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METRIC SPACES AND IDEALS OF FUNCTIONS

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0. Introduction

In each metric space (X,d) there is defined the space $\mathrm{Lip}\,X$ of complex-valued, bounded, and uniformly Lipschitzian functions. In the algebra $\mathrm{Lip}\,X$, it is natural to ask for ideals closed in various notions of convergence, and also to identify the invertible elements. In particular, are the invertible elements exactly those with no zero in X? Wiener's Tauberian Theorem in Fourier analysis is the first and most remarkable example of this harmonious state of affairs. A moment's reflection confirms that, for the algebra $\mathrm{Lip}\,X$, this is true only for compact metric spaces X, the trivial examples in our investigation. We therefore introduce a type of convergence weaker than convergence in norm; it has already proved useful in some problems in descriptive set theory and reflects in a subtle way the metric properties of X. A sequence (f_n) in $\mathrm{Lip}\,X$ *converges strongly* to g, written $s-\lim f_n=g$, if $\|f_n\|\leqq C$ in the Banach space $\mathrm{Lip}\,X$ and $\lim f_n(x)=g(x)$ for each element x of X. In Section 3 we explain how this is really a type of convergence in the dual space of a certain Banach space L_0^*. This brings us to the edge of some recondite questions about iterated (or even transfinite) limits, and we have adhered to the notion of strong limits to avoid these questions. To illustrate the differences between these two approaches, we mention this problem: which maximal ideals of $\mathrm{Lip}\,X$ are closed with respect to strong convergence of sequences? This is *not* the problem studied in Section 1.

In Section 2 we investigate a representation of $\mathrm{Lip}\,X$ as operators in the space l_1, and some applications to set theory. These applications explain our interest in Lipschitz spaces over Polish spaces.

1. A Tauberian property of Lip X

An element f of $\mathrm{Lip}\,X$ is called *total* if there is a sequence (g_n) in $\mathrm{Lip}\,X$ such that $s-\lim fg_n=1$. Two obvious problems are presented:

(1) Identify the total elements f of $\mathrm{Lip}\,X$.

(2) Characterize the metric spaces with the property that each element f of $\mathrm{Lip}\,X$, such that $|f|>0$ everywhere in X, is total. Borrowing from Fourier analysis, we say then that $\mathrm{Lip}\,X$ is *Tauberian*.

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Theorem 1. *For each element f of* $\operatorname{Lip} X$, *f is total if and only if: (t) there is a $\delta>0$ so that on each open ball $B(x,\delta)$ in X, $|f|$ has a positive infimum.*

Theorem 1 is obtained by solving a more general problem: for each f in $\operatorname{Lip} X$, we identify the set of strong limits $g=s-\lim h_m f$. This has no interest unless $\inf|f|=0$, so that each set $A_r \equiv (|f| \leqq r^{-1})$ is non-empty. We then define

$$u(x)=\lim_r d(x, A_r),$$

so that $0 \leqq u(x) \leqq +\infty$.

Theorem 1′. *An element g of* $\operatorname{Lip} X$ *is a strong limit $s-\lim h_m f$ if and only if $|g(x)| \leq Cu(x)$ for some $C<+\infty$.*

Proof. The necessity is easily verified. Indeed, suppose that $g_m = h_m f$ satisfies $|g_m(x)-g_m(y)| \leqq Cd(x,y)$ and $|h_m| \leqq a_m < +\infty$. When $y \in A_r$, then $|g_m(x)| \leqq |g_m(y)| + Cd(x,y) \leqq a_m r^{-1} + Cd(x,y)$. Therefore $|g_m(x)| \leqq a_m r^{-1} + Cd(x, A_r)$. Taking the limit as $r \to +\infty$, we get $|g_m(x)| \leq Cu(x)$, and so $|g(x)| = \lim |g_m(x)| \leqq Cu(x)$.

