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REDUCING ENERGY DURING SPECIFIED TIME INTERVALS FOR MULTIPLE TRAINS USING SIMPLE TRAIN MODELS

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Abstract

Electricity supply operators offer financial incentives to encourage large energy users to reduce their power demand during declared periods of increased demand from energy users such as residential homes. This demand flexibility enables electricity system operators to ensure adequate power supply and avoid the construction of peaking power plants.

Railway operators can sometimes reduce their power demand during specified peak demand periods without disrupting the train schedules. For trains with infrequent stops, such as intercity trains, it is possible to speed up trains prior to the peak demand period, slow down during the peak demand period, then speed up again after the peak demand period. We use simple train models to develop an optimal strategy that minimizes energy use for a fleet of trains subject to energy-use constraints during specified peak demand intervals. The strategy uses two sets of interacting parameters to find an optimal solution—a Lagrange multiplier for each energy-constrained time interval to control the speed of trains during each interval, and a Lagrange multiplier for each train to control the relative train speeds and ensure each train completes its journey on time.

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1. Introduction

Electric trains are an efficient means of transport, but can be large energy users that impose significant power demands on national or regional electricity supply systems.

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Electricity supply systems often experience regular peaks in power demand during morning and evening periods, when public transport use and household energy use are both high. Extreme weather events can also cause peaks in power demand. During periods of high power demand, the price of electricity increases dramatically.

The cost of electricity for large energy users, such as electric train operators, often depends, at least in part, on their peak power demand during each month, so large energy users have a financial incentive to manage their peak power demand. Furthermore, electricity suppliers often seek to cooperate with large energy users to better manage electricity supply during unusual periods of high power demand. The electricity suppliers can give notification up to 24 hours in advance of periods where overall power demand on the electricity network is predicted to be unusually high. Financial incentives are offered to large energy users provided they can successfully reduce their energy use. For example, the French transmission system operator, *Réseau de Transport d'Électricité* (RTE), offers financial incentives for companies to provide demand response capability, which they are obliged to meet at declared peak demand intervals.

The French national state-owned railway company, *Société Nationale des Chemins de fer Français* (SNCF), is the largest electricity user in France, and has been investigating how they can manage the total peak power demand of their trains on the French electricity grid [8]. They are particularly interested in reducing energy use during peak demand periods that are announced by the electricity system operator 24 hours in advance; these peak demand periods typically last for an hour, with agreed targets for energy reduction across the electricity network. Energy use is measured in 10-minute intervals.

To achieve reductions in energy use during a peak demand period, the trains collectively must slow down. Trains with frequent stops during a peak demand period will have limited capacity to slow down without disrupting arrival times at key locations, such as stations and junctions. However, adjusting train speed profiles without disrupting arrival times can be achieved on long intercity trips such as SNCF's premier high-speed *Train à Grande Vitesse* (TGV) service, where trains are already driven efficiently using driving advice from the *Energymiser* system [1, 2]. If trains speed up before the peak demand period, slow down during the peak demand period and then speed up again after the peak demand period, as suggested by Pam et al. [8], then this strategy will increase energy use before and after the peak demand period but reduce energy use during the peak demand period.

The problem is to determine how much each train should slow down for the fleet of trains to collectively achieve the required energy reduction during each 10-minute interval of a peak demand period. Ideally, this should be done in a way that minimizes the total energy use for the fleet of trains.

A feasibility study by Pam et al. [8] shows that it is possible to reduce energy use during a 1-hour peak demand period for a fleet of 105 TGV trains. They conducted experiments on SNCF high-speed lines to demonstrate that significant energy reductions can be made using their algorithm. Their heuristic method finds

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good, but not necessarily optimal, solutions to the problem. This paper starts to develop a theoretical basis for an optimal solution. Although the simple train model we use is unrealistic, the method can be applied to more realistic models and the results obtained with the simple model indicate that it should be possible to find an optimal solution to a more realistic version of the problem.

2. Relevant literature

There is very little literature directly related to the problem of controlling a fleet of trains to manage peak power demand, partly because this type of control has only recently become possible with the advent of practical driving advice systems with real-time data communication between control systems and trains. Section 2.1 reviews work directly related to the problem of managing energy use during a peak demand period. Additional literature related to power demand of trains, but not directly related to our problem, is discussed in Section 2.2.

2.1. Reducing energy during specified time intervals This section reviews work directly related to the problem of managing energy use for a fleet of trains during specified time intervals.

