Proceedings of the Edinburgh Mathematical Society (1997) 40, 261–266 ©

SPECTRAL MAPPING THEOREM FOR REPRESENTATIONS OF MEASURE ALGEBRAS

by H. SEFEROĞLU

(Received 29th April 1995)

Let G be a locally compact abelian group, $M_0(G)$ be a closed regular subalgebra of the convolution measure algebra $M(G)$ which contains the group algebra $L^1(G)$ and $\omega : M_0(G) \to B$ be a continuous homomorphism of $M_0(G)$ into the unital Banach algebra B (possibly noncommutative) such that $\omega(L^1(G))$ is without order with respect to B in the sense that if for all $b \in B, b.\omega(L^1(G)) = \{0\}$ implies $b = 0$. We prove that if $sp(\omega)$ is a synthesis set for $L^1(G)$ then the equality $\sigma_B(\omega(\mu)) = \overline{\hat{\mu}(sp(\omega))}$ holds for each $\mu \in M_0(G)$, where $sp(\omega)$ denotes the Arveson spectrum of ω, $\sigma_B(.)$ the usual spectrum in B, $\hat{\mu}$ the Fourier–Stieltjes transform of μ.

1991 *Mathematics subject classification:* Primary 46H15; Secondary 47A10.

1. Introduction and the main result

Let G be a locally compact abelian group, $\hat{G}$ its dual group, $L^1(G)$ the group algebra and $M(G)$ the Banach algebra of bounded regular complex Borel measures on G. Given any closed subalgebra $M_0(G)$ of $M(G)$ which contains $L^1(G) : L^1(G) \subset M_0(G) \subset M(G)$. $\hat{G}$ can be considered as a subset of the maximal ideal space of $M_0(G)$ and the restriction of the Gelfand transform of $\mu \in M_0(G)$ to $\hat{G}$ coincides with the Fourier–Stieltjes transform $\hat{\mu}$ of μ. This implies that $M_0(G)$ is a semisimple algebra.

Let X be a Banach space, $B(X)$ the algebra of all bounded linear operators on X and 1_X the unit element of $B(X)$. For any $T \in B(X)$ we denote by $\sigma(T)$ the spectrum of T. Now let U be a representation of G be means of isometries, i.e. a map $U : G \to B(X)$ satisfying

(i) $U(s+t) = U(s)U(t)$ for all $s, t \in G$, $U(0) = 1_X$

(ii) $\|U(s)x\| = \|x\|$ for all $s \in G, x \in X$

(iii) $s \to U(s)x$ is a continuous for each $x \in X$.

Then this representation induces a continuous homomorphism $\pi : M_0(G) \to B(X)$ given by

$$\pi(\mu) = \int_G U(g)d\mu(g).$$

The Arveson spectrum $sp(U)$ of U [2] is defined as the hull in $L^1(G)$ of the closed ideal

$I_U = \{f \in L^1(G) \mid \pi(f) = 0\}$. In this setting, A. Connes proved that for every Dirac measure μ the spectral mapping theorem $\sigma(\pi(\mu)) = \overline{\hat{\mu}(sp(U))}$ holds (see [7]). C. D'Antoni, R. Longo and L. Zsido [1], proved the spectral mapping theorem for every $\mu \in L^1(G) \oplus M_d(G)$, where $M_d(G)$ is the algebra of all discrete measures on G. Furthermore S.-E. Takahasi and J. Inoue [7] proved the spectral mapping theorem for any regular subalgebra of $M(G)$ in the case that G is compact. Since there exists a largest regular subalgebra of $M(G)$ which contains $L^1(G) \oplus M_d(G)$ [7], the Takahasi–Inoue theorem contains the D'Antoni–Longo–Zsido spectral mapping theorem for the compact case.

Now, let B be a complex Banach algebra (possibly noncommutative) with the unit element 1_B and let $\omega : M_0(G) \to B$ be a continuous homomorphism. The Arveson spectrum of ω (which we will denote by $sp(\omega)$) is defined as the hull in $L^1(G)$ of the closed ideal $I_\omega = \{f \in L^1(G) \mid \omega(f) = 0\}$. More precisely, $sp(\omega) = \{\chi \in \hat{G} \mid \omega(f) = 0 \Rightarrow \hat{f}(\chi) = 0, f \in L^1(G)\}$ where $\hat{f}$ denotes the Fourier transform of $f \in L^1(G)$. It is easily seen that $sp(\omega)$ is a closed subset of $\hat{G}$. Recall ([4, p. 13]) that a Banach algebra A is without order if for all $a \in A$, $a.A = \{0\}$ implies $a = 0$. We say that a subset $C \subset A$ is without order with respect to A, if for all $a \in A$, $a.C = \{0\}$ implies $a = 0$.