For the sufficiency we define a set B_r by the inequality $|g(x)| \leqq 2Cd(x, A_r)$, so that $B_1 \subseteq B_2 \subseteq \ldots$ and $\bigcup B_r = X$. We define g_m on $A_m \cup B_m$ by the formulas

$$g_m = g \quad \text{on} \quad B_m,\ g_m = 0 \quad \text{on} \quad A_m \quad \text{for} \quad m=1,2,3.$$

Now $|g_m| \leqq |g|$ everywhere and $|g_m(x)-g_m(y)| \leqq C'd(x,y)$ for all x, y in $A_r \cup B_r$. We can extend [7, p. 63] g_m to all of X so that the extension g_m^* satisfies $|g_m^*| \leqq \|g\|_\infty$ and $|g_m^*(x) - g_m^*(y)| \leqq 2C'd(x,y)$. Then $s-\lim g_m^* = g$, and for each m, $g_m^* = h_m f$ for a certain h_m in $\operatorname{Lip} X$; to define h_m we specify that $h_m = 0$ on $f^{-1}(0)$. This completes the proof of Theorem 1′.

To deduce Theorem 1, we apply Theorem 1′ with $g=1$. If $u(x) \geqq C^{-1}$, then for some $r \geqq 1$, the ball $B(x, 2^{-1}C^{-1})$ misses A_r, so that $|f| > r^{-1}$ on $B(x, 2^{-1}C^{-1})$. If $|f| \geqq r^{-1}$ on $B(x,\delta)$ then $u(x) \geqq d(x, A_{r+1}) \geqq \delta$.

As for problem (2), the nature of spaces X for which $\operatorname{Lip} X$ is Tauberian, the key idea is the *modulus of local compactness*: $\rho(x) \equiv \sup\{r \geqq 0 : \bar{B}(x,r) \text{ is compact}\}$. (Here $\bar{B}(x,r)$ is the closed ball of radius r, centred at x).

Theorem 2. $\operatorname{Lip} X$ *is Tauberian if and only if*

(C) each sequence (x_n) in X, such that $\rho(x_n) \to 0$, admits a convergent subsequence.

The initial (C) stands for "coercive" and follows a suggestion of Horacio Porta, who observed the analogy with the "Palais-Smale condition" in nonlinear analysis.

Proof. The sufficiency of (C) is a consequence of Theorem 1. Indeed, suppose that

$f \in \mathrm{Lip}\, X$, $|f|>0$ everywhere, but f fails (t). Then there exists a sequence (x_n) such that $|f|$ has infimum 0 on each ball $B(x_n, 1/n)$. Since $|f|>0$ on X, we have necessarily $\rho(x_n) \leqq n^{-1}$, and since $f \in \mathrm{Lip}\, X$, $f(x_n)=O(n^{-1})$. But then the sequence (x_n) has no convergent subsequence, contradicting (C).

The proof that (C) is necessary is somewhat more involved. We first observe that if $\mathrm{Lip}\, X$ is Tauberian, then X must be complete. Indeed, let $(x_n)_1^\infty$ be a Cauchy sequence in X and let $f(x) \equiv \inf\{n^{-1}+\arctan d(x, x_n), n \geqq 1\}$. Then $f \in \mathrm{Lip}\, X$ and f certainly fails condition (t) to be total. But $f>0$ everywhere in X unless the sequence (x_n) converges in X.

We suppose, therefore, that X is complete, but that (C) is violated. Then there is a sequence (x_n), such that $\rho(x_n) \to 0$ and $d(x_n, x_m) \geqq \delta > 0 (1 \leqq n < m)$. Passing to a subsequence, we can suppose that $\rho(x_n) \leqq (n+3)^{-1}\delta$ for $n \geqq 1$. Again using the completeness, we can find, for each n, a sequence $(x_{nk})_{k=1}^\infty$ such that $d(x_n, x_{nk}) \leq (n+2)^{-1}\delta$ for each k, but $d(x_{nk}, x_{nl}) \geqq \delta_n > 0$, for $l \neq k$. The double sequence, of all points x_{nk}, forms a set which is closed in X, without accumulation points. We now define

$$u(x) \equiv \inf\{k^{-1}+\arctan d(x, x_{nk}) \colon k \geqq 1, n \geqq 1\}.$$

In view of the discreteness mentioned above, $u>0$ everywhere, and clearly $u \in \mathrm{Lip}\, X$. Since $\lim_k u(x_{nk})=0$ for each n, and $d(x, x_{nk}) \leqq (n+2)^{-1}\delta, u$ fails condition (t), and the proof of Theorem 2 is complete. (The function u will be used later to illustrate another property of complete spaces without the coercive property.)

