BIPOSITIVE AND ISOMETRIC ISOMORPHISMS OF SOME CONVOLUTION ALGEBRAS

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1. Introduction and summary. Throughout this paper the term "space" will mean "Hausdorff locally compact space" and the term "group" will mean "Hausdorff locally compact group." If G is a group and $1 \leqslant p < \infty$, $L^p(G)$ denotes the usual Lebesgue space formed relative to left Haar measure on G. It is well known that $L^1(G)$ is an algebra under convolution, and that the same is true of $L^p(G)$ whenever G is compact. We introduce also the space $C_c(G)$ of complex-valued continuous functions f on G for each of which the support (supp f), is compact. The "natural" topology of $C_c(G)$ is obtained by regarding $C_c(G)$ as the inductive limit of its subspaces

$$C_A(G) = \{f \in C_c(G) : \operatorname{supp} f \subset A\},$$

where A ranges over a base for the compact subsets of G, and where each $C_A(G)$ is topologized by using the supremum norm: $||f|| = \sup\{|f(x)| : x \in G\}$. (These remarks about $C_c(G)$ apply equally well if G is any space, not necessarily a group.) It is clear that $C_c(G)$ is again an algebra under convolution.

We suppose that G and G' are two groups and, letting A stand for L^1 or C_c, or for L^p if the groups are compact, we assume the existence of an isomorphism P between $A(G)$ and $A(G')$ considered as convolution algebras. It will be assumed that P is either bipositive (in the sense that $Pf \geqslant 0$ a.e. if and only if $f \geqslant 0$ a.e., f being a general element of A) or isometric; when $A = C_c$ the isometry is meant in reference to the supremum norm.

Kawada **(3)** showed that for $A = L^1$, the existence of a bipositive isomorphism P of $A(G)$ onto $A(G')$ entails that G and G' are isomorphic (as topological groups). A similar result for the isometric case was given by Wendel **(7**, Theorem 1**)**. The aim of this paper is to show that Kawada's result extends to L^p algebras over compact groups, and that both the Kawada and Wendel theorems have analogues for C_c algebras. I do not know, however, whether Wendel's theorem extends to L^p algebras over compact groups when $p > 1$. The main results are therefore expressed by the following two theorems.

THEOREM 1. *Suppose that G and G' are compact groups, that $1 \leqslant p < \infty$, and that there exists a bipositive isomorphism P of $L^p(G)$ onto $L^p(G')$. Then G and G' are isomorphic topological groups.*

THEOREM 2. *Suppose that G and G' are groups, and that there exists an algebraic*

Received July 20, 1964.

https://doi.org/10.4153/CJM-1965-082-8 Published online by Cambridge University Press

isomorphism P of $C_c(G)$ onto $C_c(G')$ which is either (a) *bipositive or* (b) *isometric. Then G and G' are isomorphic topological groups.*

The proofs are based on some results of intrinsic interest concerning multipliers of the types of algebra involved. These are discussed separately in §2, the proofs of the main theorems occupying §§3 and 4.

2. Auxiliary results about multipliers. By a (right) multiplier of $L^p(G)$, or of $C_c(G)$, is meant a continuous endomorphism T of $L^p(G)$, or of $C_c(G)$, which commutes with right translations. (Left multipliers are defined in an analogous fashion, but we shall have no need to consider these.) It is easy to verify that a continuous endomorphism T of $L^p(G)$, or of $C_c(G)$, is a multiplier if and only if it commutes with right convolutions, that is,

$$T(f * g) = Tf * g \tag{2.1}$$

for f, g in $L^p(G)$ or $C_c(G)$, as the case may be; if $p > 1$, we are here assuming that G is compact.

Wendel's isomorphism theorem mentioned in §1 depends in part upon the fact that when $p = 1$ each multiplier of $L^1(G)$ is represented by convolution on the left by some bounded Radon measure on G; see **(8)**. This is no longer true for $p > 1$ (see Remark (2) below) but, as we shall show, remains true for positive multipliers. We shall also show that a similar representation theorem is valid for multipliers of $C_c(G)$.

PROPOSITION 1. *Suppose that G is a compact group, that $1 \leqslant p < \infty$, and that T is a positive multiplier of $L^p(G)$. Then there exists a positive Radon measure μ on G such that*

$$Tf = \mu * f \tag{2.2}$$

for $f \in L^p(G)$.

