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TAIL PROBABILITIES FOR WEIGHTED SUMS OF PRODUCTS OF NORMAL RANDOM VARIABLES.

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Weighted sums of products of independent normal random variables arise naturally as distributional limits for various statistics. This note investigates the rate at which the tail probability of these sums approaches zero.

1. INTRODUCTION

Random variables of the form $\sum_{n=1}^{\infty} \lambda_n B_n C_n$, where (B_n) and (C_n) are independent sequences of independent, zero mean, normal random variables and (λ_n) is a square summable sequence of non-negative constants occur as distributional limits for various statistics. Kallenberg [**6**] has shown that terms of this form arise in the characterisation of continuous, separately exchangeable processes and so these distributions can arise when studying processes that involve multiparameter indexing sets and certain forms of exchangeability. Examples of statistics of this type can be found in the study of generalised U-statistics [**1, 9**], in the study of symmetric statistics [**10**] and in the study of row and column exchangeable processes [**4**].

There has been considerable attention paid to the order of the tail probabilities of random variables of the form $\sum_{n=1}^{\infty} \lambda_n (B_n^2 - EB_n^2)$ that arise in the study of quadratic forms (see, for example, [**2**]) and while the distribution of the product of two normal random variables has been investigated and tabulated (see, for example, [**8**]) the weighted sum of products has not.

In this paper we shall establish the exact order of the tail probabilities for the weighted sums of products of normal random variables. This result has potential applications in a number of different areas. For example, it is used in [**5**] to establish tightness when proving invariance principles for statistics with limiting distributions of the above form, and knowledge of the exact tail behaviour is needed when calculating asymptotic relative efficiencies for such statistics.

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2. Main Result

Let (B_n) and (C_n), $n = 1, 2, \ldots$ be two independent sequences of independent $N(0,1)$ random variables. First we shall consider the finite sum of products $W = \sum_{j=1}^{n} B_j C_j$. Note that W is symmetric about 0. The following Lemma establishes the rate at which the tail probabilities of W approach 0.

Lemma. *For $n \geqslant 1$,*

$$\lim_{x\to\infty} x^{1-(n/2)} e^x P\Big(|W| > x\Big) = \Big[2^{(n/2)-1}\Gamma\Big(\frac{n}{2}\Big)\Big]^{-1}.$$

In particular, if $n = 2$ then $P\Big(|W| > x\Big) = e^{-x}$.

Proof: As noted in Kendall and Stuart [7, p.290], W is equal in distribution to the difference $U - V$, where U and V are i.i.d. gamma($n/2$, 1) random variables. Thus for $x > 0$, W has density

$$f(x) = ce^{-x}\int_0^\infty \Big(xy + y^2\Big)^{(n/2)-1} e^{-2y}dy,$$

where $c = \Gamma(n/2)^{-2}$. So

$$f(x) = cx^{n-1}e^{-x}\int_0^\infty \Big(v + v^2\Big)^{(n/2)-1} e^{-2vx}dv. \tag{1}$$

Thus if $n = 2$ then $P\Big(|W| > x\Big) = e^{-x}$.

If $n = 1$, then from (1)

$$\begin{aligned} P\Big(|W| > x\Big) &= \frac{2}{\pi}\int_x^\infty\int_0^\infty \Big(v+v^2\Big)^{-1/2} e^{-(2v+1)u}dvdu \\ &= \frac{2}{\pi}\int_0^\infty \Big(v+v^2\Big)^{-1/2}(2v+1)^{-1}e^{-(2v+1)x}dv. \end{aligned}$$

Thus,

$$\begin{aligned} \lim_{x\to\infty} x^{1/2}e^x P\Big(|W| > x\Big) &= \lim_{x\to\infty}\frac{2x^{1/2}}{\pi}\int_0^\infty \Big(v+v^2\Big)^{-1/2}(2v+1)^{-1}e^{-2vx}dv \\ &= \lim_{x\to\infty}\frac{\sqrt{2}}{\pi}\int_0^\infty t^{-1/2}\Big(1+\frac{t}{2x}\Big)^{-1/2}\Big(1+\frac{t}{x}\Big)^{-1}e^{-t}dt \\ &= \sqrt{\frac{2}{\pi}}. \end{aligned}$$