Example. Let $e_{m,n}$ be a double sequence of orthonormal vectors in a Hilbert space, and let $a_{mn}=m^{-1}e_{m1}+m^{-1}e_{m,n+1}$. Then X is the set $\{a_{m,n}, m \geqq 1, n \geqq 1\} \cup \{0\}$, and then $\rho(a_{m,n})=\sqrt{2}\, m^{-1}$, $\rho(0)=0$. The space X is coercive, but not locally compact.

Let $\sigma_s(f)$—the strong spectrum of f—be the set of complex numbers λ, such that $f-\lambda$ is not total. When X is coercive, or more generally, when the completion of X is coercive, the relation $\sigma_s(f+g) \subseteq \sigma_s(f)+\sigma_s(g)$ is always true, and (incidentally) $\sigma(f+g) \subseteq \sigma(f)+\sigma(g)$ is valid for the spectrum in a commutative Banach algebra; these assertions are proved by entirely different methods, but are analogous in spirit.

Theorem 3. *Let X be complete but not coercive. Then the relation $\sigma_s(f+g) \subseteq \sigma_s(f)+\sigma_s(f)$ fails for a certain pair of elements of* $\mathrm{Lip}\, X$.

Proof. This uses the example u constructed in the proof of Theorem 2. Let v be any real function in $\mathrm{Lip}\, X$ such that $v(x_{nk})=c_n$ $(n \geqq 1, k \geqq 1)$, with a sequence of distinct real numbers; and finally, let $f=u+iv$, $g=u-iv$. We saw that u is not total, so that 0 belongs to $\sigma_s(f+g)$. We shall prove that no purely imaginary number $i\mu$ can belong to $\sigma_s(f)$. Indeed, let B be a subset of X, of diameter $<2^{-6}\delta$, such that $i\mu$ is in the closure of $f(B)$. (We suppose $\delta<1$). Since $u(x) \geqq \inf \arctan d(x, x_{nk})$, there must be an x in B, and some x_{nk}, such that $\tan d(x, x_{nk})<2^{-6}\delta$.The first index n is therefore determined by the set B, and $\mu=c_n$. Once n is determined, it is clear that the diameter of B is at least the smaller of δ and δ_n. Therefore $\sigma_s(f)$ and $\sigma_s(g)$ are contained in the interior of the right half-plane, whence $0 \notin \phi_s(f)+\sigma_s(g)$.

2. A special representation of Lip X

A representation of Lip X means a homomorphism of Lip X into the algebra of bounded operators in some Banach space E. We write the operation as $f \cdot y$ ($f \in \text{Lip}\, X$, $y \in E$) and write $\sigma_e(f)$ for the point spectrum of the bounded linear operator $y \mapsto f \cdot y$. (No confusion is caused by these simplifications).

Theorem 4. *Let X be a complete, separable metric space. Then there is a representation of* Lip X *by operators in the space l^1, such that*

$$\sigma_e(f) = f(X) \text{ for each } f \text{ in } \text{Lip}\, X.$$

Similar results are obtained in [2], [3], [4]. In [2], the same result is obtained, with a rather elusive space depending on X. In [4] the space c_0 occurs, but the algebras are different and σ_e is determined more subtly. In [3], there is a representation of Lip X, X being a certain compact metric space (homeomorphic to an interval) which leads to spectra σ_e that are not Borel sets.

In proving Theorem 4, we use the notion of a *strongly continuous representation*: for each sequence (f_n) in Lip X, such that $s - \lim f_n = 0$, we have $\lim f_n \cdot y = 0$ for every y in E.

Lemma 1. *For every strongly continuous representation of* Lip X, $\sigma_e(f) \subseteq f(X)$.

This is a lemma of [2]; we emphasize that X is a Polish space.

