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FIXED POINT PRINCIPLES FOR CONES OF A BANACH SPACE FOR THE MULTIVALUED MAPS DIFFERENTIABLE AT THE ORIGIN AND INFINITY

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ABSTRACT. In [6] and [7], Krasnosel'skiĭ proved several fundamental fixed point principles for operators leaving invariant a cone in a Banach space. In [9], Nussbaum extended one of the results, the theorem about compression and expansion of a cone, to k-set contraction maps, $k < 1$. Other versions for completely continuous maps were given by Fournier-Peitgen [2] and G. Fournier [1].

The purpose of this paper is to generalise some of these results to upper semi continuous multivalued maps which are k-set contractions, $k < 1$, and differentiable at the origin and infinity.

0. **Notations and definitions.** Let E and E' be two real Banach spaces. In this paper, all multivalued maps $F: E \longrightarrow E'$ are assumed to satisfy $F(x)$ is compact for every $x \in E$. F is upper semi-continuous (u. s. c.) at $x \in E$ if for any neighborhood V of $F(x)$, there exists a neighborhood U of x such that $F(y) \subset V$ for any $y \in U$. F is upper semi-continuous on E if for each open subset V of E', the set $F^{-1}(V) = \{x \in E \mid F(x) \subset V\}$ is an open subset of E. F is lower semi-continuous (l. s. c.) on E if for every open set $V \subset E'$, the set $\{x \in E \mid F(x) \cap V \neq \emptyset\}$ is an open subset of E. If F is both u. s. c. and l. s. c. on E, we say that F is *continuous on E*.

PROPOSITION 0.1. *If* $F: E \longrightarrow E'$ *is* u. s. c., *the image* $F(K) = \bigcup_{x \in K} F(x)$ *of a compact set in E is also compact.*

If $F: E \longrightarrow E'$ and $G: E' \longrightarrow E''$ are two u. s. c. multivalued maps, then the composition $G \circ F: E \longrightarrow E''$ is u. s. c. A point $x \in E$ is called a *fixed point* of $F: E \longrightarrow E$ if $x \in F(x)$. A u. s. c. multivalued map $F: E \longrightarrow E'$ is *acyclic* if for every $x \in E$, $F(x)$ is acyclic with respect to the Čech homology functor H with rational coefficients *i.e.* $F(x)$ is non-empty, $H_0\big(F(x)\big) \simeq \mathbb{Q}$ and $H_q\big(F(x)\big) = 0$ for all $q \geq 1$.

A multivalued map $F: E \longrightarrow E'$ is called *homogenous* if $F(kx) = kF(x)$ for any $x \in E$ and $k \in \mathbb{R}$. We say that F is *semi-linear positive* if $F\big(\sum_{i=1}^{n} t_i x_i\big) \subset \sum_{i=1}^{n} t_i F(x_i)$ for every $x_i \in E$, $t_i \geq 0$ and $\sum_{i=1}^{n} t_i \leq 1$. In this case $F(0) = 0$ and $F(\operatorname{co} A) \subset \operatorname{co} F(A)$ where $\operatorname{co} A$ is the convex hull of $A \subset E$.

A real number λ is an eigenvalue of $F: E \longrightarrow E$ if there exists $x \in E$ such that $\lambda x \in F(x)$. x is called an *eigenvector* corresponding to λ.

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For $A \subset E$, we define the measure of non-compactness $\gamma(A)$ of A (due to Kuratowski) to be $\gamma(A) = \inf\{r > 0 \mid$ there exists a finite covering of A by subsets of diameter at most $r\}$.

We have the following properties:

(0.2) $0 \leq \gamma(A) \leq \delta(A)$ where $\delta(A)$ is the diameter of A.

(0.3) If $A \subset B \subset E$, $\gamma(A) \leq \gamma(B)$.

(0.4) $\gamma(A \cup B) = \max\{\gamma(A), \gamma(B)\}$, $\gamma(A+B) \leq \gamma(A)+\gamma(B)$ and $\gamma\big(N_\epsilon(A)\big) \leq \gamma(A)+2\epsilon$ where $N_\epsilon(A) = \{x \in E \mid d(x,A) < \epsilon\}$ and $d(x,A) = \inf_{y\in A} \|x-y\|$.