Proof. We start from the relation (2.1). Let (N_i) be a base of compact neighbourhoods of the neutral element of G. For each i, choose a non-negative function $f_i \in L^p(G)$ vanishing outside N_i and such that $\int_G f_i \, dx = 1$. For use in the proof of Proposition 2 we remark that, whether or not G is compact, we may suppose that all the N_i are subsets of some compact subset of G, and that each f_i may be chosen from $C_c(G)$. Put $h_i = Tf_i$, so that h_i is a non-negative function in $L^p(G)$. Since $f_i * g \to f * g$ in $L^p(G)$, (2.1) shows that

$$Tg = \lim h_i * g \qquad \text{in } L^p(G). \tag{2.3}$$

At the same time,

$$||h_i * g||_p = ||T(f_i * g)||_p \leqslant ||T|| \cdot ||f_i * g||_p \leqslant ||T|| \cdot ||f_i||_1 \cdot ||g||_p \leqslant ||T|| \cdot ||g||_p.$$

Taking $g = 1$, this shows that the non-negative functions h_i satisfy the inequality

$$\int_G h_i \, dx \leqslant ||T||.$$

The net $(h_i\,dx)$ of positive Radon measures therefore has a vague limiting point μ, itself a positive Radon measure on G. If g is continuous, the net $(h_i * g)$ then has $\mu * g$ as a limiting point for the topology of uniform convergence, and comparison with (2.3) shows that $Tg = \mu * g$ for such g. Since both Tg and $\mu * g$ are continuous functions of g, (2.2) follows thence for all $g \in L^p(G)$.

Remarks. (1) Proposition 1 remains true for certain categories of non-compact groups G, and in particular for all Abelian G, but I do not know whether it is true for all G.

(2) When G is infinite, compact, and Abelian, there exist for $p > 1$ multipliers T of $L^p(G)$ which are not representable in the form (2.2) with μ a Radon measure on G. Indeed, let S be any infinite Sidon subset of the character group X of G (**5**, pp. 121–130), and let b be any bounded function on X which vanishes off S. If it is supposed first that $1 < p \leqslant 2$, the inequalities on (**5**, p. 130) show that there exists a multiplier T of $L^p(G)$ such that

$$(Tf)\hat{g}(\xi) = b(\xi)\hat{f}(\xi) \qquad (\xi \in X),$$

where $\hat{g}$ denotes the Fourier transform of g. By duality, this assertion extends to values of p satisfying $2 \leqslant p < \infty$.

This multiplier T is expressible in the form (2.2) if and only if $b = \hat{\mu}$. However, it is easily deducible from (**5**, Section 5.7.7) that $b = \hat{\mu}$ for some Radon measure μ on G if and only if

$$\textstyle\sum_{\xi\in X} |b(\xi)|^2 < \infty. \tag{2.4}$$

Since S is infinite, (2.4) can be denied, in which case the multiplier T fails to admit a representation (2.2) with μ a Radon measure on G.

We now turn to the consideration of multipliers of $C_c(G)$ and an analogue of Proposition 1 for them.

A positive multiplier T of $C_c(G)$ is said to be *minimal* if each positive multiplier T_0 of $C_c(G)$, for which $T - T_0$ is again positive, is a scalar multiple of T.

We shall denote by $M(G)$ the space of Radon measures on G, and by $M_c(G)$ the subspace formed of measures having compact supports.

PROPOSITION 2. (i) *To each multiplier T of $C_c(G)$ there corresponds a unique measure $\mu \in M_c(G)$ such that* (2.2) *holds for $f \in C_c(G)$; T is positive if and only if its representative measure μ is positive.*

(ii) *A positive multiplier T of $C_c(G)$ is minimal if and only if its representative measure μ is a multiple of a Dirac measure.*

Proof. The uniqueness of μ, the second statement of (i), and (ii) are all virtually evident, once the existence of a representative measure $\mu \in M_c(G)$ is established.

If e is the neutral element of G, the mapping $f \to Tf(e)$ is a continuous linear functional on $C_c(G)$, so that there exists a measure $\lambda \in M(G)$ such that

$$Tf(e) = \int_G f \, d\lambda \tag{2.5}$$

for $f \in C_c(G)$. Using the fact that T commutes with right translations, (2.5) leads easily to (2.2), provided μ is taken to be the measure for which

$$\int_G f \, d\mu = \int_G f(x^{-1}) \, d\lambda(x)$$

for all $f \in C_c(G)$. It remains to show that μ has a compact support.