Throughout this paper we will assume that $\omega(L^1(G))$ is without order with respect to B. It can be seen that, under this condition $sp(\omega) \neq \emptyset$, whenever $\omega \neq 0$.

In the present note we prove the following.

Theorem. *Let $M_0(G)$ be a regular Banach algebra and let $\omega : M_0(G) \to B$ be a continuous homomorphism such that $\omega(L^1(G))$ is without order with respect to B. If $sp(\omega)$ is a synthesis set for $L^1(G)$, then we have*

$$\sigma_B(\omega(\mu)) = \overline{\hat{\mu}(sp(\omega))}$$

for each $\mu \in M_0(G)$.

Note that if G is a compact, then each subset of $\hat{G}$ is a synthesis set for $L^1(G)$ ([5, p. 197]). It can be seen that, the linear span of the ranges of the operators $\pi(f), f \in L^1(G)$, is dense in X. This implies that, $\pi(L^1(G))$ is without order with respect to $B(X)$. Thus, the above theorem contains the Takahasi–Inoue spectral mapping theorem.

For the proof of the theorem we need some preliminary results.

2. The theorem

Let A be a complex commutative Banach algebra, and let $\Delta(A)$ be the structure space of A. It is well known that $\Delta(A)$ is a locally compact Hausdorff space and the Gelfand transform $\hat{a}$ of any $a \in A$ is a continuous function on $\Delta(A)$ which vanishes at infinity. The hull of any ideal $I \subset A$ is $h(I) = \{h \in \Delta(A) \mid \hat{a}(h) = 0, a \in I\}$. A is regular if, for each closed $S \subset \Delta(A)$ and each $h \in \Delta(A) \backslash S$ there exist $a \in A$ with $\hat{a}(h) = 1$ and $\hat{a}(S) = 0$. If A is regular and semisimple then, $I(S) = \{a \in A \mid \hat{a}(h) = 0, h \in S\}$ is the largest closed ideal with hull equal to S, and $J(S) = \{a \in A \mid \hat{a} = 0$ near S and supp $\hat{a}$ is compact$\}$ is the smallest ideal with hull equal to S. S is a set of synthesis if and only if $\overline{J(S)} = I(S)$.

Lemma 1. *Let A be a regular semisimple Banach algebra with identity 1_A and let $\omega : A \to B$ be a continuous one-to-one homomorphism such that $\omega(1_A) = 1_B$. Then we have*

$$\sigma_B(\omega(a)) = \hat{a}(\Delta(A))$$

for each $a \in A$.

Proof. We put $B_0 = \overline{\omega(A)}$. Then the mapping $\omega^* : \Delta(B_0) \to \Delta(A)$ is one-to-one. Note that $\omega^*\Delta(B_0) = \Delta(A)$. Suppose on the contrary that there exists $h_0 \in \Delta(A)$ but $h_0 \notin \omega^*\Delta(B_0)$. Let U and V be neighbourhoods of h_0 and $\omega^*\Delta(B_0)$ respectively such that $U \cap V = \emptyset$. Then there exists a, b in A such that $\hat{a}(h_0) = 1$, $\hat{a}(\Delta(A)\backslash U) = 0$, $\hat{b}(\omega^*\Delta(B_0)) = 1$ and $\hat{b}(\Delta(A)\backslash V) = 0$. It is easily seen that $ab = 0$ which implies $\omega(a)\omega(b) = 0$. Since $\omega(b) = B_0^{-1}$ we obtain $\omega(a) = 0$ or $a = 0$. This contradicts $\hat{a}(h_0) = 1$. Now we have $\sigma_{B_0}(\omega(a)) = \hat{a}(\Delta(A))$ for each $a \in A$. It remains to prove that B_0 is a full subalgebra of B. Let $b \in B_0$ be such that $b \in B^{-1}$ and let B_1 be a smallest closed subalgebra of B containing b^{-1} and B_0. Since B_0 is a regular subalgebra of B_1, any $h_0 \in \Delta(B_0)$ may be extended to some $h_1 \in \Delta(B_1)$ by Šilov's theorem ([5, p. 249]). Since $b \in B_1^{-1}$ we have $h_0(b) = h_1(b) \neq 0$ and so $b \in B_0^{-1}$.

Lemma 2. *Let A be a regular semisimple Banach algebra without unit and let $\omega : A \to B$ be a continuous one-to-one homomorphism. Then we have*

$$\sigma_B(\omega(a)) = \overline{\hat{a}(\Delta(A))}$$

for each $a \in A$.