(0.5) $\gamma(\overline{\operatorname{co} A}) = \gamma(A)$.

(0.6) $\gamma(A) = 0$ if and only if A is relatively compact.

(0.7) If $K_1 \supset K_2 \supset \cdots \supset K_n \supset \cdots$ where K_n is closed and non empty for any n and $\lim_{n\to\infty} \gamma(K_n) = 0$, then $K_\infty = \bigcap_{n\geq 1} K_n$ is non-empty and compact and for any neighborhood V of K_∞, there exists an integer n_v such that $K_n \subset V$ for all $n \geq n_v$.

The measure of non-compactness of $F\colon E \longrightarrow E'$ is $\gamma(F) = \inf\big\{k \mid \gamma\big(F(A)\big) \leq k\gamma(A)$ for all bounded $A \subset E\big\}$.

DEFINITION 0.8. A u. s. c. multivalued map $F\colon E \longrightarrow E$ is called *k-set contraction* if there exists $k < 1$ such that $\gamma\big(F(A)\big) \leq k\gamma(A)$ for every bounded subset A of E *i.e.* if $\gamma(F) < 1$.

Let $\mathcal{F}(E)$ be the set of all non empty bounded closed sets in a Banach space E. For $A, B \in \mathcal{F}(E)$ define $\rho(A,B) = \sup\{d(x,B) \mid x \in A\}$ and let $\rho(A,B) = \sup\{\rho(A,B), \rho(B,A)\}$, d_H is called the *Hausdorff metric* in $\mathcal{F}(E)$. We write $\|A\|$ in place of $d_H(A,0) = \sup_{a\in A} \|a\|$.

We have the following property:

(0.9) $d_H(A,B) \leq \epsilon$ if and only if $A \subset N_\epsilon(B)$ and $B \subset N_\epsilon(A)$.

DEFINITION 0.10. A cone C of E is a closed convex subset of E such that $\lambda C \subset C$ for all $\lambda \geq 0$ and $C \cap (-C) = \{0\}$ where $-C = \{-x \mid x \in C\}$.

Now, we present briefly the fixed point index for composition of multivalued acyclic maps in Banach spaces. This generalises the index given by Siegberg-Skordev [10] which is defined for compositions of multivalued acyclic maps defined on open subsets of compact polyhedra. For more details, the reader can consult [3].

Let $f = (F_n, \ldots, F_0)$ be a sequence of acyclic maps $F_i\colon X_i \longrightarrow X_{i+1}$ where each X_i is Hausdorff for any $i = 0, 1, \ldots, n$ and $X_0 = U$ is an open subset of $X_{n+1} = X$. The sequence f is said to be an *acyclic decomposition* for F if $F = F_n \circ \cdots \circ F_0$. Moreover f is admissible if $\operatorname{Fix} F = \{x \in U \mid x \in F(x)\}$ is compact in U.

We shall write $X \in \mathcal{F}$ if X is a closed subset of a Banach space E from which it inherits its metric and if X has a locally finite covering by closed convex subsets of X. If in addition the covering is a finite covering, we shall write $X \in \mathcal{F}_0$.

DEFINITION 0.11. An admissible sequence $f = (F_n, \ldots, F_0)$ of acyclic maps is compacting, if there exists an open set W and a sequence $\{K_n\} \subset \mathcal{F}_0$ such that

(0.11.1) $\operatorname{Fix}(F) \subset W \subset \bar{W} \subset U$ where $F = F_n \circ \cdots \circ F_0\colon U \longrightarrow X$ and $X \in \mathcal{F}$;

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(0.11.2) $W \subset K_1 \subset X$;
(0.11.3) $F(W \cap K_n) \subset K_{n+1} \subset K_n$ for all $n \in \mathbb{N}^+$;
(0.11.4) $\lim_{n\to\infty} \gamma(K_n) = 0$.