A feasibility study by Pam et al. [8] simulated a fleet of 105 TGV trains to show that it is possible to reduce power demand that initially varies between 220 MW and 290 MW during a 1-hour peak demand period to a constant average power of 200 MW for each 10-minute interval. Experiments conducted on individual trains running on the SNCF South Europe Atlantic high-speed line confirmed that significant power reductions could be achieved in practice. Their algorithm for adjusting train schedules to reduce energy use in 10-minute intervals worked by adjusting the locations of trains at the beginning and end of each interval during a 1-hour peak demand period, but did not attempt to find an optimal solution to the problem.

Scheepmaker et al. [12] gave a comprehensive review of the background literature on optimal train control. Our previous work using Pontryagin's principle is summarized by Pudney [9], with details given in a pair of papers by Albrecht et al. [1, 2]. Pontryagin's principle states necessary conditions for an optimal control u^* that can be found by maximizing the control Hamiltonian $\mathcal{H}$ [9]. There are five possible optimal control modes for a train equipped with continuous control travelling along a track with known gradient and fixed journey time: *maximum acceleration*, *speedhold* using partial acceleration, *coast*, *speedhold* using partial regenerative braking (if available) and *maximum brake* [2]. Rao et al. [10, 11] extend this analysis for scenarios where regeneration energy can be shared between trains but not exported back to the grid, and identify two additional control modes: driving to absorb power generated by other trains and braking to supply power to other trains.

Howlett et al. [6] proposed a driving strategy for a single train on a level track to minimize energy use when there are energy constraints during specified time intervals. They used classical methods of constrained optimization to develop an optimal strategy

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that uses a period of *coasting* followed by *speedholding* at an optimal driving speed during intervals where it is necessary to decrease speed, and a period of *maximum acceleration* followed by *speedholding* at an optimal driving speed during intervals where it is necessary to increase speed.

In this paper, we extend the work of Howlett et al. [6] by considering multiple trains. Our aim is to find general characteristics for an optimal solution, so we start with simple train models with piecewise constant speed profiles (Section 3). This is equivalent to assuming that the energy use for a time interval depends on the average speed of the train during that interval. Section 4 formulates the problem of minimising total energy use of a fleet of trains with one peak demand interval. Section 5 extends the problem with multiple consecutive peak demand intervals. In Section 6, we discuss the implications for more realistic train models.

2.2. Methods for reducing peak power demand for train operations There are several papers that consider reducing peak power demand for train operations, but not the specific problem of managing power demand during specified time intervals.

Su et al. [14] evaluated general strategies for reducing energy use on railways. They considered timetable optimization, reducing train mass, improving the design of gradients along the track, increasing traction and braking forces, using regenerative braking, and improving running resistance. Although these strategies can provide energy savings, some require drastic system changes such as changing the train schedules, operating trains made of lighter materials, and with larger traction and braking forces, adjusting the slope of the track, installing regenerative braking systems, reducing aerodynamic drag, and reducing the rolling resistance of the train. Even if these strategies are implemented and trains are operating as efficiently as possible, a method for managing energy use for a fleet of trains during specified time intervals is still required.

Other literature has focused on energy-efficient scheduling to prevent high peaks in power demand. Albrecht [3] examined the possibility of using train running time control for synchronizing acceleration and braking phases to reduce power peaks and energy use, and showed that optimized timetables for multiple trains gave smaller power peaks and less energy use. Albrecht [3] also looked at the effects of stochastic station dwell times, which made it harder for the trains to run on time and preserve running time reserve, and still found that the modification of train running times can contribute to significantly reducing power peaks and energy use, and thereby reducing energy cost.

Li and Lo [7] used a genetic algorithm to jointly optimize the scheduling and speed control of metro rail systems such that the net energy use is minimized by coordinating arrival and departure times so that regenerative energy from braking trains is used by nearby accelerating trains. Their results showed that a larger headway leads to a smaller energy saving rate, and the maximum energy saving of roughly 25% is achieved when the minimum allowable headway is used.

There is much recent literature related to energy efficient train operations, but none address the problem of managing a fleet of trains during specified time intervals to reduce peak power demand.

3. Simple train model

We wish to calculate how much each train in a fleet should slow down during a peak demand interval $t \in [p_0, p_1]$ so that the total energy use during the peak demand interval does not exceed a given limit. We start with a simplified model and assume:

- trains travel on a level track with no speed limits;
- trains do not stop during their journeys;
- trains follow piecewise constant speed profiles and can change speed instantly. This is equivalent to calculating the average speed for each section of the journey;
- trains have sufficient power to travel at the required average speeds;
- the power reduction that can be achieved by slowing a train is independent of on which part of the railway electrical network the train is.

