\
&= i \otimes C\big(\operatorname{Ad} k_{C(N)} \otimes \lambda_N(w_N)\,(k_A(a) \otimes 1)\big).
\end{aligned}
$$

Next, suppose (π, μ) is a covariant representation of (A, N, ϵ), and define a representation ν of $C_0(G)$ by $\nu(f) = \mu(f|_N)$. Then (π, ν) is a covariant

representation of (A, G, δ) which preserves $1 \otimes 1$, and $\pi \times_{G/N} \nu$ has the properties required of $\pi \times \mu$ in (b). The density condition (c) follows from the corresponding property of $A \times_{G/N} G$, and our claim is justified.

3. The decomposition theorem

THEOREM 3.1. *Suppose $\delta : A \to M(A \otimes C_r^*(G))$ is a coaction of a locally compact group G on A, and N is a closed normal amenable subgroup of G. Then there is a twisted coaction (γ, W) of $(G, G/N)$ on $A \times_{\delta|} G/N$ such that*

$$A \times_\delta G \cong \left(A \times_{\delta|} G/N\right) \times_{\gamma, G/N, W} G;$$

formulas for γ and W are given in Lemmas 3.3 and 3.4 below.

Most of this section is devoted to the proof of this theorem, and throughout we shall use the notation in its statement. We first have to construct the coaction γ and the twist W. For this, we need the following simple lemma.

LEMMA 3.2. *Let σ denote the flip isomorphism of $C_r^*(G/N) \otimes C_r^*(G)$ onto $C_r^*(G) \otimes C_r^*(G/N)$. Then for $a \in A$ we have*

$$\delta \otimes i(\delta|(a)) = i \otimes \sigma \left[(\delta| \otimes i)(\delta(a))\right].$$

LEMMA 3.3. *Let $\pi = (j_A \otimes i) \circ \delta : A \to M((A \times_{\delta|} G/N) \otimes C_r^*(G))$ and $\mu = j_{C(G/N)} \otimes 1 : C_0(G/N) \to M((A \times_{\delta|} G/N) \otimes C_r^*(G))$. Then (π, μ) is covariant in the sense that*

$$\pi \otimes i(\delta|(a)) = \mu \otimes \lambda_{G/N}(w_{G/N}) (\pi(a) \otimes 1) \mu \otimes \lambda_{G/N}(w_{G/N})^*.$$

The resulting nondegenerate homomorphism

$$\gamma = \pi \times \mu : A \times_{\delta|} G/N \to M \left((A \times_{\delta|} G/N) \otimes C_r^*(G)\right)$$

is a coaction of G on $A \times_{\delta|} G/N$.

PROOF. We verify the covariance of (π, μ) by computing and using Lemma 3.2:

$$\pi \otimes i(\delta|(a))\mu \otimes \lambda(w_{G/N})$$
$$= j_A \otimes i \otimes i\left(\delta \otimes i(\delta|(a))\right)\mu \otimes \lambda(w_{G/N})$$

$$= j_A \otimes i \otimes i\big(i \otimes \sigma\,[(\delta| \otimes i)(\delta(a))]\big)i \otimes \sigma\,\big(j_{C(G/N)} \otimes \lambda(w_{G/N}) \otimes 1_{C_r^*(G)}\big)$$
$$= i \otimes \sigma\,\big[j_A \otimes i \otimes i\big(\delta| \otimes i\,(\delta(a))\big)\,\big(j_{C(G/N)} \otimes \lambda(w_{G/N}) \otimes 1\big)\big]$$
$$= i \otimes \sigma\,\big[\big(j_{C(G/N)} \otimes \lambda(w_{G/N}) \otimes 1\big)i \otimes \sigma\big(j_A \otimes i(\delta(a)) \otimes 1\big)\big]$$
$$= \mu \otimes \lambda(w_{G/N})(\pi(a) \otimes 1).$$

It now follows from the universal property of $A \times_\delta G$ that there is a nondegenerate homomorphism $\gamma = \pi \times \mu$ satisftying $\pi = \gamma \circ j_A$, $\mu = \gamma \circ j_{C(G/N)}$. Since the elements of the form $j_A(a)j_{C(G/N)}(f)$ span a dense subspace of $A \times_{\delta|} G/N$, it is enough to check the coaction properties on them. First, note that for $\lambda(z) \in C_r^*(G)$, we have

$$(1 \otimes \lambda(z))\big(\gamma\big(j_A(a)j_{C(G/N)}(f)\big)\big) = (1 \otimes \lambda(z))j_A \otimes i(\delta(a))\,\big(j_{C(G/N)}(f) \otimes 1\big)$$
$$= j_A \otimes i\big((1 \otimes \lambda(z))\delta(a)\big)\,\big(j_{C(G/N)}(f) \otimes 1\big)$$
$$\in j_A \otimes i\,\big(A \otimes C_r^*(G)\big)\big(j_{C(G/N)}(C_0(G/N)) \otimes 1\big)$$
$$\subset M\big((A \times_{\delta|} G/N) \otimes C_r^*(G)\big).$$

Now we can verify the coaction identity:

$$\gamma \otimes i\,\big(\gamma\,\big(j_A(a)j_{C(G/N)}(f)\big)\big) = \gamma \otimes i\big(j_A \otimes i(\delta(a))j_{C(G/N)}(f) \otimes 1\big)$$
$$= j_A \otimes i \otimes i\big(\delta \otimes i(\delta(a))\big)\,\big(j_{C(G/N)}(f) \otimes 1 \otimes 1\big)$$
$$= j_A \otimes i \otimes i(i \otimes \delta_G(\delta(a)))\,\big(j_{C(G/N)}(f) \otimes 1 \otimes 1\big)$$
$$= i \otimes \delta_G\,\big(j_A \otimes i(\delta(a))(j_{C(G/N)}(f) \otimes 1)\big)$$
$$= i \otimes \delta_G\,\big(\gamma(j_A(a)j_{C(G/N)}(f))\big)$$