We have the following theorem:

THEOREM 0.12. *If $f = (F_n, \ldots, F_0)$ is a compacting sequence, then the fixed point index,* $\mathrm{ind}(U, f, X)$, *is defined.*

Although this index depends on the decomposition f, it has nevertheless the following property:

(0.13) $\mathrm{ind}(U, f, X) = \mathrm{ind}\big(U, (F_n, \ldots, F_{i+1}, F_i \circ F_{i-1}, F_{i-2}, \ldots, F_0), X\big)$, provided that $F_i \circ F_{i-1}$ is still acyclic.

REMARK. If $F: C \longrightarrow C$ is a u. s. c. multivalued map with convex values such that F is a k-set contraction, then F is compacting. Hence the index of F is defined and unique by (0.13).

This fixed point index has the usual properties of the index (see [3]), including a homotopy property which is a new result even in the single-valued case. We have also a Lefschetz theorem.

Finally, we will recall our notion of differentiable multivalued maps (see [4]).

Let E, E' be two real Banach spaces and U an open subset of E.

DEFINITION 0.14. A multivalued map $T: U \longrightarrow E'$ is differentiable at the point $x \in U$ if there exists a u. s. c. multivalued map $S_x: T(x) \times E \longrightarrow E'$, $(z, h) \longrightarrow S_x(z, h)$, such that the map $S_{x,z}: E \longrightarrow E'$ defined by $S_{x,z}(h) = S_x(z, h)$ is u. s. c., homogenous and semi-linear positive and such that for any $\epsilon > 0$, there exists $\delta = \delta(\epsilon, x) > 0$ such that if $\|h\| < \delta$, then

$$d_H\Big(T(x+h), \bigcup_{z \in T(x)} \big(z + S_x(z, h)\big)\Big) \leq \epsilon \|h\|.$$

The map S_x is called a *differential* of T at x. If T is differentiable at every point of U, T is said to be *differentiable* on U. We don't have the uniqueness of the differential at a point. However, our differential is not necessarily a map with convex values.

REMARK 0.15. The map $S_{x,z}: E \longrightarrow E'$ is continuous in h for all (x, z).

PROOF. Since $S_{x,z}$ is u. s. c. at $0 \in E$, then for any $\epsilon > 0$, there exists $\delta > 0$ such that $\|h\| < \delta$ implies that $S_{x,z}(h) \subset N_\epsilon(0)$. Since $S_{x,z}$ is homogeneous and semi-linear positive, $S_{x,z}(x' + h) \subset S_{x,z}(x') + S_{x,z}(h)$ where $x' \in U$ and then $\|h\| < \delta$ implies $S_{x,z}(x' + h) \subset S_{x,z}(x') + N_\epsilon(0) \subset N_\epsilon\big(S_{x,z}(x')\big)$. Moreover, if $y \in S_{x,z}(x') \cap V$, where V is open, there exists $\epsilon > 0$ such that $N_\epsilon(y) \subset V$. As above (replacing x' by $x' - h$ and then h by $-h$), there exists $\delta > 0$ such that $\|h\| < \delta$ implies that $S_{x,z}(x') \subset N_\epsilon\big(S_{x,z}(x'+h)\big)$ and thus $S_{x,z}(x' + h) \cap V \neq \emptyset$.

EXAMPLE 0.16. Let $T: U \longrightarrow \mathbb{R}^n$ be the map defined by $T(x) = \mathrm{co}\big(T_1(x), \ldots, T_n(x)\big)$ where $T_i: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a single-valued differentiable map on an open subset U of $\mathbb{R}^n$. Then

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T is differentiable on U and the map $S_x \colon T(x) \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ defined by

$$S_x(z,h) = \left\{ \sum_{i=1}^{n} a_i DT_i(x)(h) \,\middle|\, \sum_{i=1}^{n} a_i T_i(x) = z, a_i \geq 0 \text{ and } \sum_{i=1}^{n} a_i = 1 \right\}$$

is a differential of T at $x \in U$.

The differentiable multivalued maps have the following properties:

(0.17) Let $T \colon U \longrightarrow E'$ be a multivalued map differentiable at $x \in U$. Then there exists $\delta > 0$ and there exists $k > 0$ such that $\|S_x(z,h)\| \leq k\|h\|$ for $\|h\| < \delta$ and for every $z \in T(x)$.

