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A generalized Banach-Mazur theorem

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For every infinite cardinal a we let C_a be the set of all real-valued continuous functions on a product of a closed unit intervals with the supmetric. It is shown that C_a has separability degree a . Further, the classical theorem of Banach and Mazur is generalized by showing that every metric space of separability degree a is isometric to a subspace of C_a .

The classical theorem of Banach and Mazur states that the space C_1 of all real-valued continuous functions defined on the closed unit interval $I = [0 , 1]$ is a universal separable metric space; i.e., every separable metric space is isometric with some subset of C_1 . In this paper we shall obtain universal metric spaces for metric spaces which are not separable; in particular, we shall obtain a family of spaces which are generalizations of C_1 and are universal for metric spaces with fixed separability degrees.

By a direct generalization of the proof of the fact that the space C_1 is separable (see [2], p. 158) one can see that the space C_n of all real-valued continuous functions on the product of n closed unit intervals with the supmetric is separable. Let a be an infinite cardinal number and let C_a be the set of all real-valued continuous functions on the product of a closed unit intervals with the supmetric.

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THEOREM A *$C_{\mathfrak{a}}$ has separability degree* $\mathfrak{a}$.

Proof Let A be a set of cardinality $\mathfrak{a}$ and for each finite subset F of A let $I_F = \{t \in I^{\mathfrak{a}} : \pi_i(t) = 0$ for all $i \in A - F\}$. Thus I_F can be identified with the product of n closed unit intervals where n is the cardinality of F . By our above remarks, we may let D_n be a countable dense subset of C_n . If $g \in D_n$, then g can be extended to a continuous function g^* on $I^{\mathfrak{a}}$ by setting $g^* = g \circ \pi_F$ where π_F is the projection onto I_F defined by $\pi_F(t) = t^*$ where $\pi_i(t^*) = 0$ for $i \in A - F$ and $\pi_i(t^*) = \pi_i(t)$ for $i \in F$. Let G denote the set of all g^* so obtained. Now there are $\mathfrak{a}$ finite subsets of A and for each of these finite subsets F the corresponding D_n is countable. Thus the set G has cardinality $\mathfrak{a}.\aleph_0 = \mathfrak{a}$. We shall show that G is dense in $C_{\mathfrak{a}}$. Let $f \in C_{\mathfrak{a}}$ and ε be a given positive number. Since $I^{\mathfrak{a}}$ is a compact uniform space with the product uniformity, f is uniformly continuous. Therefore there exists a finite subset F of A and a positive real number δ with the property that if $s , t \in I^{\mathfrak{a}}$ are such that $| \pi_i(s) - \pi_i(t) | < \delta$ for $i \in F$ then $| f(s) - f(t) | < \varepsilon/2$. Now let $t \in I^{\mathfrak{a}}$ and let $t^* = \pi_F(t)$. Then $| f(t) - f(t^*) | < \varepsilon/2$. But $t^* \in I_F$ and $f|I_F \in C_n$ where $n = \overline{\overline{F}}$. Therefore there exists an element $g \in D_n$ such that $| f(t^*) - g(t^*) | < \varepsilon/2$ and so $| f(t) - f^*(t) | + | f(t^*) - g(t^*) | < 2(\varepsilon/2) = \varepsilon$. But we have $| f(t) - g^*(t) | = | f(t) - g(t^*) | < \varepsilon$ which shows that $\sup | f - g^* | \leqq \varepsilon$.

We may now show that the space $C_{\mathfrak{a}}$ is universal for metric spaces of separability degree $\mathfrak{a}$.

THEOREM B *Every metric space of separability degree* $\mathfrak{a}$ *is isometric to a subspace of* $C_{\mathfrak{a}}$.

