1
Topological Spaces

Basic mathematical notions useful in the rest of this book are given in this chapter. For conciseness, the definitions and results are not always given in full. They are restricted to the simplest version necessary to follow and understand the results and proofs in this book.

1.1 Topological Spaces

This section lists a few basic notions and definitions from general topology. Most of the topological objects encountered in this book are metric spaces whose definition is also recalled.

Definition 1.1 (Topological space) A *topology* on a set X is a family $\mathcal{O}$ of subsets of X that satisfies the three following conditions:

1. the empty set $\emptyset$ and X are elements of $\mathcal{O}$,
2. any union of elements of $\mathcal{O}$ is an element of $\mathcal{O}$,
3. any finite intersection of elements of $\mathcal{O}$ is an element of $\mathcal{O}$.

The set X together with the family $\mathcal{O}$, whose elements are called open sets, is a *topological space*. A subset C of X is *closed* if its complement is an open set. If $Y \subset X$ is a subset of X, then the family $\mathcal{O}_Y = \{O \cap Y : O \in \mathcal{O}\}$ is a topology on Y, called the *induced topology*.

Definition 1.2 (Closure, interior and boundary) Let S be a subset of a topological space X. The *closure* $\bar{S}$ of S is the smallest closed set containing S. The *interior* $\mathring{S}$ of S is the largest open set contained in S. The *boundary* ∂S of S is the set difference $\partial S = \bar{S} \setminus \mathring{S}$.

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Definition 1.3 (Metric space) A *metric (or distance)* on a set X is a map $d : X \times X \to [0, +\infty)$ such that:

1. for any $x, y \in X$, $d(x, y) = d(y, x)$,
2. for any $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$,
3. for any $x, y, z \in X$, $d(x, z) \leq d(x, y) + d(y, z)$.

The set X together with d is a *metric space*. The smallest topology containing all the open balls $B(x, r) = \{y \in X : d(x, y) < r\}$ is called the *metric topology* on X induced by d.

Definition 1.4 (Continuous map) A map $f : X \to X'$ between two topological spaces X and X' is *continuous* if and only if the pre-image $f^{-1}(O') = \{x \in X : f(x) \in O'\}$ of any open set $O' \subset X'$ is an open set of X. Equivalently, f is continuous if and only if the pre-image of any closed set in X' is a closed set in X.

Definition 1.5 (Compact space) A topological space X is a *compact space* if any open cover of X admits a finite subcover, i.e. for any family $\{U_i\}_{i \in I}$ of open sets such that $X = \cup_{i \in I} U_i$ there exists a finite subset $J \subseteq I$ of the index set I such that $X = \cup_{j \in J} U_j$.

For metric spaces, compacity is characterized using sequences: a metric space X is compact if and only if any sequence in X has a convergent subsequence.

Definition 1.6 (Connected spaces) A topological space X is connected if it is not the union of two disjoint open sets: if O_1, O_2 are two disjoint open sets such that $X = O_1 \cup O_2$ then $O_1 = \emptyset$ or $O_2 = \emptyset$.

A topological space X is path-connected if for any $x, y \in X$ there exists a continuous map $\gamma : [0, 1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

A path-connected space is always connected, but the reverse is not true in general. See Exercise 1.1.

Euclidean spaces. The space $\mathbb{R}^d$, $d \geq 1$ and its subsets are examples of particular interest. Throughout the book, for $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$

$$\|x\| = \sum_{i=1}^{d} x_i^2$$

denotes the *Euclidean norm* on $\mathbb{R}^d$. It induces the *Euclidean metric* on $\mathbb{R}^d$: $d(x, y) = \|x - y\|$. The standard topology on $\mathbb{R}^d$ is the one induced by the Euclidean metric.

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A subset $K \subset \mathbb{R}^d$ (endowed with the topology induced from the Euclidean one) is compact if and only if it is closed and bounded (Heine–Borel theorem).