LEMMA 3.4. *Let γ be as in Lemma* 3.3. *Then* $W = j_{C(G/N)} \otimes \lambda_{G/N}(w_{G/N})$ *is a twist for γ relative to G/N.*

PROOF. We verify properties (a), (b) and (c) of Definition 2.1. The first is easy: we just observe that both sides of (a) are the image of the function $sN \to \lambda_{G/N}(sN) \in M(C_0(G/N) \otimes C_r^*(G/N) \otimes C_r^*(G/N))$ under the homomorphism $j_{C(G/N)} \otimes i \otimes i$. It is enough to check (b) on elements of the form $b = j_A(a)j_{C(G/N)}(f)$:

$$\gamma|(b) = i \otimes q(\gamma(b))$$
$$= i \otimes q\,\big(j_A \otimes i(\delta(a))(j_{C(G/N)}(f) \otimes 1)\big)$$
$$= j_A \otimes i(\delta|(a))(j_{C(G/N)}(f) \otimes 1)$$
$$= j_{C(G/N)} \otimes \lambda_{G/N}(w_{G/N})(j_A(a) \otimes 1)j_{C(G/N)} \otimes \lambda_{G/N}(w_{G/N})^*(j_{C(G/N)}(f) \otimes 1)$$
$$= j_{C(G/N)} \otimes \lambda_{G/N}(w_{G/N})(j_A(a)j_{C(G/N)}(f) \otimes 1)j_{C(G/N)} \otimes \lambda_{G/N}(w_{G/N})^*,$$

where the last step works because $C_0(G/N)$ is commutative. Finally, we have

$$
\begin{aligned}
\gamma \otimes i(W) &= \gamma \otimes i \left(j_{C(G/N)} \otimes \lambda_{G/N}(w_{G/N}) \right) \\
&= i \otimes \sigma \left(j_{C(G/N)} \otimes \lambda_{G/N}(w_{G/N}) \otimes 1 \right) \\
&= i \otimes \sigma(W \otimes 1),
\end{aligned}
$$

so (c) holds.

PROOF OF THEOREM 3.1. Let

$$
l_A = k_{A \times G/N} \circ j_A : A \to M((A \times_{\delta|} G/N) \times_{\gamma, G/N} G);
$$

we shall prove that $((A \times_{\delta|} G/N) \times_{\gamma, G/N} G, l_A, k_{C(G)})$ is a crossed product for (A, G, δ), in the sense that it satisfies conditions (a), (b), (c) of Theorem 1.1. First of all, we have

$$
\begin{aligned}
l_A \otimes i(\delta(a)) &= k_{A \times G/N} \otimes i \left(j_A \otimes i(\delta(a)) \right) \\
&= k_{A \times G/N} \otimes i \left(\gamma \left(j_A(a) \right) \right) \\
&= k_{C(G)} \otimes i \, (w_G) \left(k_{A \times G/N} \left(j_A(a) \right) \otimes 1 \right) k_{C(G)} \otimes i \, (w_G)^*,
\end{aligned}
$$

so (a) is easy. Next suppose that (π, μ) is a covariant representation of (A, G, δ), and let $\nu = \mu|_{C_0(G/N)}$. We want to show that (π, ν) is covariant, and then that $(\pi \times \nu, \mu)$ is a covariant representation of $(A \times_{\delta|} G/N, G, \gamma)$ which preserves W. We have

$$
\begin{aligned}
\pi \otimes i(\delta|(a)) &= \pi \otimes i \left(i \otimes q(\delta(a)) \right) \\
&= i \otimes q \left(\mu \otimes \lambda_G \, (w_G) \, (\pi(a) \otimes 1) \mu \otimes \lambda_G \, (w_G)^* \right) \\
&= \mathrm{Ad} \, (\mu \otimes (q \circ \lambda_G) \, (w_G)) \, (\pi(a) \otimes 1).
\end{aligned}
$$

Now $i \otimes (q \circ \lambda_G)(w_G)$ is the multiplier of $C_0(G, C_r^*(G/N))$ given by the function $s \to \lambda_{G/N}(sN)$, which is constant on N-cosets; thus

$$
\mu \otimes (q \circ \lambda) \, (w_G) = \nu \otimes \lambda_{G/N}(w_{G/N}),
$$

and we have shown that (π, ν) is covariant. To show that $(\pi \times \nu, \mu)$ is covariant, we compute:

$$
\begin{aligned}
(\pi \times \nu) \otimes i \left(\gamma \left(j_A(a) j_{C(G/N)}(f) \right) \right) \\
= (\pi \times \nu) \otimes i \left(j_A \otimes i(\delta(a))(j_{C(G/N)}(f) \otimes 1) \right)
\end{aligned}
$$

$$= \pi \otimes i(\delta(a))(\nu(f) \otimes 1)$$
$$= \mu \otimes i\,(w_G)\,(\pi(a) \otimes 1)\mu \otimes i\,(w_G)^*\,(\nu(f) \otimes 1)$$
$$= \mu \otimes i\,(w_G)\,(\pi(a)\nu(f) \otimes 1)\,\mu \otimes i\,(w_G)^*$$
$$= \mu \otimes i\,(w_G)\,\big(\pi \times \nu(j_A(a)j_{C(G/N)}(f)) \otimes 1\big)\,\mu \otimes i\,(w_G)^*.$$

It is also easy to see that $(\pi \times \nu, \mu)$ preserves W:

$$(\pi \times \nu) \otimes i(W) = (\pi \times \nu) \otimes i\,\big(j_{C(G/N)} \otimes \lambda_{G/N}(w_{G/N})\big)$$
$$= \nu \otimes \lambda_{G/N}(w_{G/N}),$$

which is just $\mu \otimes \lambda_{G/N}(w_{G/N})$ by definition of ν. Thus we obtain a representation $\rho = (\pi \times \nu) \times_{G/N} \mu$ of $(A \times G/N) \times_{\gamma, G/N} G$, which immediately satisfies $\rho \circ k_{C(G)} = \mu$, and also

$$\rho \circ l_A = \rho \circ k_{A \times G/N} \circ j_A = (\pi \times \nu) \circ j_A = \pi.$$

Thus our triple $((A \times G/N) \times_{G/N} G, l_A, k_{C(G)})$ also satisfies (b).