Proof Let (X , d) be a metric space of separability degree $\mathfrak{a}$ and let P be a dense subset of X with $\overline{\overline{P}} = \mathfrak{a}$. Since $\mathfrak{a} = \aleph_0.\mathfrak{a}$ we may let

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A be a set of cardinality $\mathfrak{a}$ and denote the elements of P by p_i^n with $i \in A$ and n a positive integer. If X is not discrete we may choose a fixed point $q \in X - P$ and let $w_i^n(p_j^k) = d(p_i^n, p_j^k) - d(p_j^k, q)$. Using the triangle inequality we obtain $|w_i^n(p_j^k)| \leqq d(p_i^n, q)$ so that $0 \leqq \left[1 + w_i^n(p_j^k) / d(p_i^n, q)\right]/2 \leqq 1$. If in the middle expression in this last inequality we fix i, j, and k and let n vary over the positive integers, we have a sequence in I. It can be shown (see [2], p. 150) that there exists a sequence of continuous functions $v_n : I \to I$ such that if $\{c_n\}$ is a sequence in I then there exists a point $t \in I$ such that $v_n(t) = c_n$ for every n. Therefore we can say that there exists a point $t_{i,j}^k \in I$ such that $v_n(t_{i,j}^k) = \left[1 + w_i^n(p_j^k)/d(p_i^n, q)\right]/2$. Solving this equation we have $w_i^n(p_j^k) = d(p_i^n, q)\left[2v_n(t_{i,j}^k) - 1\right]$. Let T be the set of all $t_{i,j}^k$. We then define for every n a real-valued function f^n on T such that $f^n(t_{i,j}^k) = w_i^n(p_j^k)$. Since v_n is continuous on I, f^n can be extended to the closure of T and then extended linearly to the entire interval I; we denote this function again by f^n. We define $f_i^n = f^n \circ \pi_i$ where π_i is the i-th projection from $I^{\mathfrak{a}}$. We shall now show that $\sup |f_i^n - f_j^k| = d(p_i^n, p_j^k)$ for each $i, j \in A$ and positive integers n, k. Let $t_r^m \in I^{\mathfrak{a}}$ be such that $\pi_h(t_r^m) = t_{h,r}^m$ for every $h \in A$. Then we have

$$\begin{aligned} f_i^n(t_r^m) - f_j^k(t_r^m) &= f^n \circ \pi_i(t_r^m) - f^k \circ \pi_j(t_r^m) \\ &= f^n(t_{i,r}^m) - f^k(t_{j,r}^m) \\ &= w_i^n(p_r^m) - w_j^k(p_r^m) \\ &= d(p_i^n, p_r^m) - d(p_j^k, p_r^m) \\ &\leqq d(p_i^n, p_j^k) . \end{aligned}$$

On the other hand

$$\begin{aligned} f_i^n(t_j^k) - f_j^k(t_j^k) &= f^n(t_{i,j}^k) - f^k(t_{j,j}^k) \\ &= w_i^n(p_j^k) - w_j^k(p_j^k) \\ &= d(p_i^n , p_j^k) . \end{aligned}$$

Finally, if $t \in I^a$ is not of the form t_r^m , then because of the linearity of f^n and f^k we also have $| f_i^n(t) - f_j^k(t) | \leq d(p_i^n , p_j^k)$. Now let u be the mapping such that $u(p_i^n) = f_i^n$. Then u is an isometry from P to the set of all the f_i^n . We wish to extend u from P to X . If $x \in X$, there exists a net $\{p_\delta : \delta \in \Delta\}$ in P converging to x . Since $\{p_\delta\}$ converges, it is Cauchy. Therefore $\{u(p_\delta)\}$ is Cauchy since u is an isometry on P . But $u(p_\delta)$ is always uniformly continuous so $\{u(p_\delta)\}$ converges to a continuous function $f_x \in C_a$. It is easy to see that f_x does not depend on the choice of net converging to x . Finally, $\sup | f_x - f_y | = d(x , y)$ for if $\{p_\delta\} \to x$ and $\{p_\lambda\} \to y$ then $\{\sup | u(p_\delta) - u(p_\lambda) | \} = \{d(p_\delta , p_\lambda)\}$ converges to $\sup | f_x - f_y |$. Thus the isometry u can be extended from P to X .

If X is a separable metric space, then A in the above proof can be taken to be a singleton. This would yield as a corollary the classical theorem of Banach and Mazur (see [1], p.187).

References

[1] Stefan Banach, *Théorie des opérations linéaires,* (Monografie Matematyczne, Warszawa, 1932).

[2] W. Sierpinski, *General Topology,* (University of Toronto Press, Toronto, 1952).

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