Finally, if $n > 2$ then

$$P\Big(|W| > x\Big) = 2c\int_0^\infty \Big(v+v^2\Big)^{(n/2)-1}\Big(\int_x^\infty u^{n-1}e^{-(2v+1)u}du\Big)\,dv.$$

Using integration by parts it is easy to show that

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$$\lim_{x\to\infty} x^{1-(n/2)} e^x P\Big(|W| > x\Big) = \lim_{x\to\infty} 2c \int_0^\infty \Big(v + v^2\Big)^{(n/2)-1} x^{n/2} (2v+1)^{-1} e^{-2vx} dv$$
$$= 2^{1-(n/2)} \Big/ \Gamma\Big(\frac{n}{2}\Big).$$

□

Let (λ_n), $n = 1, 2, 3, \ldots$ be a nonincreasing sequence of positive real numbers. Let $\xi_1 = \lambda_1 = \ldots = \lambda_{n_1} > \lambda_{n_1+1}$. Similarly, we define ξ_j to be the j^{th} largest of the λ_n and n_j its multiplicity. Let $S = \sum_{i=1}^{\infty} \lambda_i B_i C_i$. Thus, $S = \sum_{j=1}^{\infty} \xi_j W_j$ where $\xi_1 > \xi_2 > \ldots \geqslant 0$ and $W_j = \sum_{i=n_1+\ldots+n_{j-1}+1}^{n_1+\ldots+n_j} B_i C_i$.

THEOREM. *Let (λ_n), $n = 1, 2, 3, \ldots$ be a nonincreasing sequence of positive real numbers such that $\sum_{i=1}^{\infty} \lambda_i^2 < \infty$. If $S = \sum_{i=1}^{\infty} \lambda_i B_i C_i$ then $P\Big(|S| < \infty\Big) = 1$ and*

$$P\Big(|S| > x\Big) < KP\Big(|W_1| > x/\lambda_1\Big),$$

where K is a finite, positive constant.

PROOF: That $P\Big(|S| < \infty\Big) = 1$ is shown in [**6**]. It is easily shown that S has moment generating function $\prod_{i=1}^{\infty} \Big(1 - t^2 \xi_i^2\Big)^{-n_i/2}$ for $t < \xi_1^{-1}$.

We observe first that S has a density f_S and we shall use the method exploited by Hoeffding [**3**] to show that

$$\frac{f_S(x)}{f(x)} = \int_{-\infty}^{\infty} \frac{f(x-z)}{f(x)} dG(z) \leqslant K, \ \forall x \neq 0, \tag{2}$$

for some finite constant K, where f is the density of $\xi_1 W_1$ and G is the distribution function of $\sum_{j=2}^{\infty} \xi_j W_j$. Thus, for $x > 0$,

$$P\Big(|S| > x\Big) \leqslant KP\Big(|\xi_1 W_1| > x\Big). \tag{3}$$

In what follows, the K_i's are finite, positive constants although their values are not always explicitly stated.

To prove Equation (2), we again use the observation in [**7**] that $\xi_1 W_1$ is equal in distribution to the difference of two independent gamma$(n_1/2, 1/\lambda_1)$ random variables. Let $\kappa = \lambda_1^{-n_1} \Gamma(n_1/2)^{-2}$. Then

$$f(x) = \kappa e^{-x/\lambda_1} \int_{-x\vee 0}^{\infty} \Big(y^2 + xy\Big)^{(n_1/2)-1} e^{-2y/\lambda_1} dy$$

and so

$$\frac{f(x-z)}{f(x)} = e^{z/\lambda_1} \cdot \frac{\int_{(z-x)\vee 0}^{\infty} (y^2 + xy - zy)^{(n_1/2)-1} e^{-2y/\lambda_1} dy}{\int_0^{\infty} (y^2 + xy)^{(n_1/2)-1} e^{-2y/\lambda_1} dy} \tag{4}$$

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since we may assume (by symmetry) that $x > 0$. Thus when $n_1 = 2$

$$\int_0^\infty \frac{f(x-z)}{f(x)} dG(z) \leqslant \int_{-\infty}^\infty e^{z/\lambda_1} dG(z) = \prod_{i=2}^\infty \left(1 - \frac{\xi_i^2}{\xi_1^2}\right)^{-n_i/2} = \zeta. \tag{5}$$

That ζ is finite follows from the fact that $\xi_1 > \xi_i$, $\forall i \geqslant 2$ and that $\sum_{i=2}^\infty n_i \xi_i^2 = \sum_{i=n_1+1}^\infty \lambda_i^2 < \infty$.