It remains to check (c). We know that $\{j_A(a)j_{C(G/N)}(g)\}$ spans a dense subspace of $A \times G/N$, and hence that the elements of the form

$$k_{A \times G/N}\big(j_A(a)j_{C(G/N)}(g)\big)k_{C(G)}(f) = l_A(a)k_{A \times G/N} \circ j_{C(G/N)}(g)k_{C(G)}(f)$$

span a dense subspace of $(A \times G/N) \times_{G/N} G$. But if $g \in A(G/N)$, which is dense in $C_0(G/N)$, we have

$$j_{C(G/N)}(g) = j_{C(G/N)}\big(S_g(w_{G/N})\big) = S_g\big(j_{C(G/N)} \otimes \lambda_{G/N}(w_{G/N})\big) = S_g(W),$$

so

$$l_A(a)k_{A \times G/N} \circ j_{C(G/N)}(g)k_{C(G)}(f) = l_A(a)k_{A \times G/N}\big(S_g(W)\big)k_{C(G)}(f)$$
$$= l_A(a)k_{C(G)}(g)k_{C(G)}(f)$$
$$= l_A(a)k_{C(G)}(gf).$$

Thus each of our set of generators for which $g \in A(G/N)$ is in the set $\{l_A(a)k_{C(G)}(f)\}$, and hence the latter span a dense subspace of $(A \times_{\delta_|} G/N) \times_{G/N} G$. We have now shown that

$$\big((A \times_{\delta_|} G/N) \times_{\gamma, G/N} G, l_A, k_{C(G)}\big)$$

is a crossed product for (A, G, δ), and hence is isomorphic to $(A \times_\delta G, j_A, j_{C(G)})$.

REMARK 3.5. In applications, one would hope to use this theorem inductively to decompose a crossed product along a composition series for G: if $\{e\} = N_0 \vartriangleleft N_1 \vartriangleleft \ldots \vartriangleleft N_k = G$, we should have

$$A \times_\delta G \cong \left(\ldots \left(\left(A \times_{\delta_1} G/N_{k-1} \right) \times_{G/N_{k-1}} G/N_{k-2} \right) \times \ldots \times_{G/N_1} G \right).$$

To justify this, we need a slightly stronger version of Theorem 3.1, in which we start with a normal subgroup M containing N and a twisted coaction (δ, Y) of $(G, G/N)$, and deduce that

$$A \times_{\delta, G/M, Y} G \cong \left(A \times_{\delta_1, G/M, Y} G/N \right) \times_{\gamma, G/N, W} G.$$

Although the notation gets a little messy, it is quite easy to extend our argument, and we shall now outline the extra steps involved.

First, one has to verify that twisted actions can be restricted to the quotient G/N: in fact, the same Y, viewed as an element of $UM(A \otimes C_r^*((G/N)/(M/N)))$, is a twist for $\delta|_{G/N}$ relative to M/N. The homomorphism $\gamma_1 = ((k_A \otimes i) \circ \delta) \times (k_{C(G/N)} \otimes 1)$ of $A \times_{\delta_1} G/N$ into $M((A \times_{G/M} G/N) \otimes C_r^*(G))$ preserves the twist Y, and induces a coaction γ of G on $A \times_{\delta_1, G/M, Y} G/N$, as in Lemma 3.3. As before, $W = k_{C(G/N)} \otimes \lambda_{G/N}(w_{G/N})$ is a twist for γ relative to G/N, and we prove the theorem by showing that $((A \times_{\delta_1, G/M} G/N) \times_{\gamma, G/N, W} G, l_A, k_{C(G)})$ is a twisted crossed product for $(A, (G, G/M), (\delta, Y))$. The old verifications of (a) and (c) carry over, giving (a) and (d) of Proposition 2.11. The new condition (b) is verified by applying its analogues to the pairs $(k_A, k_{C(G/N)})$ and $(k_{A \times_{G/M} G/N}, k_{C(G)})$, respectively:

$$\begin{aligned} l_A \otimes i(Y) &= \left(k_{A \times_{G/M} G/N} \otimes i \right) \circ (k_A \otimes i)(Y) \\ &= \left(k_{A \times_{G/M} G/N} \otimes i \right) \left(k_{C(G/N)} \otimes \lambda_{G/N}(w_{G/N}) \right) \\ &= k_{C(G)} \otimes \lambda_{G/N}(w_{G/N}). \end{aligned}$$

In checking (c), we now also have to show that, if (π, μ) preserves Y, then $\left(\pi, \mu|_{C_0(G/N)} \right)$ preserves the twist Y for $\delta|$, and $(\pi \times \nu, \mu)$ preserves the twist W for γ. However, this is quite straightforward, and hence the theorem generalises to twisted crossed products, as claimed.

4. Duality for twisted crossed products

THEOREM 4.1. *Let G be a locally compact group, N a closed normal amenable subgroup, and (δ, W) a twisted coaction of $(G, G/N)$ on a C^*-algebra A.*

There is an action $\hat{\delta}$ of N on $A \times_{\delta, G/N, W} G$ such that

$$\hat{\delta}_n \left(k_A(a) k_{C(G)}(f) \right) = k_A(a) k_{C(G)} \left(\sigma_n(f) \right) \quad \text{for } a \in A, \ f \in C_0(G),$$

and then $(A \times_{\delta, G/N, W} G) \times_{\hat{\delta}} N$ is Morita equivalent to A.