1.2 Comparing Topological Spaces

There are many ways of measuring how close two objects are. We distinguish between topological and geometric criteria.

1.2.1 Homeomorphism, Isotopy and Homotopy Equivalence

In topology, two topological spaces are considered to be the same when they are *homeomorphic*.

Definition 1.7 (Homeomorphism) Two topological spaces X and Y are homeomorphic if there exists a continuous bijective map $h : X \to Y$ such that its inverse h^{-1} is also continuous. The map h is called a homeomorphism.

As an example, a circle and a simple closed polygonal curve are homeomorphic. By contrast, a circle and a segment are not homeomorphic. See Exercise 1.6.

The continuity of the inverse map in the definition is automatic in some cases. If U is an open subset of $\mathbb{R}^d$ and $f : U \to \mathbb{R}^d$ is an injective continuous map, then $V = f(U)$ is open and f is a homeomorphism between U and V by Brower's invariance of domain.[1] The domain invariance theorem may be generalized to manifolds: If M and N are topological k-manifolds without boundary and $f : U \to N$ is an injective continuous map from an open subset of M to N, then f is open and is an homeomorphism between U and $f(U)$.

If X is homeomorphic to the standard unit ball of $\mathbb{R}^d$, X is called a *topological ball*.

The notions of compacity and connexity are preserved by homeomorphism. See Exercise 1.4.

Let h be a map between two topological spaces X and Y. If h is a homeomorphism onto its image, it is called an *embedding* of X in Y.

When the spaces X and Y are subspaces of $\mathbb{R}^d$, the notion of *isotopy* is stronger than the notion of homeomorphism to distinguish between spaces.

Definition 1.8 (Ambient isotopy) An ambient isotopy between $X \subset \mathbb{R}^d$ and $Y \subset \mathbb{R}^d$ is a map $F : \mathbb{R}^d \times [0, 1] \to \mathbb{R}^d$ such that $F(., 0)$ is the identity map on $\mathbb{R}^d$, $F(X, 1) = Y$ and for any $t \in [0, 1]$, F is a homeomorphism of $\mathbb{R}^d$.

[1] See T. Tao's blog https://terrytao.wordpress.com/2011/06/13/brouwers-fixed-point-and-invariance-of-domain-theorems-and-hilberts-fifth-problem/

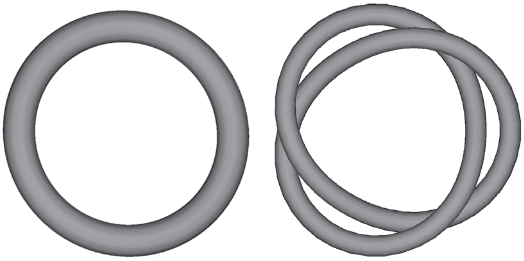

Figure 1.1 Two surfaces embedded in $\mathbb{R}^3$ homeomorphic to a torus that are not isotopic.

Intuitively, the previous definition means that X can be continuously deformed into Y without creating any self-intersection or topological changes. The notion of isotopy is stronger than the notion of homeomorphism in the sense that if X and Y are isotopic, then they are obviously homeomorphic. Conversely, two subspaces of $\mathbb{R}^d$ that are homeomorphic may not be isotopic. This is the case for a knotted and an unknotted torus embedded in $\mathbb{R}^3$ as the ones in Figure 1.1. Note that, although intuitively obvious, proving that these two surfaces are not isotopic is a nonobvious exercise that requires some background in algebraic topology.

In general, deciding whether two spaces are homeomorphic is a very difficult task. It is sometimes more convenient to work with a weaker notion of equivalence between spaces called *homotopy equivalence*.

Given two topological spaces X and Y, two maps $f_0, f_1 : X \to Y$ are *homotopic* if there exists a continuous map $H : [0,1] \times X \to Y$ such that for all $x \in X$, $H(0,x) = f_0(x)$ and $H(1,x) = f_1(x)$. Homotopy equivalence is defined in the following way.

