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Can a Fibonacci group be a unique products group?

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We show that a certain class of Fibonacci groups can not be right ordered. A question remaining is:

Are the torsion-free members of this class unique products groups?

The Fibonacci groups in which we are interested are the groups

$$F(2, n) = \langle x_1, x_2, \ldots, x_{n+2} : x_i x_{i+1} = x_{i+2},\ i = 1, 2, \ldots, n,\ x_{n+1} = x_1,\ x_{n+2} = x_2 \rangle$$

(see Brunner [2] and Johnson [3]).

The pair $(G, \leq)$ is a *right ordered group* if G is a group, $\leq$ is a linear order on the set G, and for all $a, b, c \in G$, $a \leq b$ implies $ac \leq bc$. A group G is called a *right orderable group* if there is a linear order $\leq$ on the set G such that $(G, \leq)$ is a right ordered group.

The group G is a *unique products group* if for all $a_1, a_2, \ldots, a_m$, $b_1, b_2, \ldots, b_n \in G$, there exist a_i and b_j $(1 \leq i \leq m$ and $1 \leq j \leq n$) such that

$$a_i b_j \notin \{a_r b_s : 1 \leq r \leq m,\ 1 \leq s \leq n, \text{ and } (i, j) \neq (r, s)\} .$$

We denote by $\underline{\underline{R}}$ and $\underline{\underline{U}}$ the classes of right orderable groups and unique products groups respectively. It is known that $\underline{\underline{R}} \subseteq \underline{\underline{U}}$ (Botto Mura and Rhemtulla [1]) while I know of no group in $\underline{\underline{U}} \setminus \underline{\underline{R}}$. (Also recall that

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the question "Is a torsion-free group a unique products group?" is still open.)

In this note we show (see the theorem below) that $F(2, n) \notin \underline{\underline{R}}$ for all integers $n \geq 3$. However, it is possible that $\underline{\underline{U}}$ contains some of these Fibonacci groups. One contender is $F(2, 6)$ which is isomorphic to the torsion-free group $\langle x, y : x^{-1}y^2x = y^{-2}, y^{-1}x^2y = x^{-2}\rangle$ (Brunner [2], Passman [4]). So we leave the following question for the reader:

Is $F(2, 6)$ a unique products group?

It is worth noting that the integral group ring $\mathbb{Z}(F(2, 6))$ has no zero divisors (Passman [4]). So if the answer to our question is "no" then $F(2, 6)$ provides a negative answer to the question:

If $\mathbb{Z}(G)$ has no zero divisors, is G a unique products group?

We conclude with a theorem which, probably, can be generalized to the groups $F(r, n)$ (Johnson [3]). Observe that $n \geq 3$ is a necessary restriction on n since $F(2, 0)$ is the free group on two generators and both $F(2, 1)$ and $F(2, 2)$ are the trivial group.

THEOREM. *For all integers* $n \geq 3$, $F(2, n)$ *is not a right orderable group.*

Proof. Suppose $n \geq 3$ is an integer. We rewrite the presentation for $F(2, n)$ as

$$\langle x_1, x_2, \ldots, x_n : x_1x_2 = x_3, \ldots, x_{n-1}x_n = x_1, x_nx_1 = x_2\rangle .$$

Assume $\leq$ is a right order for $F(2, n)$. Without loss of generality $x_1 > 1$ (where 1 is the identity of $F(2, n)$) and x_1 is the generator with greatest magnitude. That is $x_1 \geq x_i$ and $x_1 \geq x_i^{-1}$ for all $i = 2, 3, \ldots, n$. So we have

$$\begin{aligned} x_3 \leq x_1 &\Rightarrow x_1x_2 \leq x_1 \\ &\Rightarrow x_1x_nx_1 \leq x_1 \\ &\Rightarrow x_1x_n \leq 1 , \end{aligned}$$

while

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$$x_{n-1} \le x_1 \Rightarrow x_{n-1}x_n \le x_1x_n$$
$$\Rightarrow x_1 \le x_1x_n$$
$$\Rightarrow 1 < x_1x_n$$

a contradiction.

So no $F(2, n)$, $n \ge 3$, can be right ordered.

References

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