Let $B = A \times_{\delta, G/N, W} G$, and denote by σ_s the automorphism of $C_0(G)$ defined by $\sigma_s(f)(t) = f(ts)$. As in [22, 2.14], one verifies that, for any $s \in G$, $(B, k_A, k_{C(G)} \circ \sigma_s)$ is a crossed product for (A, G, δ), and it is easy to check that, for $n \in N$, the pair $(k_A, k_{C(G)} \circ \sigma_n)$ preserves W. Thus the existence of $\hat{\delta}_n$ follows from the uniqueness of the crossed product (Section 2.12). The continuity of the action σ on $C_0(G)$ implies that $\hat{\delta}$ is a strongly continuous action of N on B.

The imprimitivity bimodule will be the quotient of the $(A \times_\delta G) \times_{\hat{\delta}} N - A \times G/N$ bimodule X of Mansfield [13] corresponding to the ideal

$$I_W = \bigcap \left\{ \ker \pi \times \mu \ \middle| \ \begin{array}{l} (\pi, \mu) \text{ is a covariant representation of} \\ \quad\quad (A, G/N, \delta|) \text{ preserving } W \end{array} \right\}$$

in $A \times G/N$. This gives an equivalence between the quotient $(A \times G/N)/I_W = A \times_{G/N} G/N$, which is isomorphic to A, and a quotient of $(A \times_\delta G) \times_{\hat{\delta}} N$, which we shall have to identify with $(A \times_{G/N} G) \times N$. The key observation is that Mansfield's inducing process is compatible with the twists.

If (π, μ) is a covariant representation of $(A, G/N, \delta|)$, we shall denote by $\operatorname{Ind}^{A \times G} \pi \times \mu$ the representation of $A \times_\delta G$ induced via Mansfield's bimodule X, and by $\operatorname{Ind}^E \pi \times \mu$ the corresponding induced representation of the imprimitivity algebra $E = (A \times_\delta G) \times_{\hat{\delta}} N$. Since the original action of $A \times_\delta G$ on the bimodule X agrees with that obtained from the canonical embedding of $A \times_\delta G$ in $M(E) \cong B(X)$ (c.f. [13, Section 5]), there is a unitary representation V of N such that

$$\operatorname{Ind}^E \pi \times \mu = \left(\operatorname{Ind}^{A \times G} \pi \times \mu \right) \times V.$$

LEMMA 4.2. *Suppose (π, μ) a covariant representation of $(A, G/N, \delta|)$. Then (π, μ) preserves W (as a twist for $\delta|$) if and only if $\operatorname{Ind}^{A \times G} \pi \times \mu$ preserves W.*

PROOF. The bimodule X is constructed in [13] as the completion of a subalgebra $\mathscr{D}$ of $A \times_\delta G$, which roughly speaking consists of norm limits of sums of elements of the form $\delta(S_v(\delta(a)))(1 \otimes M_f)$, where $v \in A_c(G)$ is fixed and the support of $f \in C_c(G)$ lies in a fixed compact subset E of G. (For convenience,

we use here notation like that of [13], so that $A \times_\delta G$ is defined concretely as an algebra of operators on $\mathcal{H} \otimes L^2(G)$, and $j_A = \delta$, $j_{C(G)} = 1 \otimes M$.) To define the induced representation $\mathrm{Ind}^{A \times G} \pi \times \mu$ we need the $A \times G/N$-valued inner product on $\mathcal{D}$, and we recall how it was constructed in [13, Section 4]. We view $A \times_{\delta|} G/N$ as an algebra of operators on $\mathcal{H} \otimes L^2(G)$ via the representation $(\pi \otimes i) \circ \delta \times (1 \otimes M_G)$ (c.f. [13, Proposition 7]), so that $\delta|(b)(1 \otimes M_{G/N}(f))$ is carried into $\delta(b)(1 \otimes M_G(f))$; in [13], this image algebra is denoted $A \times_\delta G/N$. Mansfield then proves that the map

$$\Psi : \delta(\delta_v(a))\,(1 \otimes M(f)) \rightarrow \delta(\delta_v(a))\,(1 \otimes M(\varphi(f)))\,,$$

where $\varphi(f)(sN) = \int f(xn)\,dn$ and $\delta_v(a) = S_v(\delta(a))$ extends to a well defined map of $\mathcal{D}$ into a subalgebra $\mathcal{D}_N$ of $A \times_\delta G/N$ [13, Proposition 16]. He also gives an alternative characterization of Ψ as a map from $\mathcal{D}$ to $M(A \times_\delta G/N)$: for $x, y \in \mathcal{D}$, we have

$$\Psi(x)y = \int_N \hat{\delta}_n(x)y\,dn, \qquad y\Psi(x) = \int_N y\hat{\delta}_n(x)\,dn$$

[13, Lemma 18]. From this characterization we can deduce that Ψ satisfies $\Psi(x^*) = \Psi(x)^*$, and hence has the following expectation-like properties:

(4.1) $\Psi\,(\delta(\delta_u(a))x) = \delta(\delta_u(a))\Psi(x)$ for $a \in A$, $u \in A_c(G)$

(4.2) $\Psi((1 \otimes M(g))x) = (1 \otimes M(g))\Psi(x)$ for $g \in C_c(G/N)$.

(The adjoint of (4.2) holds because $\varphi(fg) = \varphi(f)g$ if g is constant on N-orbits.) Finally, the inner product on $\mathcal{D}$ is defined by

$$\langle x, y \rangle = \varphi(x^*y),$$

but we have to remember that the latter really belongs to the concrete realisation $A \times_\delta G/N$ of $A \times_{\delta|} G/N$ on $\mathcal{H} \otimes L^2(G)$.