(0.18) If $T \colon U \longrightarrow E'$ is a multivalued map differentiable at the point $x \in U$, then T is continuous at that point.

We have also a mean-value theorem and a remainder theorem (see [4]).

1. **Expansion and compression of cones.** We need the following results to prove the principal theorems of this paper. In the following and for all the sections, E is a real Banach space and U is an open subset of E.

PROPOSITION 1.1. *Let $F \colon U \longrightarrow E$ be a differentiable multivalued map at the point $x \in U$ such that $x \in F(x)$. If F is a k-set contraction, $k < 1$, then $S_{x,x}$ is a k-set contraction.*

PROOF. By hypothesis, there exists $k < 1$ such that $\gamma\big(F(A)\big) \leq k\gamma(A)$ for every bounded subset A of E. Let $\epsilon > 0$ be such that $2\epsilon < 1 - k$. Since F is differentiable at $x \in U$, there exists $\delta > 0$ such that $\|h\| < \delta$ implies that $x+S_{x,x}(h) \subset \bigcup_{z\in F(x)}\big(z+S_{x,z}(h)\big) \subset N_{\epsilon\|h\|}\big(F(x+h)\big)$.

Let $y \in A \subset N_\delta(0)$, then by the positive semi-linearity of $S_{x,x}$, $S_{x,x}(A) \subset S_{x,x}(A-y) + S_{x,x}(y)$. Hence

$$\begin{aligned}
\gamma\big(S_{x,x}(A)\big) &\leq \gamma\big(S_{x,x}(A-y)\big) + 0 = \gamma\big(x + S_{x,x}(A-y)\big)\\
&\leq \gamma\Big(N_{\epsilon\delta(A)}\big(F(x+A-y)\big)\Big)\\
&\leq \gamma\big(F(x+A-y)\big) + 2\epsilon\delta(A)\\
&\leq k\gamma(A) + 2\epsilon\delta(A)\\
&\leq (k+2\epsilon)\delta(A) = k'\delta(A)
\end{aligned}$$

where $k' = k + 2\epsilon < 1$.

If $A = \bigcup_{i=1}^{n} A_i$ is such that $\delta(A_i) < \frac{2}{k'+1}\gamma(A)$, we have $\gamma\big(S_{x,x}(A)\big) \leq \max_{i=1,\ldots,n}\gamma\big(S_{x,x}(A_i)\big) \leq \max_{i=1,\ldots,n} k'\delta(A_i) \leq k''\gamma(A)$ where $k'' = \frac{2k'}{k'+1} < 1$.

Then $\gamma\big(S_{x,x}(A)\big) \leq k''\gamma(A)$ on $N_\delta(0)$ and by homogeneity of $S_{x,x}$ we have $\gamma\big(S_{x,x}(A)\big) \leq k''\gamma(A)$.

DEFINITION 1.2. Let $T \colon U \longrightarrow E'$ a multivalued map such that $T(x)$ is compact for every $x \in U$ where $U = \{x \in E \mid \|x\| > M > 0\}$. T is differentiable at infinity if for any $\epsilon > 0$, there exists $\delta > 0$ such that $\|h\| > \delta$ implies

$$d_H\big(T(h), S(h)\big) \leq \epsilon\|h\|$$

where $S \colon E \longrightarrow E'$ is a u. s. c., homogenous and semi-linear positive map.

The map S is called a *differential of T at infinity*.

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PROPOSITION 1.3. *If $F\colon U \longrightarrow E$ is a differentiable multivalued map at infinity such that F is a k-set contraction, then the differential S is a k-set contraction.*

PROOF. Let $0 < k < 1$ such that $\gamma\big(F(A)\big) \leq k\gamma(A)$ where A is an arbitrary bounded subset of E. We must show that $\gamma\big(S(A)\big) \leq k'\gamma(A)$ where $k' = \frac{k+1}{2} < 1$. We distinguish two cases:

PARTICULAR CASE. If $0 < r = d(0, A)$, it is sufficient to show that there exists $R > 0$ such that $\gamma\big(S(RA)\big) \leq k'\gamma(RA)$ since S is homogenous and $\gamma(RA) = R\gamma(A)$.