The focus of this study is to determine the optimal average speed required for each section of each journey, and how much each train will contribute to demand reduction during each peak demand interval. Further work will be required to determine optimal strategies that incorporate real train dynamics.

Figure 1 shows four trains with constant speed profiles. The peak demand interval (grey shaded region) starts at time $p_0 = 1800$ seconds and finishes at time $p_1 = 5400$ seconds. The first train (green) starts its journey before the peak demand interval and finishes after the peak demand interval. The second train (orange) starts its journey before the peak demand interval and finishes within the peak demand interval. The third train (blue) starts and finishes its journey within the peak demand interval, and the fourth train (pink) starts its journey within the peak demand interval and finishes after the peak demand interval.

The force required to drive a train at constant speed v on a level track is the force required to overcome aerodynamic and rolling resistance forces, which are commonly modelled using the Davis formula [5]:

$$R(v) = r_0 + r_1 v + r_2 v^2,$$

where r_0, r_1 and r_2 are known constants. The power required to drive at constant speed v is therefore

$$\varphi(v) = vR(v) = r_0 v + r_1 v^2 + r_2 v^3.$$

In general, φ will be a monotonic increasing convex function of speed with monotonic increasing derivative

$$\begin{aligned}\varphi'(v) &= R(v) + vR'(v)\\ &= r_0 + 2r_1 v + 3r_2 v^2.\end{aligned}$$

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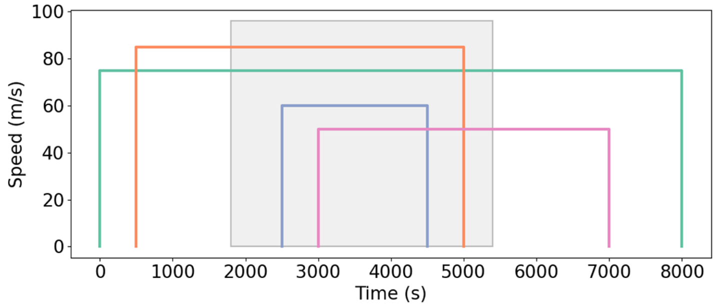

FIGURE 1. Initial constant piecewise speed profiles for four trains. The shaded region is the peak demand period.

4. Problem formulation with a single peak demand interval

In this section, we consider the case where the peak demand period comprises a single interval with a target energy use. In Section 5, we consider the more general case where the peak demand period comprises multiple consecutive intervals, each with a target energy use.

Consider a fleet of n trains. Each train i is given a start time s_i and a finish time f_i, with $s_i < f_i$. The length of journey i is X_i. The most energy-efficient driving strategy for each train i, on a level track with no speed limits and no stops, is to drive at a constant speed

$$v_i = \frac{X_i}{d_i},$$

where $d_i = f_i - s_i$ is the journey duration. Suppose every train has a part of its journey during the peak demand interval $[p_0, p_1]$, so that a journey is starting before the end of the peak period $s_i < p_1$ and finishing after the start of the peak period $f_i > p_0$ for all trains $i \in \{1, \ldots, n\}$. Assume that each train i will travel at a constant optimal speed v_i throughout its journey then, during the peak demand interval $[p_0, p_1]$, the trains will have a combined total energy use of

$$E_0 = \sum_{i=1}^{n} h_i \varphi_i(v_i),$$

where $h_i = \min(f_i, p_1) - \max(s_i, p_0)$ is the duration of the overlap between intervals $[s_i, f_i]$ and $[p_0, p_1]$. Suppose we wish to reduce this to $E_1 = (1 - \alpha)E_0$, where $0 < \alpha < 1$ and 100α is the percentage energy reduction required during the peak demand interval. To achieve the required energy reduction, we must find a new speed w_i for each train i during the peak demand interval so that

$$\sum_{i=1}^{n} h_i \varphi_i(w_i) = E_1 = (1 - \alpha)E_0. \tag{4.1}$$

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To compensate for a changed speed in the peak demand interval $[p_0, p_1]$, each train that starts before the peak demand period ($s_i < p_0$) or finishes after the peak demand period ($f_i > p_1$) must change speed in the intervals $[s_i, p_0]$ and $[p_1, f_i]$ so that the average journey speed and hence journey distance remain unchanged. We know from Howlett et al. [6] that each train i should have the same speed z_i before and after the peak demand period. To meet the distance constraint for each train, we must have

