Canad. Math. Bull. Vol. 19 (1), 1976

A GLOBAL EXISTENCE AND UNIQUENESS THEOREM FOR ORDINARY DIFFERENTIAL EQUATIONS

BY
W. DERRICK AND L. JANOS

As observed by A. Bielecki and others ([1], [3]) the Banach contraction principle, when applied to the theory of differential equations, provides proofs of existence and uniqueness of solutions only in a local sense. S. C. Chu and J. B. Diaz ([2]) have found that the contraction principle can be applied to operator or functional equations and even partial differential equations if the metric of the underlying function space is suitably changed. The purpose of this note is to apply the ideas of Chu and Diaz to the differential equation

$$y'(x) = f(x, y), \tag{1}$$

where $f(x, y)$ is a continuous function from $(-a, a)\times E^k$ into E^k, $0<a\leq\infty$, which satisfies the global Lipschitz condition

$$|f(x, y_2)-f(x, y_1)| \leq z(x)\,|y_2-y_1|, \tag{2}$$

for all $x\in(-a, a), y_1, y_2\in E^k$, and some non-negative continuous function $z(x)$ defined on $(-a, a)$. Here $|\cdot|$ denotes the usual norm in E^k.

Let $\{\mathscr{I}_n \mid n\geq 1\}$ be an increasing family of compact intervals which contain zero and satisfy $\bigcup_n\mathscr{I}_n=(-a, a)$. Denote by $C(\mathscr{I}_n)$ the Banach space of continuous vector functions $g:\mathscr{I}_n\to E^k$ with the norm

$$\|g\|_{(n,\lambda)} = \sup_{x\in\mathscr{I}_n}\left\{\exp\left(-\lambda\left|\int_0^x z(t)\,dt\right|\right)|g(x)|\right\}, \tag{3}$$

where λ is an arbitrary parameter. The Fréchet space $C(-a, a)$ may be topologized by the family of seminorms $\{\|g\|_{(n,\lambda)} \mid n\geq 1\}$. If $\lambda=0$, the spaces $C(\mathscr{I}_n)$ have the usual sup norm $\|\cdot\|_0$ on $\mathscr{I}_n$.

THEOREM. *If the right side $f(x,y)$ of the differential equation* (1) *satisfies the condition* (2), *then the initial value problem $y(0)=y_0$ has a unique solution y for every $y_0\in E^k$. Furthermore if g is an arbitrary function in $C(-a, a)$ then the sequence $\{g_n\}$ of iterates of g, under the operator*

$$T: g(x)\to y_0+\int_0^x f(t, g(t))\,dt, \tag{4}$$

converges to y uniformly on each compact subinterval of $(-a, a)$.

https://doi.org/10.4153/CMB-1976-015-7 Published online by Cambridge University Press

Proof. Let $\mathscr{I}$ be a compact subinterval containing 0 of $(-a, a)$ and for simplicity, denote the norm of $g \in C(\mathscr{I})$ by $\|g\|_\lambda$. By (3), it is clear that the norms $\|g\|_\lambda$, for arbitrary real λ, are all equivalent to the standard supremum norm $\|g\|$. Denote the restriction of T to $C(\mathscr{I})$ again by the same letter. We prove that

$$(5) \qquad \|Tg_2 - Tg_1\|_\lambda \leq \frac{1}{\lambda}\|g_2 - g_1\|_\lambda,$$

for all $g_1, g_2 \in C(\mathscr{I})$ and $\lambda > 0$. It is easy to verify that the identity

$$(6) \qquad \left|\int_0^x \exp\left(\lambda\left|\int_0^t z(s)\,ds\right|\right)z(t)\,dt\right| = \frac{1}{\lambda}\left(\exp\left(\lambda\left|\int_0^x z(t)\,dt\right|\right)-1\right),$$

is valid for every $x \in (-a, a)$. Using (2) and the definition of $\|g\|_\lambda$ we have:

$$\|Tg_2 - Tg_1\|_\lambda \leq \sup_{x\in\mathscr{I}}\left\{\exp\left(-\lambda\left|\int_0^x z(t)\,dt\right|\right)\cdot\left|\int_0^x z(t)\,|g_2(t)-g_1(t)|\,dt\right|\right\},$$

which in turn is majorized by

$$\|g_2 - g_1\|_\lambda \sup_{x\in\mathscr{I}}\left\{\exp\left(-\lambda\left|\int_0^x z(t)\,dt\right|\right)\cdot\left|\int_0^x \exp\left(\lambda\left|\int_0^t z(s)\,ds\right|\right)z(t)\,dt\right|\right\}.$$

By identity (6) this last term equals

$$\frac{1}{\lambda}\|g_2 - g_1\|_\lambda \sup_{x\in\mathscr{I}}\left\{\exp\left(-\lambda\left|\int_0^x z(t)\,dt\right|\right)\left(\exp\left(\lambda\left|\int_0^x z(t)\,dt\right|\right)-1\right)\right\} \leq \frac{1}{\lambda}\|g_2 - g_1\|_\lambda.$$

The proof of our theorem now follows from the classical Banach contraction principle applied to T and the distance function $\|g_2 - g_2\|_\lambda$, if we choose $\lambda > 1$.

For arbitrary $g \in C\,(-a, a)$, we easily obtain the classical error estimate

$$(7) \qquad \|T^n g - y\|_\lambda \leq \frac{1}{\lambda^n - \lambda^{n-1}}\|Tg - g\|_\lambda.$$

We now focus our attention on a given point $x^* \in (-a, a)$ and determine the "error" of the n-th iterate T^n at this point. Choosing the subinterval $\mathscr{I}$ so as to contain x^* we obtain from (7) and the definition of $\|.\|_\lambda$ the inequality

$$(8) \qquad |T^n g(x^*) - y(x^*)| \leq \exp\left(\lambda\left|\int_0^{x^*} z(t)\,dt\right|\right)\|T^n g - y\|_\lambda \leq \frac{\exp(\lambda a)}{\lambda^n - \lambda^{n-1}}\|Tg - g\|_0,$$

where $a = |\int_0^{x^*} z(t)\,dt|$. Selecting $n > ea$ and $\lambda = n/a$, we obtain from (8) the final estimate

$$|T^n g(x^*) - y(x^*)| \leq \frac{1}{1-(a/n)}\left(\frac{n}{ea}\right)^{-n}\|Tg - g\|_0,$$

showing a rapid diminishing of the error with increasing n. The choice $\lambda = n/a$ is almost optimal for large values of n.

https://doi.org/10.4153/CMB-1976-015-7 Published online by Cambridge University Press

References

1. A. Bielicki, *Une remarque sur le methode de Banach–Cacciopoli–Tikhonov dans la théorie des equations differentielles ordinaires*, Bull. Acad. Polon. Sci. **4**, 1956, pp. 261–264.

2. S. C. Chu and J. B. Diaz, *A Fixed point theorem for "in large" applications of the contraction principle*, A. D. Ac. di Torino, Vol. **99** (1964–65), pp. 351–363.

3. L. Janos, *Contraction property of the operator of integration*, Can. Math. Bull. (to appear).

University of Montana