By considering the family (f_i) introduced in the proof of Proposition 1, and noticing that μ is the vague limit of the measures $(\mu * f_i)dx$, it is seen that the desired conclusion will follow as soon as the following assertion is established:

$$\begin{cases} \text{To each compact subset } K \text{ of } G \text{ corresponds a compact subset } K^* \text{ of } G \\ \text{such that supp } Tf \subset K^* \text{ whenever } f \in C_c(G) \text{ and supp} f \subset K. \end{cases} \tag{2.6}$$

Now the truth of (2.6) is almost obvious if T is positive: one may then take $K^* = \text{supp } Tf_0$, where f_0 is any non-negative function in $C_c(G)$ which takes the value 1 at all points of K. The validity of (2.6) follows for any T, if G is σ-compact, from (**1**, Proposition 4), since in this case $C_c(G)$ is a strict inductive limit of a denumerable sequence of Banach spaces and, thanks to continuity of T, the set

$$\{Tf : f \in C_K(G), ||f|| \leqslant 1\}$$

is bounded in $C_c(G)$ and therefore contained in some subspace $C_{K^*}(G)$. The proof that (2.6) is valid in general will come from some lemmas. These are stated in terms more general than are needed here, partly because they are useful for discussing other problems concerning multipliers.

If Y is a space (see §1), we denote by $C(Y)$ the space of all continuous functions on Y, endowed with the topology of locally uniform convergence. The space Y is said to possess the *Nachbin–Shirota property* if, whenever S is a non-relatively compact subset of Y, there exists a function $g \in C(Y)$ which is unbounded on S. It was proved by Nachbin and Shirota (independently) that $C(Y)$ is barrelled if and only if Y possesses the Nachbin–Shirota property; see (**4**; **6**).

LEMMA 1. *If Y is a group, then Y possesses the Nachbin–Shirota property, so that $C(Y)$ is barrelled.*

Proof. Let S be a subset of Y which is not relatively compact. Let N and N' be symmetric compact neighbourhoods of the neutral element in Y such that $N'^2 \subset N$. One can then construct by recurrence a sequence (y_n) of points of S such that the sets $y_n N$ are disjoint. For each n one may choose a non-negatives function $g_n \in C_c(Y)$ such that $g_n(y_n) = n$ and supp $g_n = S_n \subset y_n N'$. The set $S_n N'$ are then disjoint, and it follows thence that the set

$$A = \cup \{S_n : n = 1, 2, \ldots\}$$

https://doi.org/10.4153/CJM-1965-082-8 Published online by Cambridge University Press

is closed in Y. Consider the series

$$\sum_{n=1}^{\infty} g_n. \tag{2.7}$$

It is easily verified that each point y_0 of Y has a neighbourhood throughout which all but at most one term of the series (2.7) vanish (consider separately the cases $y \in A$ and $y \notin A$). As a consequence, the sum-function g of the series (2.7) is continuous on Y. Evidently, $g(y_n) \geqslant g_n(y_n) = n$, so that g is unbounded on S.

Remark. A similar, slightly simpler, construction shows that any σ-compact space Y possesses the Nachbin–Shirota property. However, if Y is non-compact and such that every countable subset thereof is relatively compact (as is the case when Y is the well-known space of countable ordinals), then Y evidently does not possess the Nachbin–Shirota property.

LEMMA 2. *Let Y be a space possessing the Nachbin–Shirota property. Let F be a barrelled locally convex space and T a linear map of F into $C_c(Y)$ which is continuous for the vague topology $\sigma(M_c(Y), C_c(Y))$. Let B be any bounded subset of F. Then there exists a compact subset K of Y such that* supp $Tf \subset K$ *for each* $f \in B$.

Proof. Let g be any non-negative function in $C(Y)$. Since, by integration theory,

$$|Tf|(g) = \text{Sup}\{|Tf(ug)| : u \in C_c(Y), \|u\| \leqslant 1\},$$

where $|Tf|$ denotes the total variation of the measure Tf, the assumed continuity of T ensures that $f \to |Tf|(g)$ is a lower semicontinuous seminorm on F. Since F is barrelled, this seminorm is continuous on F. Consequently,

$$N(g) = \text{Sup}\{|Tf(g)| : f \in B\}$$

is finite for $g \in C(Y)$, and is a lower semicontinuous seminorm on $C(Y)$. By Lemma 1, $C(Y)$ is barrelled, so that N is continuous on $C(Y)$. This signifies that there exists a compact subset K of Y such that

$$N(g) \leqslant \text{const. Sup}\{|g(y)| : y \in K\},$$

whence it follows at once that supp $Tf \subset K$ whenever $f \in B$.