Definition 1.9 (Homotopy equivalence) Two topological spaces X and Y have the same homotopy type (or are homotopy equivalent) if there exist two continuous maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ is homotopic to the identity map in X and $f \circ g$ is homotopic to the identity map in Y.

As an example, the unit ball in a Euclidean space and a point are homotopy equivalent but not homeomorphic. A circle and an annulus are also homotopy equivalent: see Figure 1.2 and Exercise 1.8.

Definition 1.10 (Contractible space) A contractible space is a space that has the same homotopy type as a single point.

For example, a segment or more generally any ball in a Euclidean space $\mathbb{R}^d$ is contractible: see Exercise 1.7.

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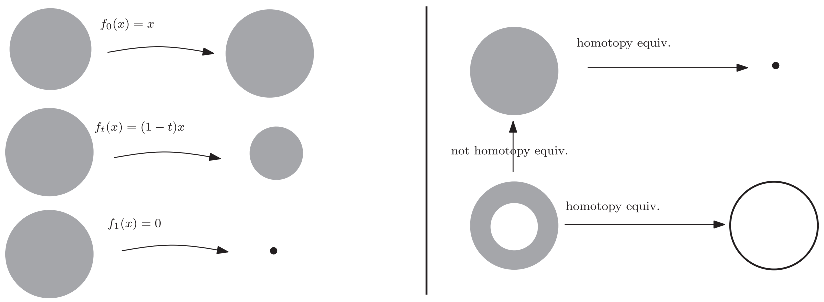

Figure 1.2 An example of two maps that are homotopic (left) and examples of spaces that are homotopy equivalent, but not homeomorphic (right).

It is often difficult to prove homotopy equivalence directly from the definition. When Y is a subset of X, the following criterion is useful to prove homotopy equivalence between X and Y.

Proposition 1.11 *If $Y \subset X$ and if there exists a continuous map $H : [0,1] \times X \to X$ such that:*

1. $\forall x \in X$, $H(0,x) = x$,
2. $\forall x \in X$, $H(1,x) \in Y$,
3. $\forall y \in Y$, $\forall t \in [0,1]$, $H(t,y) \in Y$,

then X and Y are homotopy equivalent.

Definition 1.12 (Deformation retract) If, in Proposition 1.11, the last property of H is replaced by the following stronger one

$$\forall y \in Y, \quad \forall t \in [0,1], \quad H(t,y) = y,$$

then H is called a *deformation retract* of X to Y.

A classical way to characterize and quantify topological properties and features of spaces is to consider their *topological invariants*. These are mathematical objects (numbers, groups, polynomials, …) associated to each topological space that have the property of being the same for homeomorphic spaces. The homotopy type is clearly a topological invariant: two homeomorphic spaces are homotopy equivalent. The converse is false: for example, a point and a segment are homotopy equivalent but are not homeomorphic. See Exercise 1.7. Moreover, most of the topological invariants considered in the sequel are indeed homotopy invariants, i.e. they are the same for spaces that are homotopy equivalent.

1.2.2 Hausdorff Distance

The set of compact subsets of a metric space can be endowed with a metric, called the Hausdorff distance, that allows to measure how two compact subsets are from each other. We give the definition for compact subspaces of $\mathbb{R}^d$ here but this immediately adapts to the compact subsets of any metric space.

Definition 1.13 (Offset) Given a compact set X of $\mathbb{R}^d$, the *tubular neighborhood* or *offset* X^ε of X of radius ε, i.e., the set of all points at distance at most ε from X:

$$X^\varepsilon = \left\{ y \in \mathbb{R}^d : \inf_{x \in X} \|x - y\| \leq \varepsilon \} = \bigcup_{x \in X} \bar{B}(x, \varepsilon) \right\}$$

where $\bar{B}(x, \varepsilon)$ denotes the closed ball $\{y \in \mathbb{R}^d : \|x - y\| \leq \varepsilon\}$.