$$X_i = k_i z_i + h_i w_i \quad \text{for all } i \in \{1, \ldots, n\}, \tag{4.2}$$

where $k_i = \max(p_0 - s_i, 0) + \max(f_i - p_1, 0)$ is the journey duration before and after the peak demand interval for train i. So when $k_i > 0$,

$$z_i = \frac{X_i - h_i w_i}{k_i}. \tag{4.3}$$

When train i has its journey entirely within the peak demand period, then it is not possible for the train to both reduce energy use and meet the journey time constraints. In this case, the total duration before and after the peak demand period is $k_i = 0$, and so the speed during the peak demand interval remains unchanged with $w_i = v_i$. The new total energy for all train journeys will be

$$E = \sum_{i=1}^{n} k_i \varphi_i(z_i) + E_1. \tag{4.4}$$

4.1. Minimizing total energy We can now define an optimization problem. We wish to find constant speeds w_i and z_i for each train $i \in \{1, \ldots, n\}$ to minimize the total energy use E defined by (4.4), such that the energy use E_1 during the peak demand period, defined by the summation in (4.1), is constrained to the pre-defined level $(1 - \alpha)E_0$. Speed w_i is the optimal speed of train i during the peak demand interval, and speed z_i is the optimal speed of train i before and after the peak demand period. Equation (4.2) must be satisfied for each train i to ensure that the journey covers the correct distance X_i. To solve this problem, we form a Lagrangian function $\mathcal{L} = \mathcal{L}(z_i, w_i)$ given by

$$\begin{aligned}\mathcal{L} = &\sum_{i=1}^{n} k_i \varphi_i(z_i) + E_1 \\ &+ \lambda\Big(\sum_{i=1}^{n} h_i \varphi_i(w_i) - E_1\Big) \\ &+ \sum_{i=1}^{n} \delta_i (X_i - k_i z_i - h_i w_i),\end{aligned}$$

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where λ and δ_i are Lagrange multipliers. The optimal solution must satisfy the Karush–Kuhn–Tucker conditions [4]:

$$\frac{\partial \mathcal{L}}{\partial w_i} = \frac{\partial \mathcal{L}}{\partial z_i} = 0 \quad \text{for all } i \in \{1, \dots, n\},$$

and the complementary slackness conditions

$$\lambda\Big(\sum_{i=1}^{n} h_i \varphi_i(w_i) - E_1\Big) = 0$$

and

$$\delta_i(X_i - k_i z_i - h_i w_i) = 0 \quad \text{for all } i = \{1, \dots, n\}.$$

The partial derivatives with respect to the new speeds w_i and z_i are

$$\frac{\partial \mathcal{L}}{\partial w_i} = h_i(\lambda \varphi_i'(w_i) - \delta_i), \tag{4.5}$$

$$\frac{\partial \mathcal{L}}{\partial z_i} = k_i(\varphi_i'(z_i) - \delta_i). \tag{4.6}$$

Equation (4.6) can equal zero when either $k_i = 0$ or when $\delta_i = \varphi_i'(z_i)$. If a train journey starts and finishes within the peak demand period, then $k_i = 0$, and then (4.2) gives $w_i = v_i$. In other words, journeys entirely within the peak demand interval remain unchanged. Otherwise, if journey i starts before or finishes after the peak demand period where $k_i > 0$, then setting the partial derivatives from (4.5) and (4.6) to zero and rearranging gives

$$\delta_i = \lambda \varphi_i'(w_i) = \varphi_i'(z_i). \tag{4.7}$$

Substituting z_i from (4.3) gives

$$\lambda \varphi_i'(w_i) = \varphi_i'\Big(\frac{X_i - h_i w_i}{k_i}\Big). \tag{4.8}$$

Equation (4.8) defines the relationship between the Lagrange multiplier λ and the speed w_i for each train i during the peak demand interval. Notice that the optimal speed w_i for each train depends only on λ and not on the speeds of the other trains. Furthermore, we have the following.