Let $\epsilon > 0$ be such that $2\epsilon\|A\| < (k' - k)\gamma(A) + \frac{(1-k)}{2}\gamma(A)$. Since F is differentiable at infinity, there exists $\delta > 0$ such that $\|h\| > \delta$ implies that $S(h) \subset N_{\epsilon\|h\|}\big(F(h)\big)$. Choose $R > 0$ such that $rR > \delta$. Then $S(RA) \subset N_{\epsilon R\|A\|}\big(F(RA)\big)$ and

$$\begin{aligned}\gamma\big(S(RA)\big) &\leq 2\epsilon R\|A\| + \gamma\big(F(RA)\big)\\ &\leq 2\epsilon R\|A\| + k\gamma(RA)\\ &\leq \frac{(1-k)}{2}\gamma(RA) + k\gamma(RA) = k'\gamma(RA).\end{aligned}$$

GENERAL CASE. Since S is u. s. c., there exists $\delta > 0$ such that $S\big(N_\delta(0)\big) \subset N_{\frac{k'}{2}\gamma(A)}(0)$. Then

$$\begin{aligned}\gamma\big(S(A)\big) &\leq \max\Big\{\gamma\big(S\big(A \setminus N_\delta(0)\big), \gamma\Big(S\big(N_\delta(0) \cap A\big)\Big)\Big\}\\ &\leq \max\big\{k'\gamma\big(A \setminus N_\delta(0)\big), k'\gamma(A)\big\} = k'\gamma(A) \text{ by the first case,}\end{aligned}$$

and we get the conclusion.

LEMMA 1.4. *If C is a cone of E, there exists $y \in C$ such that $\|x + \lambda y\| \geq \|x\|$ for every $x \in C$ and for all $\lambda \geq 0$.*

PROOF. See [8].

LEMMA 1.5. *Let C be a cone in E and $S\colon C \longrightarrow C$ a homogeneous multivalued map such that S is a k-set contraction and all eigenvalues of S are different from 1. There exists $\epsilon > 0$ such that $d\big(x, S(x)\big) > \epsilon\|x\|$ for $\|x\| \neq 0$.*

PROOF. Since S is homogeneous, it is sufficient to show that there exists $\epsilon > 0$ such that $d\big(x, S(x)\big) > \epsilon$ for $\|x\| = 1$. We proceed by contradiction and we assume that there exists a sequence $\{x_n\} \subset C$, $\|x_n\| = 1$, such that $d\big(x_n, S(x_n)\big)$ goes to 0 for n large enough. Let $z_n \in S(x_n)$ such that $d(x_n, z_n)$ tends to 0, then $\gamma(\{x_n\}) = \gamma(\{z_n\}) \leq \gamma\big(S(x_n)\big) \leq k\gamma(\{x_n\})$ where $k < 1$ since S is a k-set contraction. But this is true only if $\gamma(\{x_n\}) = 0$. In this case $\{x_n\}$ has a subsequence which converges to $y \in \overline{\{x_n\}}$ and so $\|y\| = 1$ and $y \in S(y)$, a contradiction since 1 is not an eigenvalue of S.

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LEMMA 1.6. *Let C be a cone in E and let $F\colon C \longrightarrow C$ be a multivalued map which is a k-set contraction and differentiable at 0 such that $F(0) = 0$. If S_0 is a differential of F at 0 and all eigenvalues of $S_{0,0}$ are different from 1, then 0 is an isolated fixed point of F.*

PROOF. We proceed by contradiction. If 0 is not an isolated fixed point of F, then there exists a sequence $\{x_n\} \subset C$ which converges to 0 such that $x_n \in F(x_n)$. Since F is differentiable at 0, then $x_n \in F(x_n) \subset N_{\epsilon_n \|x_n\|}\big(S_{0,0}(x_n)\big)$ where ϵ_n goes to zero if n is big enough. This implies that $\frac{x_n}{\|x_n\|} \in N_{\epsilon_n}\Big(S_{0,0}\big(\frac{x_n}{\|x_n\|}\big)\Big)$. Let $y_n = \frac{x_n}{\|x_n\|}$, $\|y_n\| = 1$ and $d\big(y_n, S_{0,0}(y_n)\big)$ goes to 0 for n large enough, which is impossibe by the Lemma 1.5.