We proceed to construct a *tree* T whose elements are certain finite sequences $(x_1, x_2, \ldots, x_k)$ in X. This is the main improvement over [2] and allows us to construct representations of Lip X in l^1. Let X_0 be a countable dense subset of X, and let T consist of all sequences $(x_1, \ldots, x_k)$ from X_0, such that (for $k \geqq 2$), $d(x_1, x_2) \leqq 2^{-2}$, $d(x_2, x_3) \leqq 2^{-3}, \ldots$. We form a vector space V of countable dimension, attaching to each element $(x_1, \ldots, x_k)$ of T a vector $[x_1, \ldots, x_k]$, and making a basis of these elements. We define the operation of Lip X over V as follows:

$$f \cdot [x_1] = f(x_1) \cdot [x_1] \text{ for each } x_1 \text{ in } X_0,$$

$$f \cdot [x_1, \ldots, x_k] = f(x_k) \cdot [x_1, \ldots, x_k] + (f(x_k) - f(x_{k-1})) \cdot ([x_1] + \cdots + [x_1, \ldots, x_{k-1}]).$$

The basis elements are now turned into a basis for l^1, with a variant of the usual norm: $\|[x_1]\| = 1$, $\|[x_1, x_2]\| = 1/2, \ldots, \|[x_1, \ldots, x_k]\| = 2^{1-k}$. Therefore $\|f \cdot [x_1, \ldots, x_k]\| \leqq 2^{1-k}\|f\|_\infty + 2|f(x_k) - f(x_{k-1})|$. Since $d(x_{k-1}, x_k) \leqq 2^{-k}$, the representation of Lip X in V can be extended by continuity to all of l^1, and is easily seen to be strongly continuous. By Lemma 1, $\sigma_e(f) \subseteq f(X)$. To obtain the reverse inclusion let x be any element of X, and let x_k in X_0 be chosen so that $d(x_k, x) \leqq 2^{-k-2}$. Therefore $d(x_{k-1}, x_k) \leqq 2^{-k}$ and the sequence

$$y = [x_1] + [x_1, x_2] + [x_1, x_2, x_3] + \ldots$$

converges absolutely to a limit different from 0, such that $f\cdot y=f(x)\cdot y$ for every f in $\operatorname{Lip} X$. Therefore $\sigma_e(f)\supseteq f(X)$ and Theorem 4 is proved.

Let $\mathcal{N}$ be the space of sequences of sequences of positive integers $\mathbf{n}=(n_1,n_2,\dots,n_k,\dots)$ with $d(\mathbf{n},\mathbf{n}')=2^{-k}$, k being the least integer at which $n_k\neq n'_k$. For each Polish space X of finite diameter, there is a Lipschitz mapping of $\mathcal{N}$ onto X.

In the following two corollaries σ_e refers to the representation in Theorem 4.

Corollary 1. *For each bounded analytic (or Souslin) set S in the plane, there is an element f of* $\operatorname{Lip}\mathcal{N}$, *such that* $\sigma_e(f)=f(\mathcal{N})=S$.

Proof. It is a classical fact that $S=g(\mathcal{N})$, g being a continuous map of $\mathcal{N}$ onto S. Denoting by Γ the graph of g, a closed bounded subset of $\mathcal{N}\times R^2$, we have a Lipschitz mapping f_1 of $\mathcal{N}$ onto Γ; to obtain f we follow this with the projection of Γ into $\mathbb{R}^2$. (We learned of this construction after writing [**2**]).

The next result requires no new ideas about operators but is more striking from the viewpoint of set theory. When E is a closed subspace of l^1, invariant under the representation, we write $\sigma_e(f|E)$ for the point spectrum of f, as an operator in E.

Corollary 2. *Let S be a bounded analytic set in the plane. Then there is an element g of* $\operatorname{Lip}\mathcal{N}$ *such that*

(i) $g(\mathcal{N})=S$

(ii) *for each analytic set $S_1\subseteq S$, there is an invariant subspace E, such that* $\sigma_e(g|E)=S_1$.

The first tool in the proof of Corollary 2 is the notion of a *universal analytic* set $\sum$: this means (for the moment) an analytic set $\sum$ in $\mathbb{R}^{-2}\times[0,1]$ such that the various sections $\sum_t$ exhaust all the analytic sets in S. Let ϕ be a Lipschitz mapping of $\mathcal{N}$ onto $\sum$, written in the form $(u(x)+iv(x),w(x))$, and let $g=u+iv$. Defining $X_t=\{t\in X\colon w(x)=t\}$ we obtain $g(X_t)=\sum_t$, with the advantage that X_t is closed. For the existence of the set $\sum$, see [**5**] and [**6**, pp. 252–255].