- If $\lambda = 1$, then from (4.7), we have $\varphi_i'(w_i) = \varphi_i'(z_i) \implies w_i = z_i$, which implies that the speed during the peak demand interval w_i is the same as the speed before and after the peak demand interval z_i for train i. That is, if $\lambda = 1$, then the optimal journey for each train is unchanged.
- Now, suppose $\lambda > 1$. For trains that have some part of their journey before or after the peak demand period (that is, $k_i > 0$), (4.7) implies $w_i < z_i$, since φ_i' is an increasing function (Figure 2). That is, each train that has some part of its

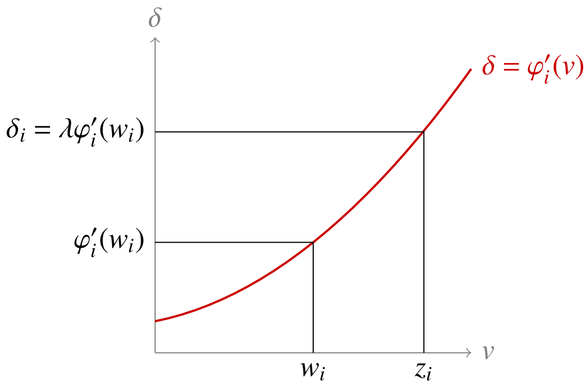

FIGURE 2. Relationship between w_i and z_i for a given $\lambda > 1$.

journey before or after the peak demand period speeds up before the peak demand period, slows down during the peak demand period and speeds up after the peak demand period. Increasing λ will decrease the speed of each train during the peak demand period and, hence, decrease the energy use during the peak demand period.

- If $\lambda < 1$, then trains will speed up during the peak demand period.

If we pick a value for λ, then we can calculate w_i by numerically solving (4.8) for each train $i \in \{1, \ldots, n\}$. If (4.1) is not satisfied, then we simply adjust λ and try again; if the energy use during the peak demand interval is too high, then λ must increase, otherwise λ must decrease.

4.2. Single peak demand interval example Consider four trains, with:

- journey distances $\mathbf{X} = [600\,000, 382\,500, 114\,000, 200\,000]$;
- start times $\mathbf{s} = [0, 500, 2500, 3000]$;
- finish times $\mathbf{f} = [8000, 5000, 4400, 7000]$.

The peak demand period starts at time $p = 1800$ and finishes at time $q = 5400$. All units are SI. The initial train speeds are $\mathbf{v} = [75, 85, 60, 50]$. If $\varphi_i(v) = v^3$, then the energy use during the peak demand interval is $E_0 \approx 4.19 \times 10^9$. If we want to reduce this by 10%, then $\alpha = 0.1$ and the target energy use during the peak demand interval is $E_1 \approx 3.77 \times 10^9$. Numerical solution of (4.8) gives the Lagrange multipliers $\lambda \approx 1.21$ and $\delta \approx [0, 0, 13\,070, 0]$. We note that $\delta_i > 0$ only when a train journey starts and finishes within the peak demand period ($k_i = 0$). The new speeds within the peak demand period are $\mathbf{w} \approx [71.09, 82.61, 60.00, 48.08]$, and the new speeds before and after the peak demand period are $\mathbf{z} \approx [78.20, 90.87, 0, 52.88]$. This solution is shown in Figure 3 and was calculated in approximately 0.005 seconds. Total energy increases from 7.049×10^9 to 7.091×10^9. This is a 0.60% increase.

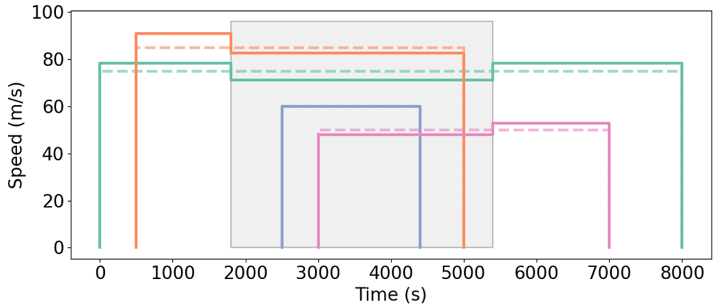

FIGURE 3. Initial (pale, dashed) and adjusted (bold) speed profiles for four trains. Shaded region is the peak demand period.

5. Problem formulation with multiple peak demand intervals

Constraining energy use during the peak demand period will manage the average power demand during the period, but not instantaneous power. In practice, energy use is usually measured at intervals of 5–15 minutes, so railways will want to manage average power in these shorter intervals.