Remark. The continuity of T implies, of course, that $T(B)$ is vaguely bounded; but this in itself yields the stated conclusion only if Y is compact. Lemma 2 thus exhibits a property peculiar to images of bounded subsets of barrelled spaces.

By combining Lemmas 1 and 2 we obtain a result more than adequate to establish the truth of (2.6) and thus complete the proof of Proposition 2.

LEMMA 3. *Let X be a space and Y a group. Let T be a continuous linear map of $C_c(X)$ into $C_c(Y)$. Then* (2.6) *is true.*

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Proof. We inject $C_c(Y)$ into $M_c(Y)$ by identifying the function $f \in C_c(Y)$ with the measure $f\,dy$, where dy is a left Haar measure on Y, noting that $\operatorname{supp} f = \operatorname{supp}(f\,dy)$. Evidently, the natural topology on $C_c(Y)$ is stronger than that induced on it by the vague topology on $M_c(Y)$. Thus we may apply Lemma 2, taking for B the set of $f \in C_H(X)$ satisfying $\|f\| \leqslant 1$, H being any given compact subset of X, to conclude that, for a suitable compact subset K of Y, one has $\operatorname{supp} Tf \subset K$ whenever $f \subset B$. But then, clearly, $\operatorname{supp} Tf \subset K$ whenever $f \in C_c(X)$ and $\operatorname{supp} f \subset H$. This establishes (2.6), with K and K^* replaced by H and K respectively.

3. Proof of Theorem 1. Take any positive $u' \in L^1(G')$ and consider the endomorphism T of $L^p(G)$ defined by

$$Tf = P^{-1}(u' * Pf). \tag{3.1}$$

Since P is bipositive, T is positive. Also, since $P(f * g) = Pf * Pg$, therefore

$$T(f * g) = Tf * g.$$

On account of the continuity of T (itself a consequence of positivity), this last is equivalent to saying that T commutes with right translations. Thus T is a positive multiplier of $L^p(G)$. By Proposition 1, therefore, we conclude that to each $u' \in L^1(G)$ there corresponds a measure μ on G such that

$$\mu * f = P^{-1}(u' * Pf). \tag{3.2}$$

This μ is obviously unique when u' is given, so that one has a map $A : u' \to \mu$ of $L^1(G')$ into $M(G)$, the space of measures on G. Evidently, A is linear and positive. Furthermore, if $u' \in L^p(G)$, reference to (3.2) shows that $Au' = P^{-1}u'$, whence it appears that

$$A(u' * v') = Au' * Av' \tag{3.3}$$

for $u', v' \in L^p(G')$. Now A, being positive, is continuous, and so (3.3) must in fact hold for all $u', v' \in L^1(G')$. Again, since A maps $L^p(G')$ into $L^p(G')$, continuity shows that it maps $L^1(G')$ into $L^1(G)$ (and not merely into $M(G)$).

Thus A is a positive isomorphism of $L^1(G')$ into $L^1(G)$.

Inverting the roles of G and G' we see likewise that there exists a positive isomorphism B of $L^1(G)$ into $L^1(G')$ such that

$$Bu * f' = P(u * P^{-1}f') \tag{3.4}$$

for $u \in L^1(G)$ and $f' \in L^p(G')$. Comparing this with the defining property of A, that is, with

$$Au' * f = P^{-1}(u' * Pf), \tag{3.5}$$

holding for $u' \in L^1(G')$ and $f \in L^p(G)$, it is seen at once that AB and BA are the identity endomorphisms of $L^1(G')$ and of $L^1(G)$ respectively. Either of A or B thus realizes a bipositive isomorphism between these L^1-algebras, and Kawada's theorem ensures the isomorphy of G and G'.

Remark. Kawada's result in its sharpened form given by Wendel (7) shows that there exists an isomorphism (topological and algebraic) t of G onto G' such that $(Pf)(tx) = c.f\,(x)$, where $c \geqslant 0$ is a constant.