LEMMA 1.7. *Let C be a cone in E and let $F\colon C \longrightarrow C$ a multivalued map which is a k-set contraction and differentiable at infinity. If S is a differential of F at infinity and all eigenvalues of S are different from 1, then F has no fixed point if $\|x\|$ is big enough.*

PROOF. Since S is a differential of F at infinity, for any $\epsilon > 0$, there exists $\delta > 0$ such that $\|x\| > \delta$ implies that $F(x) \subset N_{\epsilon\|x\|}\big(S(x)\big)$. By Lemma 1.5, $x \notin N_{\epsilon\|x\|}\big(S(x)\big)$ if $\|x\|$ is big enough and so $x \notin F(x)$.

Now we will give and prove the principal theorems of this paper.

THEOREM 1.8 (EXPANSION AT THE ORIGIN). *Let $C \subset E$ be a cone. Let $F\colon C \longrightarrow C$ be a* u. s. c. *multivalued map with convex values such that $F(0) = 0$, F is a k-set contraction and differentiable at $0 \in C$. Let S_0 be a differential of F at 0 such that*

(1.8.1) $S_{0,0}(h)$ is convex for every h,

(1.8.2) $d\Big(0, \overline{S_{0,0}\big(\partial N_r(0)\big)}\Big) > \gamma(S_{0,0})$ where $r > 0$,

(1.8.3) the eigenvalues of $S_{0,0}$ are strictly greater than 1 ($\Rightarrow$ 0 is an isolated fixed point of $S_{0,0}$).

Then $\mathrm{ind}\big(N_r(0), F, C\big) = 0$ *if r is small enough.*

PROOF. There are three important steps to do to show this result.

i) We will show that F is homotopic to $S_{0,0}$ on $N_r(0)$ if r is small enough.

Let G be the homotopy given by $G(h,t) = \Big(H_h\big(F(h), t\big), t\Big)$ where $H_h(y,t) = H(y,h,t) = (1-t)y + t\rho_h(y)$, $y \in F(h)$, $h \in C$, $t \in [0,1]$ and ρ_h is the projection on the convex compact $F(0) + S_{0,0}(h) = S_{0,0}(h)$ (*i.e.* $\rho_h(y)$ is the set of elements of $S_{0,0}(h)$ which are nearest to y). ρ_h is a multivalued map with convex values and it is u. s. c. at the point $(y,h) \in F(h) \times C$ (see [5]). The sequence $(H \times 1_{[0,1]}, g \times 1_{[0,1]})$ where g is defined by $g(h) = F(h) \times \{h\}$, is an acyclic decomposition for G.

By the Lemma 1.6, G has no fixed point on $\partial N_r(0)$ if r is small enough and by Lemma 1.10, G is compacting. By the homotopy property of the fixed point index,

$$\mathrm{ind}\big(N_r(0), F|_{N_r(0)}, C\big) = \mathrm{ind}\big(N_r(0), (\rho, g|_{N_r(0)}), C\big)$$

where ρ is defined by $\rho(y,h') = \rho_{h'}(y)$ for all $(y,h') \in F(h) \times C$.

Now the homotopy G' represented by

$$G'(h,t) = (1-t)\rho_h\big(F(h)\big) + tS_{0,0}(h) \subset S_{0,0}(h)$$

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has only one fixed point which is 0. By (1.1), $S_{0,0}$ is a k-set contraction so $S_{0,0}$ is compacting and its only fixed point is 0. G' has for acyclic decomposition $(L, g \times 1_{[0,1]})$ where $L(y,h,t) = (1-t)\rho_h(y) + tS_{0,0}(h)$.