Suppose the peak demand period $[p_0, \ldots, p_q]$ is made up of q consecutive peak demand intervals. During each peak demand interval j, the trains have combined energy use

$$E_{0j} = \sum_{i=1}^{n} h_{ij}\varphi_i(v_i) \quad \text{for all } j \in \{1, \ldots, q\},$$

where $h_{ij} = \max(0, \min(f_i, p_j) - \max(s_i, p_{j-1}))$ is the duration of the overlap between intervals $[s_i, f_i]$ and $[p_{j-1}, p_j]$ for train $i \in \{1, \ldots, n\}$. Suppose we wish to reduce the energy use during each peak demand interval to $E_{1j} = (1 - \alpha_j)E_{0j}$, where $100\alpha_j$ is the target percentage energy reduction during the peak demand interval j. To achieve the required energy reduction, we must find new speed w_{ij} for each train i during each peak demand interval j so that

$$E_{1j} = \sum_{i=1}^{n} h_{ij}\varphi_i(w_{ij}) = (1 - \alpha_j)E_{0j} \quad \text{for all } j \in \{1, \ldots, q\}. \tag{5.1}$$

To compensate for speed change within a peak demand interval $[p_{j-1}, p_j]$, each train's speed must change in another interval so that the average journey speed and hence journey distance remain unchanged. Once again, we know from Howlett et al. [6] that each train i should have the same speed z_i before and after the peak demand period $[p_0, \ldots, p_q]$. To meet the journey distance constraint for each train i, we must have

$$X_i = k_i z_i + \sum_{j=1}^{q} h_{ij} w_{ij} \quad \text{for all } i \in \{1, \ldots, n\}, \tag{5.2}$$

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where $k_i = \max(p_0 - s_i, 0) + \max(f_i - p_q, 0)$ is the journey duration before and after the peak demand period $[p_0, \ldots, p_q]$. When $k_i > 0$,

$$z_i = \frac{X_i - \sum_{j=1}^{q} h_{ij} w_{ij}}{k_i} \quad \text{for all } i \in \{1, \ldots, n\}. \tag{5.3}$$

When the journey duration is entirely within the peak demand period $[p_0, \ldots, p_q]$, then $k_i = 0$ and so

$$X_i = \sum_{j=1}^{q} h_{ij} w_{ij} \quad \text{for all } i \in \{1, \ldots, n\}. \tag{5.4}$$

The new total energy for all train journeys will be

$$E = \sum_{i=1}^{n} k_i \varphi_i(z_i) + \sum_{j=1}^{q} E_{1j}. \tag{5.5}$$

5.1. Minimizing total energy For the problem with multiple peak demand intervals, the optimization problem is to find constant speeds w_{ij} and z_i for each peak demand interval $j \in \{1, \ldots, q\}$ and for each train $i \in \{1, \ldots, n\}$ to minimize total energy use E defined by (5.5) such that the energy use E_{1j} during peak demand interval j, defined by the summation in (5.1), is constrained to the pre-defined level $(1 - \alpha_j)E_{0j}$ for each peak demand interval $j \in \{1, \ldots, q\}$. Speed w_{ij} is the optimal speed of train i during peak demand interval j, and speed z_i is the optimal speed of train i before and after the peak demand period. Equation (5.2) must be satisfied for each train i to ensure that the journey covers the correct distance X_i. To solve this problem, we once again form a Lagrangian function $\mathcal{L} = \mathcal{L}(z_i, w_{ij})$, this time given by

$$\begin{aligned}\mathcal{L} = &\sum_{i=1}^{n} k_i \varphi_i(z_i) + \sum_{j=1}^{q} E_{1j} \\ &+ \sum_{j=1}^{q} \lambda_j \Big(\sum_{i=1}^{n} h_{ij} \varphi_i(w_{ij}) - E_{1j} \Big) \\ &+ \sum_{i=1}^{n} \delta_i \Big(X_i - k_i z_i - \sum_{j=1}^{q} h_{ij} w_{ij} \Big),\end{aligned}$$

where λ_j and δ_i are Lagrange multipliers. The optimal solution must satisfy the Karush–Kuhn–Tucker conditions

$$\frac{\partial \mathcal{L}}{\partial w_{ij}} = \frac{\partial \mathcal{L}}{\partial z_i} = 0 \quad \text{for all } i \in \{1, \ldots, n\}, j \in \{1, \ldots, q\},$$

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and the complementary slackness conditions

$$\lambda_j\Big(\sum_{i=1}^{n} h_{ij}\varphi_i(w_{ij}) - E_{1j}\Big) = 0 \quad \text{for all } j \in \{1, \dots, q\}$$

and

$$\delta_i\Big(X_i - k_i z_i - \sum_{j=1}^{q} h_{ij} w_{ij}\Big) = 0 \quad \text{for all } i \in \{1, \dots, n\}.$$