Now suppose (π, μ) is a covariant representation of $(A, G/N)$. The induced representation $\mathrm{Ind}^{A \times G} \pi \times \mu$ acts, via left multiplication on $\mathcal{D}$, in the completion of $\mathcal{D} \odot \mathcal{H}_\pi$ in the norm defined by the inner product

$$\left(x \otimes \xi \mid y \otimes \eta\right) = \left(\pi \times \mu(\langle y, x\rangle)\xi \mid \eta\right),$$

and we therefore have

$$\left(\mathrm{Ind}\,\pi \times \mu(x)(y \otimes \xi) \mid z \otimes \eta\right) = \left(\pi \times \mu\,(\Psi\,(z^*xy))\,\xi \mid \eta\right)$$
$$\text{for } x \in \mathcal{D} \subset A \times_\delta G.$$

This induced representation preserves the twist W when

$$\operatorname{Ind} \pi \times \mu(\delta(j(f))) = \operatorname{Ind} \pi \times \mu\,(1 \otimes M_G(f)) \quad \text{for } f \in C_0(G/N).$$

Note that, since $\delta(j(f)) = j(f) \otimes 1$, we can write $j(f) = \delta_u(j(f))$ for any $u \in A_c(G)$ satisfying $u(e) = 1$, and hence we can apply the expectation property (4.1) with $a = j(f)$. To verify that two operators S, T on $\mathcal{H}_{\operatorname{Ind} \pi \times \mu}$ are equal, it is enough to show

$$\bigl(S(y \otimes \xi) \mid z \otimes \eta\bigr) = \bigl(T(y \otimes \xi) \mid z \otimes \eta\bigr)$$

for z of the form $(1 \otimes M(k))\delta\,(\delta_v(a))$, and hence $\operatorname{Ind} \pi \times \mu$ preserves W if and only if

$$\pi \times \mu\bigl(\Psi(\delta(\delta_v(a))(1 \otimes M(k))\delta(j(f)))y)\bigr) = \pi \times \mu\bigl(\Psi(\delta(\delta_u(a))(1 \otimes M(kf))y)\bigr).$$

The expectation properties of Ψ show that this is equivalent to

$$\pi \times \mu\bigl(\delta(\delta_v(a))j(f)\Psi((1 \otimes M(k))y)\bigr)$$
$$= \pi \times \mu\bigl(\delta(\delta_v(a))(1 \otimes M(f))\Psi((1 \otimes M(k))y)\bigr).$$

Finally, we recall that the identification of $A \times_{\delta|} G/N$ with $A \times_\delta G/N$ carries $\delta|(b)$ into $\delta(b)$, and hence this in turn is equivalent to

$$\pi\,(\delta_v(a))\,\pi(j(f))\pi \times \mu\,(\Psi((1 \otimes M(k))y))$$
$$= \pi\,(\delta_v(a))\,\mu(f)\pi \times \mu\,(\Psi((1 \otimes M(k))y)).$$

Thus $\operatorname{Ind} \pi \times \mu$ preserves W if and only if $\pi(j(f)) = \mu(f)$ for $f \in C_0(G/N)$, that is, if and only if (π, μ) preserves W.

PROOF OF THEOREM 4.1. We now let

$$J_W = \bigcap \left\{ \ker \rho \times v \,\middle|\, \begin{array}{l} (\rho, v) \text{ is a covariant representation} \\ \text{of } (A, G, \delta) \text{ preserving } W \end{array} \right\}$$

and let J be the ideal in $(A \times G) \times N$ corresponding via Mansfield's bimodule X to the ideal I_W in $A \times G/N$; in other words,

$$J = \bigcap \left\{ \ker \operatorname{Ind}^E \rho : \rho \text{ is a representation of } A \times G/N \text{ with } \rho(I_W) = 0 \right\}$$
$$= \bigcap \left\{ \ker \operatorname{Ind}^E \pi \times \mu \,\middle|\, \begin{array}{l} (\pi, \mu) \text{ is a covariant representation} \\ \text{of } (A, G/N, \delta|) \text{ preserving } W \end{array} \right\}.$$

We claim that $J = J_W \times N$.

To see that $J_W \times N \subset J$, we have to show $\mathrm{Ind}^E \pi \times \mu(J_W \times N) = 0$ whenever (π, μ) preserves W. But the lemma says that then $\mathrm{Ind}^{A \times G} \pi \times \mu$ also preserves W, and hence vanishes on J_W, which in turn implies that

$$\mathrm{Ind}^E \pi \times \mu = (\mathrm{Ind}^{A \times G} \pi \times \mu) \times V$$

vanishes on $J_W \times N$. To see that $J \subset J_W \times N$, we recall that $J_W \times N$ is the kernel of the natural homomorphism

$$q : (A \times_\delta G) \times_{\hat\delta} N \to ((A \times_\delta G)/J_W) \times_{\hat\delta} N$$

[7, Proposition 12]. If we now take a faithful representation of $((A \times G)/J_W) \times N$, and compose it with q, we obtain a representation $\theta \times U$ of $(A \times G) \times N$ with $\ker \theta \times U = J_W \times N$ and $\ker \theta = J_W$. (Any faithful representation of an ordinary crossed product $B \times N$ is automatically faithful on B.) Now the imprimitivity theorem implies that $\theta \times V = \mathrm{Ind}^E \pi \times \mu$ for some covariant representation (π, μ) of $(A, G/N)$, and since $\ker \theta \supset J_W$ implies that the representation $\theta = \mathrm{Ind}^{A \times G} \pi \times \mu$ preserves W (Corollary 2.10), it follows from Lemma 4.2 that (π, μ) preserves W. Thus $J \subset \ker \theta \times U = J_W \times N$, and the claim is established.

It now follows from [23, Section 3], [7, Proposition 12] and Example 2.13 that

$$\big((A \times_\delta G) \times_{\hat\delta} N\big) / (J_W \times N) \cong ((A \times_\delta G)/J_W) \times N = (A \times_{\delta, G/N, W} G) \times_{\hat\delta} N$$

is Morita equivalent to $(A \times_{\delta|} G/N)/I_W = A \times_{G/N} G/N \cong A$.