By excision, homotopy and (0.13),

$$\begin{aligned}\operatorname{ind}\big(N_r(0), (\rho, g|_{N_r(0)}), C\big) &= \operatorname{ind}\big(N_r(0), (S_{0,0}, g|_{N_r(0)}), C\big)\\ &= \operatorname{ind}\big(N_r(0), S_{0,0}, C\big)\end{aligned}$$

ii) We show now that $S_{0,0}$ is homotopic to $\lambda S_{0,0}$ if $\frac{r}{\|S_{0,0}(\partial N_r(0))\|} < \lambda < \frac{1}{\gamma(S_{0,0})}$.

By the choice of λ, the map $\lambda S_{0,0}$ is a k-set contraction since $\gamma(\lambda S_{0,0}) = \lambda\gamma(S_{0,0}) < 1$.

Let H' be the homotopy defined by $H'(h,s) = sS_{0,0}(h)$ for $1 \le s \le \lambda$. H' has no fixed point on $\partial N_r(0)$. For, if there exists $s \in [1,\lambda]$ and $h \in \partial N_r(0)$ such that $h \in sS_{0,0}(h)$, it follows that $\frac{1}{s}h \in S_{0,0}(h)$ and $\frac{1}{s} > 1$ by (1.8.3) which is a contradiction since $\frac{1}{s} \le 1$. Then the homotopy H' is admissible and by homotopy property,

$$\operatorname{ind}\big(N_r(0), S_{0,0}, C\big) = \operatorname{ind}\big(N_r(0), \lambda S_{0,0}, C\big).$$

iii) By (1.4), let y_0 be a point belonging to C such that $d\big(0, Ry_0 + \lambda S_{0,0}(h)\big) \ge \lambda d\big(0, S_{0,0}(h)\big)$ for all $R > 0$. Choose $R > \frac{r+\lambda\|S_{0,0}(\partial N_r(0))\|}{\|y_0\|}$; then $d\big(0, Ry_0 + \lambda S_{0,0}(h)\big) \ge R\|y_0\| - \lambda\big\|S_{0,0}\big(\partial N_r(0)\big)\big\| > \frac{(r+\lambda\|S_{0,0}(\partial N_r(0))\|)}{\|y_0\|}\|y_0\| - \lambda\big\|S_{0,0}\big(\partial N_r(0)\big)\big\| = r$.

Let H'' be the homotopy defined by $H''(h,t) = tRy_0 + \lambda S_{0,0}(h)$ for all $t \in [0,1]$. By the choice of y_0 and R, H'' has no fixed point on $\overline{N_r(0)}$. Then $\operatorname{ind}\big(N_r(0), \lambda S_{0,0}, C\big) = \operatorname{ind}\big(N_r(0), Ry_0 + \lambda S_{0,0}, C\big) = 0$.

This completes the proof.

THEOREM 1.9 (COMPRESSION AT THE ORIGIN). *Let $C \subset E$ be a cone. Let $F\colon C \longrightarrow C$ be a* u. s. c. *multivalued map with convex values such that $F(0) = 0$, F is a k-set contraction and differentiable at $0 \in C$. Let S_0 be a differential of F at 0 such that*

(1.9.1) $S_{0,0}(h)$ is convex for every h,

(1.9.2) the eigenvalues of $S_{0,0}$ belong to the interval $[0,1)$.

Then $\operatorname{ind}\big(N_r(0), F, C\big) = 1$ if r is small enough.

PROOF. As in the part (i) of (1.8), we can show that $\operatorname{ind}\big(N_r(0), F, C\big) = \operatorname{ind}\big(N_r(0), S_{0,0}, C\big)$ if r is small enough.

The homotopy H' defined by $H'(h,t) = tS_{0,0}(h)$, $t \in [0,1]$, has no fixed point on $\partial N_r(0)$. For, if there exists $t \in (0,1)$ and $h \in \partial N_r(0)$ such that $h \in tS_{0,0}(h)$, it follows that $\frac{1}{t}h \in S_{0,0}(h)$ and thus $\frac{1}{t} \in (0,1)$ by (1.9.2) which is a contradiction since $\frac{1}{t} > 1$. By the homotopy property, $\operatorname{ind}\big(N_r(0), S_{0,0}, C\big) = \operatorname{ind}\big(N_r(0), 0, C\big) = 1$.