The partial derivatives with respect to the new speeds w_{ij} and z_i are

$$\frac{\partial \mathcal{L}}{\partial w_{ij}} = h_{ij}(\lambda_j \varphi_i'(w_{ij}) - \delta_i) \quad \text{for all } i \in \{1, \dots, n\}, \text{for all } j \in \{1, \dots, q\},$$
$$\frac{\partial \mathcal{L}}{\partial z_i} = k_i(\varphi_i'(z_i) - \delta_i) \quad \text{for all } i \in \{1, \dots, n\}.$$

For the problem with multiple peak demand intervals, we now have a Lagrange multiplier λ_j for each peak demand interval j, and the speed for each train in peak demand interval j is determined by the choice of λ_j and not by the speeds of the other trains. The optimization problem can be solved as follows.

(1) Pick an initial value for each λ_j.
(2) Find values w_{ij} for each train i as follows:
 (a) if $k_i > 0$, then calculate w_{ij} numerically by solving
$$\lambda_j \varphi_i'(w_{ij}) = \varphi_i'(z_i), \tag{5.6}$$
 where z_i is given by (5.3);
 (b) otherwise, if $k_i = 0$, then pick a value for δ_i and calculate w_{ij} numerically by solving
$$\lambda_j \varphi_i'(w_{ij}) = \delta_i. \tag{5.7}$$
 If (5.4) is not satisfied, then adjust δ_i and try again.
(3) If (5.1) is not satisfied, then adjust the λ_j values and try again from Step (2).

The outer loop, which finds values for λ_j, can be solved using a multi-dimensional root-finding method. We used the SciPy `fsolve` function [13].

5.2. Multiple peak demand interval example Consider the same four trains from our single peak demand interval example (Section 4.2). As before, the peak demand period starts at time $p_0 = 1800$ and finishes at time $p_q = 5400$, and the initial train speeds are $\mathbf{v} = [75, 85, 60, 50]$. The peak demand period is divided into 12 equal intervals of 5-minutes, so we have 12 initial energy values E_{0j} and 12 target energy values $E_{1j} = 0.9\,E_{0j}$. Numerical solution of (5.6) and (5.7) give 12 Lagrange multipliers λ_j each in the range [1.173, 1.221] and $\delta \approx [0, 0, 13\,160, 0]$. Similarly to the case with

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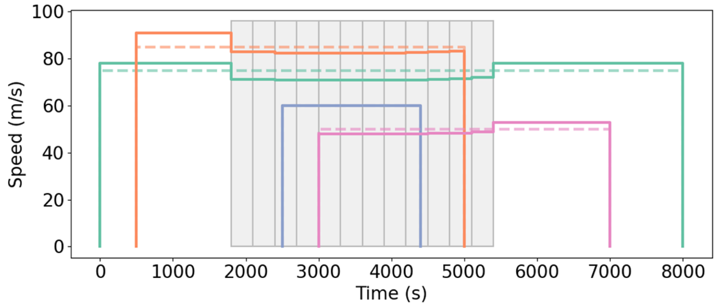

FIGURE 4. Initial (pale, dashed) and adjusted (bold) speed profiles for four trains. The shaded region is the peak demand period divided into 5-minute peak demand intervals.

a single peak demand interval, $\delta_i > 0$ only when a train journey starts and finishes within the peak demand period ($k_i = 0$). We calculate new speeds w_{ij} within each peak demand interval j. The average speeds across the entire peak demand period are $\overline{\mathbf{w}} \approx [71.11, 82.60, 60.01, 48.10]$, and the new speeds before and after the peak demand period are $\mathbf{z} = [78.18, 90.94, 0, 52.86]$. This solution is shown in Figure 4 and was calculated in approximately 0.08 seconds. Total energy increases from 7.049×10^9 to 7.0913×10^9. This is a 0.60% increase.

Notice that the average speeds $\overline{\mathbf{w}}$ across the entire peak demand period are different to the holding speeds $\mathbf{w}$ shown in the example with a single peak demand interval in Section 4.2. Within the peak demand period, whenever a journey starts or finishes, the speeds w_{ij} of the other trains change, including journeys completely within the peak demand period. However, this change is usually small.