4. Proof of Theorem 2. Let P be an algebraic isomorphism of $C_c(G)$ onto $C_c(G')$ such that either (a) P is bipositive or (b) P is isometric. In either case, P is bicontinuous for the inductive limit topologies: this is almost evident in case (a); in case (b) it is deducible from Lemma 2 by an argument similar to that used in the proof of Lemma 3. For one sees in this way that P maps each subspace $C_K(G)$, where K is a compact subset of G, into a subspace $C_{K'}(G')$, where K' is some compact subset of G'. So, since P is continuous for the normed topologies, it follows that $P|C_K(G)$ is continuous for the normed topology on $C_K(G)$ and the inductive limit topology on $C_c(G')$. Hence P is continuous for the inductive limit topologies. A similar argument applies to P^{-1}.

By using Proposition 2(i) and arguments like those in §3, we obtain the existence of linear maps

$$A: M_c(G') \to M_c(G), \qquad B: M_c(G) \to M_c(G')$$

such that

$$P^{-1}(\mu' * Pf) = A\mu' * f \tag{4.1}$$

for $f \in C_c(G)$ and $\mu' \in M_c(G')$, and

$$P(\mu * P^{-1}f') = B\mu * f' \tag{4.2}$$

for $f' \in C_c(G')$ and $\mu \in M_c(G)$. The map A is continuous in the following sense: if (μ_i') is a net in $M_c(G')$ such that the μ_i' are supported by a fixed compact set and $\mu_i' \to \mu'$ vaguely, then $A\mu_i' \to A\mu'$ vaguely. This follows from (4.1) if one uses Ascoli's theorem and the isometric nature of P. The map B possesses an analogous continuity property. Moreover, (4.1) and (4.2) combine to show that A and B are mutually inverse.

If we inject C_c into M_c (see Lemma 3), (4.1) shows that

$$Af' = P^{-1}f' \tag{4.3}$$

for $f' \in C_c(G')$. Consequently, the relation

$$A(\lambda' * \mu') = A\lambda' * A\mu' \tag{4.4}$$

holds for λ', μ' in $C_c(G')$. But then the above continuity property of A shows that (4.4) holds for arbitrary λ' and μ' in $M_c(G')$. Similarly,

$$B(\lambda * \mu) = B\lambda * B\mu \tag{4.5}$$

for λ and μ in $M_c(G)$. Thus A and B are mutually inverse algebraic isomorphisms between $M_c(G)$ and $M_c(G')$. regarded as convolution algebras.

If (a) is true, it is evident that A and B are positive, and Proposition 2(ii) shows that

$$A\epsilon_{x'} = c'(x')\epsilon_{\alpha(x')} \tag{4.6}$$

https://doi.org/10.4153/CJM-1965-082-8 Published online by Cambridge University Press

and

$$B\epsilon_x = c(x)\epsilon_{\beta(x)}, \tag{4.7}$$

where ϵ_p denotes the Dirac measure at the point p, $c'(x') > 0$, $c(x) > 0$, and α and β are maps of G' into G and of G into G' respectively. Since A and B are mutually inverse, so too are α and β. Furthermore, the relation

$$\epsilon_{x'y'} = \epsilon_{x'} * \epsilon_{y'}$$

combines with (4.4) to show that α is an algebraic isomorphism. The same is true of β, thanks to (4.5). The continuity properties of A and B show that α and β are continuous. Thus α and β are mutually inverse isomorphisms between G and G' considered as topological groups.

Now consider further case (b). If on M_c we use the customary norm (total variation) defined on the set M_b of bounded Radon measures, the isometric nature of P shows that $||A|| \leqslant 1$ and $||B|| \leqslant 1$. It follows that each of A and B is in fact isometric. Each can be extended from M_c to M_b so as to remain isometric. Since (4.3) shows that A maps $C_c(G')$ into $C_c(G)$, it follows that this extension of A maps $L^1(G')$ into $L^1(G)$ (and not merely into $M_b(G)$). Similarly the extension of B maps $L^1(G)$ into $L^1(G')$. So the extensions of A and B define mutually inverse isometric isomorphisms between $L^1(G)$ and $L^1(G')$ and Wendel's theorem implies the isomorphy of G and G'. This completes the proof.

Remark. Having shown that the extension of A defines an isometric isomorphism of $M_b(G')$ onto $M_b(G)$, one could close the proof by appeal to a theorem proved recently by Johnson **(2)**, according to which the existence of such a correspondence between the M_b-algebras entails the isomorphy of the underlying groups.

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