REMARK 4.3. As we pointed out in the introduction, the case $N = G$ is not quite as strong as Katayama's duality theorem [10], which gives an isomorphism of $(A \times_\delta G) \times_{\hat\delta} G$ onto $A \otimes \mathscr{K} (L^2(G))$. While our Morita equivalence, or the stable isomorphism which can be deduced from it using [3], should be enough for most purposes, it would definitely be preferable to have a duality theorem like Katayama's. However, his theorem appears to be intrinsically spatial (see the comments in [21, 22]), and we have been unable to extend it because we lack a concrete regular representation for $A \times_{G/N} G$. We do know, from our proof of Theorem 4.1, how to construct one abstractly: given a representation π of A on $\mathscr{H}$, we also have a representation $\pi \times_{G/N} (\pi \circ j)$ of $A \times_{G/N} G/N$ on $\mathscr{H}$ (see Example 2.13), and then Lemma 4.2 says that Mansfield's induced representation $\mathrm{Ind} \pi \times (\pi \circ j)$ factors through a representation of $A \times_{G/N} G$,

which should be our analogue of the regular representation induced from π. It may not be possible to directly generalise Katayama's theorem until we have a more manageable construction of induced representations, and this would certainly be of considerable independent interest.

5. Concluding remarks

Section 5(a).   One of our main reasons for wanting to decompose crossed products by coactions was to study their K-theory. If, for example, N is abelian, we can use duality and standard facts about K-theory for crossed products by actions of N to compare $K_*(A \times_\delta G)$ and

$$(5.1)\quad K_*((A \times_\delta G) \times N) = K_* \left(((A \times G/N) \times_{G/N} G) \times N \right) \cong K_*(A \times_{\delta|} G/N).$$

Thus we would hope that, for solvable groups at least, an inductive procedure will yield information about $K_*(A \times_\delta G)$ in terms of $K_*(A)$. While we still believe this to be a potentially useful approach, we must point out that similar information can often be obtained by first using nonabelian duality, and then decomposing the dual crossed product — indeed, we have been quite surprised at just how often this works.

Consider, for example, the crossed product $A \times_\delta G$ by a coaction of a simply-connected solvable Lie group G. Such a group is an iterated semidirect product by copies of $\mathbb{R}$, and thus we can apply (5.1) repeatedly with $N = \mathbb{R}$. Connes' Thom isomorphism [5] asserts that $K_*((A \times_\delta G) \times N)$ is then isomorphic to $K_{*+1}(A \times_\delta G)$, and hence we can deduce that $K_*(A \times_\delta G) \cong K_{*-\dim G}(A)$. Alternatively, we can first apply Katayama's duality theorem [10]

$$(A \times_\delta G) \times_{\hat\delta} G \cong A \otimes \mathcal{K}(L^2(G)),$$

then decompose the dual crossed product into ones by actions of $\mathbb{R}$, and use Connes' theorem repeatedly to obtain

$$K_*(A) \cong K_*(A \otimes \mathcal{K}) \cong K_*((A \times_\delta G) \times_{\hat\delta} G) \cong \cdots \cong K_{*-\dim G}(A \times_\delta G).$$

(The last two steps are just Connes' proof of [5, Corollary 7].) Since K-theory is periodic of order 2, the conclusion is the same.

On the face of it, the first argument is more general, since it does not depend on the splitting of the successive group extensions (that they do split for simply-connected solvable Lie groups is a well-known theorem of Iwasawa [9, 3.6]).

However, even if the decomposition of $(A \times_\delta G) \times_{\hat\delta} G$ required twisted crossed products, K-theory could not tell the difference: every twisted crossed product $A \times_{\alpha,u} G$ is stably isomorphic to an ordinary one of the form $(A \otimes \mathcal{K}) \times_\beta G$ [17, Corollary 3.7], and hence has the same K-theory. The potential advantage of our new decomposition is that it allows us to exploit the existence of nice subgroups, whereas the dual approach works when there are well-behaved quotients (see Example 5.3 below).

For example, consider the case where G has a normal subgroup isomorphic to $\mathbb{T}$. For any action β of $\mathbb{T}$ on B, there is a six-term exact sequence relating $K_*(B \times \mathbb{T})$ and $K_*(B)$, dual to the usual Pimsner-Voiculescu sequence [2, Section 10.6]. From this and the isomorphism (5.1), we obtain:

PROPOSITION 5.1. *Let δ be a coaction of a locally compact group G on a C^*-algebra A, and suppose G has a normal subgroup N isomorphic to $\mathbb{T}$. Then there is an exact sequence*

$$
\begin{array}{ccccc}
K_0(A \times_{\delta|} G/N) & \longrightarrow & K_0(A \times_{\delta|} G/N) & \longrightarrow & K_0(A \times_\delta G) \\
\uparrow & & & & \downarrow \\
K_1(A \times_\delta G) & \longleftarrow & K_1(A \times_{\delta|} G/N) & \longleftarrow & K_1(A \times_{\delta|} G/N).
\end{array}
$$

Ordinarily, the exact sequence for crossed products by $\mathbb{T}$ is less useful than the Pimsner-Voiculescu sequence, because it involves $K_*(A \times \mathbb{T})$ twice. For us, this problem arises when we apply (5.1) and the Pimsner-Voiculescu sequence to a subgroup $N \cong \mathbb{Z}$.

Here are some examples illustrating these points.