5.3. One hundred trains Ultimately, we want to calculate optimal journey profiles for a fleet of 100 trains or more that reduce their energy use in each 10-minute interval for a 1-hour peak demand period. As a final example with simple trains, we generated 100 trains with random journey distances $X_i \in [39\,100, 736\,500]$, start times $s_i \in [0, 5000]$, finish times $f_i \in [2300, 10800]$, and initial train speeds $v_i \in [60, 85]$. The peak demand period was divided into six equal intervals of 10 minutes:

$$\mathbf{p} = [1800, 2400, 3000, 3600, 4200, 4800, 5400]$$

so we have six initial energy values E_{0j} and six target energy values $E_{1j} = 0.9\,E_{0j}$. Numerical solution of (5.6) and (5.7) give the Lagrange multipliers

$$\lambda \approx [1.235, 1.142, 1.135, 1.133, 1.133, 1.133]$$

and $\delta_i \in [0, 1\,596\,000]$, from which we can calculate new speeds within each peak demand interval $w_{ij} \in [0, 83.99]$ and new speeds outside the peak demand period $z_i \in [0, 89.38]$. This solution is shown in Figure 5 and was calculated in approximately 0.97 seconds. Total energy increases from 168.86×10^9 to 169.22×10^9. This is a 0.21% increase.

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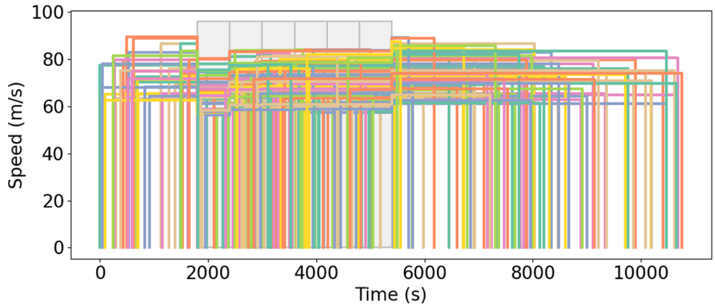

FIGURE 5. Adjusted (bold) speed profiles for 100 trains. The shaded region is the peak demand period containing six consecutive 10-minute peak demand intervals.

6. Towards realistic journey profiles

The analysis and examples in Sections 3–5 are a first step towards solving the theoretical problem of managing energy use for a fleet of trains. We have shown that, at least for a very simple train model, we can formulate and solve the optimization problem. We have also shown the following.

- There is a Lagrange multiplier λ_j for each peak demand interval $j \in \{1, \ldots, q\}$. Increasing λ_j will decrease the speed of the trains in peak demand interval j and, hence, decrease the total energy use by the fleet of trains during peak demand interval j.
- There is a Lagrange multiplier δ_i for each train journey entirely within the peak demand period ($k_i = 0$). Increasing δ_i (without changing λ) will increase the speed of train i during each of the peak demand intervals.

That is, λ_j controls the energy use of all trains during peak demand interval j, and δ_i controls the contribution of train i to energy use across all peak demand intervals.

This problem is not as simple as solving for each individual peak demand interval. The speed of train i during peak demand interval j is influenced by the speed train i will travel during other peak demand intervals as well as the speed train i will travel before and after the peak demand period. This means that the optimization problem is multidimensional with the number of dimensions depending on the number of peak demand intervals q and the number of trains n.

Real trains cannot change speed instantly and, because of gradients and speed limits, they cannot always travel at constant speed. We investigated whether we could simply use the hold speeds calculated with our simple model as hold speeds for realistic train models, but the differences between the real train speed profiles and the simple speed profiles were large enough to mean that the energy and distance constraints were not met. The next step will be to incorporate acceleration, coasting and braking dynamics into our multi-train model, similar to the approach used by Howlett et al. [6] for a single train.

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Another practical issue to be considered is that on the day of operation, some trains might be running late and so unable to drive according to a pre-planned peak-demand schedule, and it may not be feasible to calculate a new exact solution in real time. Understanding the structure of an optimal solution will be useful when trying to calculate an approximate solution in real time.

7. Conclusion

The peak demand of a fleet of trains can be reduced during specified time intervals by slowing the trains during these intervals. We have formulated the problem of achieving target peak demand reductions in specified intervals as an optimal control problem, and found the structure of an optimal solution for a fleet of trains with piecewise-constant speed profiles. The problem is a multidimensional problem that can be solved using classical methods of constrained optimisation. The number of dimensions of the problem is the number of peak demand intervals plus the number of trains. The optimal solution can be found by adjusting:

- Lagrange multipliers $\{\lambda_j\}$ to adjust the speed of the trains during each peak demand interval j;
- Lagrange multipliers $\{\delta_i\}$ to adjust the average speed of train i to ensure that it completes its journey on time.

Although the simple train model we use is unrealistic, we have identified general characteristics of the optimal solution that we expect to find when incorporating realistic train models.

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