LEMMA 1.10. *Under the same hypothesis of Theorem (1.8), the map G defined in 1.8(i) is a k-set contraction.*

PROOF. By definition of the map G, $G(h,t) \subset \operatorname{co}\big(F(h), S_{0,0}(h)\big)$ for every $h \in C$ and for all $t \in [0,1]$ and so $G(A \times [0,1]) \subset \operatorname{co}\big(F(A), S_{0,0}(A)\big)$ for every bounded subset A of E. It follows that $\gamma\big(G(A \times [0,1])\big) \le \max\Big(\gamma\big(F(A)\big), \gamma\big(S_{0,0}(A)\big)\Big)$ and thus G is a k-set contraction since F and $S_{0,0}$ are.

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THEOREM 1.11 (EXPANSION AT INFINITY). *Let $C \subset E$ be a cone. Let $F: C \longrightarrow C$ be a* u. s. c. *multivalued map with convex values such that $F(0) = 0$, F is a k-set contraction and differentiable at the infinity. Let S be a differential of F at infinity such that*

(1.11.1) $S(h)$ is convex for all h,

(1.11.2) $d\Big(0, \overline{S\big(\partial N_r(0)\big)}\Big) > \gamma(S)$, $r > 0$,

(1.11.3) the eigenvalues of S are strictly greater than 1.

Then $\operatorname{ind}\big(N_r(0), F, C\big) = 0$ *if r is big enough.*

PROOF. The proof is the same as in (1.8) but we must substitute $S_{0,0}$ by S and use Lemma 1.7 and Proposition 1.3.

THEOREM 1.12 (COMPRESSION AT INFINITY). *Let $C \subset E$ be a cone. Let $F: C \longrightarrow C$ be a* u. s. c. *multivalued map with convex values such that $F(0) = 0$, F is a k-set contraction and differentiable at the infinity. Let S be a differential of F at infinity such that*

(1.12.1) $S(h)$ is convex for all h,

(1.12.2) the eigenvalues of S belong to the interval $[0, 1)$.

Then $\operatorname{ind}\big(N_r(0), F, C\big) = 1$ if r is big enough.

PROOF. Similar to the proof of (1.9).

If we combined the results of the previous theorems, we obtain the following theorems.

THEOREM 1.13 (EXPANSION AT THE ORIGIN AND COMPRESSION AT INFINITY). *Let $C \subset E$ be a cone and let $F: C \longrightarrow C$ be a* u. s. c. *multivalued map with convex values such that $F(0) = 0$, F is a k-set contraction and differentiable at the origin 0 and infinity. Assume furthermore that conditions (1.8.1) to (1.8.3) are satisfied for $R_1 > 0$ and conditions (1.12.1)—(1.12.2) are satisfied for $R_2 > 0$. Let $U = \{x \in C \mid R_1 < \|x\| < R_2\}$. Then* $\operatorname{ind}(U, F, C) = 1$ *and thus F has a non-trivial fixed point in U.*

PROOF. By the additivity property, $\operatorname{ind}\big(N_{R_2}(0), F, C\big) = \operatorname{ind}\big(N_{R_1}(0), F, C\big) + \operatorname{ind}(U, F, C)$. It follows that $\operatorname{ind}(U, F, C) = 1 - 0 = 1$ by (1.8) and (1.12).

THEOREM 1.14 (COMPRESSION AT THE ORIGIN AND EXPANSION AT INFINITY). *Let $C \subset E$ be a cone and let $F: C \longrightarrow C$ be a* u. s. c. *multivalued map with convex values such that $F(0) = 0$, F is a k-set contraction and differentiable at the origin 0 and infinity. Assume furthermore that conditions (1.9.1)—(1.9.2) are satisfied for $R_1 > 0$ and conditions (1.11.1) to (1.11.3) are satisfied for $R_2 > 0$. Let $U = \{x \in C \mid R_1 < \|x\| < R_2\}$. Then* $\operatorname{ind}(U, F, C) = -1$*; and thus F has a non-trivial fixed point in U.*

PROOF. Similar to the proof of (1.13).

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