We shall define a closed subspace E_t, such that $\sigma_e(f|E_t)\equiv f(X_t)$ identically, and in particular $\sigma_e(g|E_t)=\sum_t$. The sets $\sum_t$ exhaust all the analytic sets in S, and the corollary follows from this. Now E_t is the subspace of elements y such that $F\cdot y=0$ for every F that vanishes on X_t. Clearly E_t is a closed, invariant subspace of l^1. Suppose that $f\cdot y=0$, with y in E_t and $y\neq 0$. Let $F\geqq 0$ be a function in $\operatorname{Lip} X$ whose zero-set is exactly X_t, for example $\arctan d(x,X_t)$. Then $(|f|^2+F)\cdot y=0$, whence $|f|^2+F$ must have a zero in X, and therefore a zero in X_t. This shows that $\sigma_e(f|E_t)\subseteq f(X_t)$. We saw before, in the proof of Theorem 4, that to each x in X, there is an element $y\neq 0$ in l^1, such that $f\cdot y=f(x)y$ for all f in $\operatorname{Lip} X$. Thus $\sigma(f|E_t)\supseteq f(X_t)$, and finally $\sigma_e(f|E_t)\equiv f(X_t)$. As a special case we obtain $\sigma_e(g|E_t)=g(X_t)=\sum_t$.

3. Conclusion

In conclusion we explain how the Tauberian concept is related to the Krein–Smulian Theorem ([**1**, V5]). Until now the norm in $\operatorname{Lip} X$ has not been specified precisely; we define it as the larger of the quantities $\|f\|_\infty$ and $\sup\{|f(x)-f(y)|/d(x,y)\colon x\neq y\}$. Let $\varepsilon(x)$ be the evaluation at x, construed as an element of the dual space L^* of $\operatorname{Lip} X$. Then (by definition) $\|\varepsilon(x)\| \leqq 1$ for every x and $\|\varepsilon(x)-\varepsilon(y)\| \leqq d(x, y)$. Denoting by L_0^* the closed linear span of the elements $\varepsilon(x)$ in L^*, we see that L_0^* determines the norm in $\operatorname{Lip} X$ exactly, and in fact $\operatorname{Lip} X$ is isometrically the dual of L_0^*. Moreover, X is separable exactly when L_0^* is separable, and in that case, the s-convergent sequences in $\operatorname{Lip} X$ are exactly the w^*-convergent sequences, i.e. L_0^*-convergent sequences. By the Krein–Smulian Theorem it follows that a convex set in $\operatorname{Lip} X$ is closed under the operation of s-limits precisely when it is w^*-closed. This makes it a natural question, to discover when the set of strong limit points of any ideal I of $\operatorname{Lip} X$ coincides with the largest ideal with the same zero-set as I. We do not answer this in great detail, but merely mention a part of the answer with interesting connections to metric spaces.

Let F be a closed subset of X and $u(x)=\arctan d(x, F)$. We define two properties of X, *modulo* F.

X is *coercive* mod F if each sequence $x_m \in X \backslash F$, such that $\rho(x_m) \leqq m^{-1} u(x_m)$ $(m=1,2,3,\ldots)$ contains a subsequence converging to a point in $X \backslash F$.

$\operatorname{Lip} X$ is *Tauberian* mod F if for each f in $\operatorname{Lip} X$, such that $|f|>0$ on $X \backslash F$, and each g in $\operatorname{Lip} X$, such that $g=0$ on F, there is a sequence h_m in $\operatorname{Lip} X$ such that $g=s-\lim h_m f$.

The equivalence of these two concepts (with no assumption of separability) is proved by an elaboration of the arguments in Section 1.

Example. We modify the example in Section 1, defining $b_{m,n}=m^{-1}e_{m1}+m^{-2}e_{m,n+1}$, $X=\{b_{m,n}, m \geqq 1, n \geqq 1\} \cup \{0\}$. Then X is coercive, but not coercive modulo $F=\{0\}$.

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