Proof. Let $A_1 = A \times C$ is the unitization of A. It is easily verified that A_1 is a regular and semisimple on the structure space $\Delta(A_1) = \Delta(A) \cup \{\infty\}$ – the one-point compactification of $\Delta(A)$. We may extend the homomorphism ω to the homomorphism $\omega_1 : A_1 \to B$ by

$$\omega_1 : (a, \lambda) \to \omega(a) + \lambda 1_B$$

Since $\|\omega_1\| \leq \max(\|\omega\|, 1)$, ω_1 is a continuous. Moreover $\omega_1 : (0, 1) \to 1_B$ and ω_1 is one-to-one, because the equality $\omega(a) + \lambda 1_B = 0$ $(a \neq 0, \lambda \neq 0)$ clearly implies that $-a/\lambda$ is the identity of A. Now applying Lemma 1 to the homomorphism $\omega_1 : A_1 \to B$ we obtain

$$\sigma_B(\omega(a)) = \hat{a}(\Delta(A_1))$$
$$= \hat{a}(\Delta(A)) \cup \{0\} = \overline{\hat{a}(\Delta(A))}.$$

Lemma 3. *Suppose the hypotheses of the theorem are satisfied. Then $\omega(\mu)=0$ if and only if $\hat{\mu}=0$ on $sp(\omega)$.*

Proof. Let $\mu \in M_0(G)$ and $\omega(\mu)=0$. Then $\omega(\mu * f)=0$ for each $f \in L^1(G)$. Since $\mu * f \in L^1(G)$ and $\mu * f \in Ker(\omega)$ it follows that $\mu * f$ belongs to the largest ideal in $L^1(G)$ with hull equal to $sp(\omega)$. Thus $\widehat{\mu * f} = \hat{\mu}.\hat{f} = 0$ on $sp(\omega)$ for each $f \in L^1(G)$. This clearly implies that $\hat{\mu}=0$ on $sp(\omega)$.

Now let $\mu \in M(G)$ and $\hat{\mu}=0$ on $sp(\omega)$. Then $\widehat{\mu * f}=0$ on $sp(\omega)$ for each $f \in L^1(G)$. Since $sp(\omega)$ is a synthesis set for $L^1(G)$, $I_\omega = \{f \in L^1(G) \mid \omega(f)=0\}$ is the unique ideal in $L^1(G)$ with hull equal to $sp(\omega)$. On the other hand we know that $\mu * f \in L^1(G)$ and hence $\mu * f \in I_\omega$. Thus we obtain $\omega(\mu)\,\omega(f)=0$ for each $f \in L^1(G)$. Since $\omega(L^1(G))$ is without order with respect to B, we conclude that $\omega(\mu)=0$.

Proof of theorem. A continuous homomorphism $\omega : M_0(G) \to B$ induces a continuous one-to-one homomorphism

$$\tilde{\omega} : M_0(G)/Ker(\omega) \to B$$

defined by

$$\tilde{\omega} : \mu + Ker(\omega) \to \omega(\mu).$$

$M_0(G)/Ker(\omega)$ is a regular Banach algebra with the structure space $h(Ker(\omega))$. Let $\overline{sp(\omega)}$ denotes the closure of $sp(\omega)$ in the usual topology of $\Delta(M_0(G))$. Recall that $I(\overline{sp(\omega)})$ is the largest closed ideal in $M_0(G)$ with hull equal to $\overline{sp(\omega)}$. Using Lemma 3 we can see that,

$$Ker(\omega) = I(\overline{sp(\omega)}).$$

Therefore, $h(Ker(\omega))$ is the closure of $\overline{sp(\omega)}$ in the hull-kernel topology. Since $M_0(G)$ is regular, the Gelfand topology coincide with the hull-kernel topology on $\Delta(M_0(G))$ and hence it follows that the structure space of $M_0(G)/Ker(\omega)$ is $\overline{sp(\omega)}$:

$$\Delta(M_0(G)/Ker(\omega)) = h(Ker(\omega)) = \overline{sp(\omega)}.$$

Now, again using Lemma 3 we can see that the algebra $M_0(G)/Ker(\omega)$ is semisimple.