We next prove Equation (2) when $n_1 > 2$. Consider the right hand side of Equation (4) and assume first that $z < 0$.

$$\begin{aligned} &\int_0^\infty \left(y^2 + xy - zy\right)^{(n_1/2)-1} e^{-2y/\lambda_1} dy \\ &\quad \leqslant \left(1 + x + |z|\right)^{(n_1/2)-1} \left[\int_0^1 y^{(n_1/2)-1} e^{-2y/\lambda_1} dy + \int_1^\infty y^{n_1-2} e^{-2y/\lambda_1} dy\right] \\ &\quad = K_1 \left(1 + x + |z|\right)^{(n_1/2)-1}. \end{aligned} \tag{6}$$

On the other hand,

$$\begin{aligned} &\int_0^\infty \left(y^2 + xy\right)^{(n_1/2)-1} e^{-2y/\lambda_1} dy \\ &\quad \geqslant (1 + x)^{(n_1/2)-1} \left[\int_0^1 y^{n_1-2} e^{-2y/\lambda_1} dy + \int_1^\infty y^{(n_1/2)-1} e^{-2y/\lambda_1} dy\right] \\ &\quad = K_2 (1 + x)^{(n_1/2)-1}. \end{aligned} \tag{7}$$

Therefore,

$$\int_{-\infty}^0 \frac{f(x-z)}{f(x)} dG(z) \leqslant \int_{-\infty}^0 \frac{K_1}{K_2} \left(1 + |z|\right)^{(n_1/2)-1} e^{z/\lambda_1} dG(z) < K_3. \tag{8}$$

The convergence of the integral in (8) is clear since G is a distribution function and $(1 + |z|)^{(n_1/2)-1} e^{z/\lambda_1}$ is bounded for $z < 0$.

Now assume that $z \geqslant 0$. Since for all $y \geqslant 0$, $(x - z)y \leqslant xy$, we have that $f(x-z)/f(x) \leqslant e^{z/\lambda_1}$, and so

$$\int_0^\infty \frac{f(x-z)}{f(x)} dG(z) \leqslant \int_{-\infty}^\infty e^{z/\lambda_1} dG(z) = \prod_{i=2}^\infty \left(1 - \frac{\xi_i^2}{\xi_1^2}\right)^{-n_i/2} = \zeta. \tag{9}$$

Thus we see that for $n_1 \geqslant 2$, (2) is satisfied with $K = K_3 + \zeta$.

The case $n_1 = 1$ is somewhat more delicate since in this case, $f(0)$ is undefined. Let g denote the density of $\sum_{j=2}^\infty \xi_j W_j$. For $x > 0$,

$$\frac{f_S(x)}{f(x)} = \frac{\int_{-\infty}^0 \int_0^\infty e^{z/\lambda_1} (y^2 + xy - zy)^{-1/2} e^{-2y/\lambda_1} g(z) dy dz}{\int_0^\infty (y^2 + xy)^{-1/2} e^{-2y/\lambda_1} dy} \tag{10}$$

$$+ \frac{\int_0^x \int_0^\infty e^{z/\lambda_1} (y^2 + xy - zy)^{-1/2} e^{-2y/\lambda_1} g(z) dy dz}{\int_0^\infty (y^2 + xy)^{-1/2} e^{-2y/\lambda_1} dy} \tag{11}$$

$$+ \frac{\int_x^\infty \int_{z-x}^\infty e^{z/\lambda_1} (y^2 + xy - zy)^{-1/2} e^{-2y/\lambda_1} g(z) dy dz}{\int_0^\infty (y^2 + xy)^{-1/2} e^{-2y/\lambda_1} dy}. \tag{12}$$

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The right hand side of (10) is bounded above by

$$\frac{\int_{-\infty}^{0} e^{z/\lambda_1} \int_0^\infty (y^2+xy)^{-1/2} e^{-2y/\lambda_1} dy\, g(z) dz}{\int_0^\infty (y^2+xy)^{-1/2} e^{-2y/\lambda_1} dy} \leqslant \zeta. \tag{13}$$