EXAMPLE 5.2. Let G be the quotient of the real Heisenberg group by the central copy of $\mathbb{Z}$: that is, $G = \mathbb{R} \times \mathbb{R} \times \mathbb{T}$ with product

$$(x_1, y_1, z_1) \cdot (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 z_2 \exp(2\pi i x_1 y_2)).$$

We take $N = Z(G) \cong \mathbb{T}$, so that $G/N = \mathbb{R}^2$, and, for any coaction δ of G, $K_*(A \times_{\delta|} G/N) \cong K_*(A \times_{\delta|} \mathbb{R}^2) \cong K_*(A)$. Thus Proposition 5.1 yields a six-term exact sequence relating $K_*(A \times_\delta G)$ to $K_*(A)$. Here similar information could be obtained by decomposing the dual crossed product, as discussed above.

EXAMPLE 5.3. Let G be the integer Heisenberg group: that is, $G = \mathbb{Z}^3$ with product

$$(m_1, n_1, p_1)(m_2, n_2, p_2) = (m_1 + m_2, n_1 + n_2, p_1 + p_2 + m_1 n_2).$$

Again, we take $N = Z(G) \cong \mathbb{Z}$. Now $G/Z \cong \mathbb{Z}^2$, so $A \times_{\delta|} G/Z$ is isomorphic to a crossed product by an action α of $\mathbb{T}^2$, and the Pimsner-Voiculescu sequence for the dual action of $\mathbb{Z}$ yields

$$
\begin{array}{ccccc}
K_0(A \times_\delta G) & \longrightarrow & K_0(A \times_\delta G) & \longrightarrow & K_0(A \times_\alpha \mathbb{T}^2) \\
\uparrow & & & & \downarrow \\
K_1(A \times_\alpha \mathbb{T}^2) & \longleftarrow & K_1(A \times_\delta G) & \longleftarrow & K_1(A \times_\delta G).
\end{array}
$$

(For this sequence to be useful, we should probably view it as giving information about $K_*(A \times_\alpha \mathbb{T}^2)$ when α is an action which extends naturally to a coaction of G.) We note that in this case dualising first gives

$$
A \otimes \mathcal{K} \cong (A \times_\delta G) \times_{\hat{\delta}} G \cong ((A \times_\delta G) \times \mathbb{Z}) \times \mathbb{Z}^2,
$$

so the central copy of $\mathbb{Z}$ is buried, and the Pimsner-Voiculescu sequence is not directly applicable.

Section 5(b).   We want to think of a twisted crossed product $A \times_{G/N} G$ as being like a crossed product of A by a coaction of N, and hence, if N is abelian, like an ordinary crossed product of A by an action of $\hat{N}$. The duality theorem helps justify this: $A \times_{G/N} G$ carries a dual action of N from which we can recover A, just as a crossed product $A \times_\beta \hat{N}$ does. Indeed, at least for separable algebras, we can deduce from the Takai duality isomorphism

$$
(A \times_{\delta,G/N} G) \otimes \mathcal{K} (L^2(N)) \cong ((A \times_{\delta,G/N} G) \times_{\hat{\delta}} N) \times_{\hat{\hat{\delta}}} \hat{N},
$$

our Theorem 4.1, and [3], that $A \times_{G/N} G$ is stably isomorphic to a crossed product of $A \otimes \mathcal{K}$ by an action of $\hat{N}$. Now every twisted crossed product $A \times_{\alpha,u} \hat{N}$ in the sense of [17] has the same property [17, 3.7], and hence it is natural to wonder whether every $A \times_{G/N} G$ has the form $A \times_{\alpha,u} \hat{N}$ for some twisted action (α, u). In fact, this is not the case, so that even for abelian N we are dealing with a genuinely new construction.

To see this, consider the decomposition of the trivial crossed product $C_0(G) = \mathbb{C} \times G$ as $C_0(G/N) \times_{\gamma,w} G$. We claim that $C_0(G)$ cannot always be a twisted crossed product $C_0(G/N) \times_{\alpha,u} \hat{N}$. First of all, since $C_0(G)$ is commutative, $\alpha = \mathrm{Ad}\, i_{\hat{N}}$ would have to be trivial. The unitary group $UM(C_0(G/N))$ is the Polish group $C(G/N, \mathbb{T})$, and the isomorphism class of $C_0(G/N) \times_{i,u} \hat{N}$ depends only on the cohomology class of the cocycle $u \in Z^2(\hat{N}, C(G/N, \mathbb{T}))$. If $\hat{N} = \mathbb{Z}$, for example, the group $H^2(\mathbb{Z}, C(G/N, \mathbb{T}))$ is trivial (the corresponding Polish extension of $\mathbb{Z}$ by $C(G/N, \mathbb{T})$ [14] must split), and hence every twisted crossed

product $C_0(G/N) \times_{i,u} \hat{N}$ is isomorphic to $C_0(G/N) \times_i \hat{N} \cong C_0(G/N \times \mathbb{T})$. However, $G = U(2)$ has a normal subgroup $N = Z(G) = \mathbb{T}1$ isomorphic to $\mathbb{T}$, and G is not homeomorphic to $(U(2)/N) \times \mathbb{T} = PU(2) \times \mathbb{T}$, so $C_0(G)$ cannot be isomorphic to any twisted crossed product of the form $C_0(PU(2)) \times_{\alpha,u} \mathbb{Z}$.

It is also natural to ask when one of our twisted crossed products $A \times_{G/N} G$ is isomorphic to an ordinary crossed product of A by a coaction of N. In the case of actions, if G is a semidirect product $N \rtimes H$, any twisted crossed product $A \times_{\alpha,N,\tau} G$ is isomorphic to an ordinary crossed product $A \times_\beta H$. (This follows, for example, from [17, 5.1], since we can choose a section $c : H \to G$ which is a homomorphism.) We do not know whether there is an analogous result for our twisted crossed products by coactions of semidirect products. Indeed, we had to work to verify that, if G is a direct product $N \times H$, then every $A \times_H G$ is isomorphic to some $A \times_\gamma N$. (In this case, the map $(n, h) \to n$ extends to a nondegenerate homomorphism $\theta : C_r^*(G) \to C_r^*(N)$, and we can take $\gamma = (i \otimes \theta) \circ \delta$.)