Assume that $h(Ker(\omega))$ is compact. Then there exist $\mu_0 \in M_0(G)$ such that $\mu_0^\vee = 1$ near $h(Ker(\omega))$ (here $\mu^\vee$ denotes the Gelfand transform of any $\mu \in M_0(G)$). Since $(\mu^* \mu_0 - \mu)^\vee$ vanishes near $h(Ker(\omega))$, it follows that $\mu^* \mu_0 - \mu$ belongs to the smallest ideal whose hull is $h(Ker(\omega))$ and so $\omega(\mu)\,\omega(\mu_0) = \omega(\mu)$ for each $\mu \in M_0(G)$. Thus $\mu_0 + Ker(\omega)$ is the identity of the algebra $M_0(G)/Ker(\omega)$. From equality

$(\omega(\mu_0) - 1_B)\omega(f) = 0$, $f \in L^1(G)$ and from the fact that $\omega(L^1(G))$ is without order with respect to B we obtain $\tilde{\omega}(\mu_0 + Ker(\omega)) = 1_B$. Now applying Lemma 1 to the homomorphism $\tilde{\omega} : M_0(G)/Ker(\omega) \to B$ we get

$$\sigma_B(\omega(\mu)) = \mu^\vee(\overline{sp(\omega)}).$$

Since $\mu^\vee(\overline{sp(\omega)})$ is closed we have

$$\overline{\hat{\mu}(sp(\omega))} = \overline{\mu^\vee(sp(\omega))} \subset \mu^\vee(\overline{sp(\omega)}).$$

On the other hand since $\mu^\vee$ is continuous on $\Delta(M_0(G))$ we get

$$\mu^\vee(\overline{sp(\omega)}) \subset \overline{\mu^\vee(sp(\omega))} = \overline{\hat{\mu}(sp(\omega))}.$$

Thus we obtain

$$\mu^\vee(\overline{sp(\omega)}) = \overline{\hat{\mu}(sp(\omega))}$$

and so

$$\sigma_B(\omega(\mu)) = \overline{\hat{\mu}(sp(\omega))}.$$

If $h(Ker(\omega))$ is noncompact, then the algebra $M_0(G)/Ker(\omega)$ has no unit element. In this case applying Lemma 2 we obtain

$$\sigma_B(\omega(\mu)) = \overline{\mu^\vee(\overline{sp(\omega)})} = \overline{\hat{\mu}(sp(\omega))}.$$

This completes the proof.

Remark. Let $\omega : M(G) \to B(L^1(G))$ be the homomorphism given by $\omega : \mu \to T_\mu$; $T_\mu f = \mu * f$. Obviously $sp(\omega) = \hat{G}$. It is also evident that, if $(f_\lambda)_{\lambda\in\Lambda}$ is a bounded approximate identity for $L^1(G)$, then $T_{f_\lambda} \to I$ strongly. This implies that, $\omega(L^1(G))$ is without order with respect to $B(L^1(G))$. Thus, from the preceding theorem we have $\sigma(T_\mu) = \hat{\mu}(\hat{G})$ for every measure μ from the largest regular subalgebra of $M(G)$. On the other hand, since $\omega(M(G))$ is a multiplier algebra for $L^1(G)$ and ω is isometry ([**4**, p. 6]) we have seen that $M(G)$ may be considered as a full subalgebra of $B(L^1(G))$ ([**4**, p. 15]). Therefore, $\sigma(T_\mu) = \sigma_{M(G)}(\mu) = \mu^\vee(\Delta(M(G)))$ for every $\mu \in M(G)$. It is well known ([**6**, p. 107]) that for non-discrete G there exists a measure $\mu \in M(G)$ such that, $\hat{\mu}(\hat{G}) = 0$ but $\mu^\vee(\Delta(M(G))) \neq 0$ (in other words $\hat{G}$ is not dense in $\Delta(M(G))$). Consequently, for such measures $\sigma(T_\mu) \neq \hat{\mu}(\hat{G})$ (see also [**1**, Remark 1]).

Acknowledgement. The author would like to thank TUBITAK for their financial support.

REFERENCES

1. C. D'Antoni, R. Longo and L. Zsido, A spectral mapping theorem for locally compact groups of operators, *Pacific J. Math.* **103** (1981), 17–24.

2. W. Arveson, On groups of automorphisms of operator algebras, *J. Funct. Anal.* **15** (1974), 217–243.

3. N. Bourbaki, *Theories spectrales* (Hermann, 1967).

4. R. Larsen, *An introduction to the theory of multipliers* (Springer-Verlag, New York, Heidelberg, 1971).

5. R. Larsen, *Banach Algebras* (Markel Dekker, Inc., New York, 1973).

6. W. Rudin, *Fourier Analysis on Groups* (Interscience, New York, 1962).

7. S.-E. Takahasi and J. Inoue, A spectral mapping theorem for some representations of compact abelian groups, *Proc. Edinburgh Math. Soc.* **35** (1992), 47–52.

Ondokuz Mayis University
Faculty of Arts and Sciences
Department of Mathematics
55139, Kurupelit, SAMSUN
Turkey

https://doi.org/10.1017/S0013091500023701 Published online by Cambridge University Press