Definition 1.14 The *Hausdorff distance* $d_H(X, Y)$ between two closed subsets X and Y of $\mathbb{R}^d$ is the infimum of the $\varepsilon \geq 0$ such that $X \subset Y^\varepsilon$ and $Y \subset X^\varepsilon$. Equivalently,

$$d_H(X, Y) = \max \left(\sup_{y \in Y}(\inf_{x \in X} \|x - y\|), \sup_{x \in X}(\inf_{y \in Y} \|x - y\|) \right).$$

The Hausdorff distance defines a distance on the space of compact subsets of $\mathbb{R}^d$. See Exercise 1.10.

1.3 Exercises

Exercise 1.1 Let X be a path connected space. Show that X is connected. Let $X \subset \mathbb{R}^2$ be the union of the vertical closed segment $\{0\} \times [-1, 1]$ and the curve $\{(t, sin(\frac{1}{t})) \in \mathbb{R}^2 : t \in (0, 1]\}$. Show that X is compact and connected but not path-connected.

Exercise 1.2 Let S be a subset of a metric space X. Show that:

1. $x \in X \in \bar{S}$ if and only if for any $r > 0$, $B(x, r) \cap S \neq \emptyset$.
2. $x \in X \in \mathring{S}$ if and only if there exists $r > 0$ such that $B(x, r) \subset S$.

Exercise 1.3 Let X be a metric space. Given $x \in X$ and $r > 0$, show that the set $\bar{B}(x, r) = \{y \in X : d(x, y) \leq r\}$ is a closed set which is indeed the closure of the open ball $B(x, r) = \{y \in X : d(x, y) < r\}$.

Exercise 1.4 Let X, Y two homeomorphic topological spaces. Prove the following equivalences:

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1. X is compact if and only if Y is compact.
2. X is connected (resp. path-connected) if and only if Y is connected (resp. path-connected).

Exercise 1.5 Show that the Euclidean space is not compact (without using the Heine–Borel theorem).

Exercise 1.6 A continuous polygonal curve $P \subset \mathbb{R}^2$ with consecutive edges $e_1 = [p_1, p_2], e_2 = [p_2, p_3], \ldots, e_n = [p_n, p_{n+1}]$ is simple and closed if and only if $e_i \cap e_j = \emptyset$ whenever $2 \leq |i - j| \mod (n)$, $e_i \cap e_{i+1} = p_{i+1}$ for $i = 1, \ldots, n-1$ and $e_n \cap e_1 = p_1$. Show that P is homeomorphic to a circle. Show that a circle and a segment are not homeomorphic.

Exercise 1.7 Let X be a segment (i.e., a space homeomorphic to $[0, 1]$) and let Y be a point. Prove that X and Y are homotopy equivalent but not homeomorphic. More generally prove that any ball in $\mathbb{R}^d$ is contractible.

Exercise 1.8 Let X be the unit circle in $\mathbb{R}^2$ and let $Y \subset \mathbb{R}^2$ be the annulus of inner radius 1 and outer radius 2. Prove that X and Y are homotopy equivalent.

Exercise 1.9 Let X and Y be two topological spaces that are homotopy equivalent. Show that if X is path-connected, then Y is also path-connected.

Exercise 1.10 Show that the Hausdorff distance is a distance on the space of compact subsets of $\mathbb{R}^d$. Show that this is no longer true if we extend the definition to noncompact sets (give an example of two different sets that are at distance 0 from each other).

1.4 Bibliographical Notes

All the ideas introduced in this chapter are classical but fundamental, and presented with many details in the classical mathematical literature. For more details about basic topology, the reader may refer to any standard book on general topology such as, e.g. [111]. The geometry of metric spaces is a wide subject in mathematics. The reader interested in the topics may have a look at [30]. More details and results about the notions of homotopy and homotopy equivalence can be found in [86, pp. 171–172] or [110, p. 108].

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