Section 5(c).   To finish, we point out that we have considered an analogue of only one of the competing theories of twisted crossed products for actions, namely that of Green [7]. Technically, Green's has the advantage that the twisted crossed product is a quotient of the usual one, allowing an intrinsically C^*-algebraic theory. The cocycle-based versions of [4, 17] inevitably involve Borel functions with values in multiplier algebras, which can be a rather clumsy mixed-mode, but they do have the advantage of extra generality (see [17, Section 5]), and appear to be the more natural setting for some purposes (for example, [17, Section 3]). The (scalar-valued) "dual cocycles" used to classify ergodic actions in [11, 25] are unitary elements of the von Neumann algebra $L(G) \bar{\otimes} L(G)$, just as Borel cocycles in $Z^2(\hat{G}, T)$ determine unitary elements of $L^\infty(\hat{G}) \bar{\otimes} L^\infty(\hat{G}) \cong L(G) \bar{\otimes} L(G)$ in the abelian case. But whereas one can easily make sense of C^*-algebra-valued Borel functions, it is not immediately clear how to characterise a tensor product $M(A) \otimes L^\infty(G) \bar{\otimes} L^\infty(G)$ these live in, and which could then be transported to the dual setting. It would be very interesting to develop such a cocycle-based theory, at least far enough to get a stabilisation theorem like that of [17, Section 3], but this does promise to be a major undertaking. Indeed, since the scalar-valued cocycle theory has only been worked out for compact groups [11, 25], even carrying out the analogous program for coactions of locally compact groups on von Neumann algebras will likely involve substantial technical difficulties.

Acknowledgements

References

[1]   E. Bédos, 'Discrete groups and simple C^*-algebras', *Math. Proc. Cambridge Phil. Soc.* **109** (1991), 521–537.
[2]   B. Blackadar, *K-theory for operator algebras*, MSRI Publications **5** (Springer, New York, 1986).
[3]   L. G. Brown, P. Green and M. A. Rieffel, 'Stable isomorphism and strong Morita equivalence of C^*-algebras', *Pacific J. Math.* **71** (1977), 349–363.
[4]   R. C. Busby and H. A. Smith, 'Representations of twisted group algebras', *Trans. Amer. Math. Soc.* **149** (1970), 503–537.
[5]   A. Connes, 'An analogue of the Thom isomorphism for crossed products of C^*-algebras', *Adv. in Math.* **39** (1981), 31–55.
[6]   M. de Brabanter, 'Decomposition theorems for certain C^*-crossed products', *Math. Proc. Cambridge Phil. Soc.* **94** (1983), 265–275.
[7]   P. Green, 'The local structure of twisted covariance algebras', *Acta Math.* **140** (1978), 191–250.
[8]   C. Herz, 'Le rapport entre l'algèbre A_p d'un groupe et d'un sous-groupe', *C. R. Acad. Sci. Paris Sér.1 Math.* **A271** (1970), 244–246.
[9]   K. Iwasawa, 'On some types of topological groups', *Ann. of Math.* **50** (1949), 507–558.
[10] Y. Katayama, 'Takesaki's duality for a nondegenerate coaction', *Math. Scand.* **55** (1985), 141–151.
[11] M. B. Landstad, 'Ergodic actions of nonabelian compact groups', in: *Ideas and methods in mathematical analysis, stochastics and applications* (ed. S. Albeverio et. al.) (Cambridge University Press, London, 1992) pp. 365–388.
[12] M. B. Landstad, J. Phillips, I. Raeburn and C. E. Sutherland, 'Representations of crossed products by coactions and principal bundles', *Trans. Amer. Math. Soc.* **299** (1987), 747–784.
[13] K. Mansfield, 'Induced representations of crossed products by coactions', *J. Funct. Anal.* **97** (1991), 112–161.
[14] C. C. Moore, 'Group extensions and cohomology for locally compact groups, III', *Trans. Amer. Math. Soc.* **221** (1976), 1–33.
[15] Y. Nakagami and M. Takesaki, *Duality for crossed products of von Neumann algebras*, Lecture Notes in Math. **731** (Springer, Berlin, 1979).
[16] D. Olesen and I. Raeburn, 'Pointwise unitary automorphism groups', *J. Funct. Anal.* **93** (1990), 278–309.
[17] J. A. Packer and I. Raeburn, 'Twisted crossed products of C^*-algebras', *Math. Proc. Cambridge Phil. Soc.* **106** (1989), 293–311.
[18] ———, 'Twisted crossed products of C^*-algebras, II', *Math. Ann.* **287** (1990), 595–612.

[19] G. K. Pedersen, *C*-algebras and their automorphism groups* (Academic Press, London, 1979).

[20] J. C. Quigg, 'Duality for reduced twisted crossed products of *C**-algebras', *Indiana Univ. Math. J.* **35** (1986), 549–571.

[21] ———, 'Full *C**-crossed product duality', *J. Austral. Math. Soc. (Series A)* **50** (1991), 34–52.

[22] I. Raeburn, 'On crossed products by coactions and their representation theory', *Proc. London Math. Soc.* **64** (1992), 625–652.

[23] M. A. Rieffel, *Unitary representations of group extensions: an algebraic approach to the theory of Mackey and Blattner*, Adv. in Math. Suppl. Studies **4** (Academic Press, New York, 1979) pp. 43–82.

[24] M. Takesaki, 'Covariant representations of *C**-algebras and their locally compact automorphism groups', *Acta Math.* **119** (1967), 273–303.

[25] A. Wassermann, 'Ergodic actions of compact groups on operator algebras, II: classification of full multiplicity ergodic actions', *Canad. J. Math.* **40** (1988), 1482–1527.

Department of Mathematics and Statistics   Department of Mathematics
University of Victoria   University of Newcastle
P.O. Box 3045   Newcastle, New South Wales, 2308
Victoria, B.C. V8W 3P4   Australia
Canada