To bound (11) and (12) we shall apply Equation (5) to the sum

$$S' = \lambda B_2 C_2 + \lambda B_3 C_3 + \sum_{i=4}^{\infty} \lambda_i B_i C_i,$$

where $\lambda_1 > \lambda > \lambda_2$, so that $f_{S'}(z) \leqslant \zeta'(2\lambda)^{-1} e^{-|z|/\lambda}$, where $\zeta' = \prod_{i=4}^{\infty} \left(1 - (\xi_i^2/\lambda^2)\right)^{-n_i/2}$. However, since $\lambda > \lambda_2 \geqslant \lambda_3$, the density of $\lambda_i B_i C_i$ is uniformly bounded above by the density of $\lambda B_i C_i$ multiplied by λ/λ_i, $i = 2, 3$. Thus, there exist finite constants K_4, K_5 such that $g(z) \leqslant K_4 f_{S'}(z) \leqslant K_5 e^{-|z|/\lambda}$. We observe also that

$$\left[\int_0^\infty \left(y^2 + xy\right)^{-1/2} e^{-2y/\lambda_1} dy\right]^{-1} \leqslant K_6 \max\left(1, \sqrt{x}\right)$$

for some constant K_6. Thus, defining $\alpha = (1/\lambda) - (1/\lambda_1) > 0$ and $K_7 = K_5 K_6$,

$$\begin{aligned}
(11) \;&\leqslant K_7 \max\left(1, \sqrt{x}\right) \int_0^x \int_0^\infty e^{-\alpha z}\left(y^2 + xy - zy\right)^{-1/2} e^{-2y/\lambda_1} dy dz \\
&\leqslant K_7 \max\left(1, \sqrt{x}\right) e^{-\alpha x} \int_0^x e^{\alpha(x-z)} (x-z)^{-1/2} dz \int_0^\infty y^{-1/2} e^{-2y/\lambda_1} dy \\
&\leqslant K_8 \max\left(1, \sqrt{x}\right) e^{-\alpha x} \int_0^x e^{\alpha u} u^{-1/2} du \\
&\leqslant \begin{cases} K_8 \int_0^1 u^{-1/2} du & \text{if } x \leqslant 1 \\ K_8 \sqrt{x} e^{-\alpha x} \left[e^{(\alpha x/2)} \int_0^{x/2} u^{-1/2} du + \sqrt{2/x} \int_{x/2}^x e^{\alpha u} du\right] & \text{if } x > 1 \end{cases} \\
&\leqslant K_9. \qquad (14)
\end{aligned}$$

Finally, making the change of variable $u = y - (z - x)$,

$$\begin{aligned}
(12) \;&\leqslant K_7 \max\left(1, \sqrt{x}\right) \int_x^\infty \int_0^\infty e^{-\alpha z}\left(u^2 + (z-x)u\right)^{-1/2} e^{-2u/\lambda_1} e^{-2(z-x)/\lambda_1} du dz \\
&\leqslant K_7 \max\left(1, \sqrt{x}\right) e^{-\alpha x} \int_x^\infty e^{-(z-x)(1/\lambda + 1/\lambda_1)} (z-x)^{-1/2} dz \int_0^\infty u^{-1/2} e^{-2u/\lambda_1} du \\
&\leqslant K_{10} \max\left(1, \sqrt{x}\right) e^{-\alpha x} \\
&\leqslant K_{11}. \qquad (15)
\end{aligned}$$

This completes the proof. □

Corollary 1. *The tail probability* $P\left(|S| > x\right) = O\left(x^{(n_1/2)-1} e^{-x/\lambda_1}\right)$.

Corollary 2. *Using the notation of the Theorem, if m is a positive constant then*

$$E\,|S|^m \leqslant K n_1^{m+1} \lambda_1^m E\,|B_1|^{2m}.$$

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PROOF: The result follows from

$$E\,|S|^m = m\int_0^\infty x^{m-1}P\Big(|S| > x\Big)dx$$

and the inequality

$$P\Big(|W_1| > x\Big) \leqslant n_1 P\Big(|B_1C_1| > x/n_1\Big).$$

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