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THE GRADED CENTER OF A TRIANGULATED CATEGORY

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To the memory of a wonderful friend, Laci Kovács

Abstract

With applications in mind to the representations and cohomology of block algebras, we examine elements of the graded center of a triangulated category when the category has a Serre functor. These are natural transformations from the identity functor to powers of the shift functor that commute with the shift functor. We show that such natural transformations that have support in a single shift orbit of indecomposable objects are necessarily of a kind previously constructed by Linckelmann. Under further conditions, when the support is contained in only finitely many shift orbits, sums of transformations of this special kind account for all possibilities. Allowing infinitely many shift orbits in the support, we construct elements of the graded center of the stable module category of a tame group algebra of a kind that cannot occur with wild block algebras. We use functorial methods extensively in the proof, developing some of this theory in the context of triangulated categories.

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1. Introduction

The graded center of a triangulated category C is the set of natural transformations $\mathrm{Id}_C \to \Sigma^n$, $n \in \mathbb{Z}$, that commute with the shift functor Σ up to a sign $(-1)^n$. In [12], Linckelmann investigated the graded center of a block algebra of a finite group. His main result showed that the graded center of the derived category of a block is, modulo a nilpotent ideal, noetherian over the cohomology ring of the block. Along the way, Linckelmann showed that there is a large ideal of nilpotent elements in the graded center generated by elements in degree minus one that are supported on only a single Σ-orbit of modules. This result was extended by Linckelmann and Stancu to obtain elements in all degrees each of which is supported on only a single module that is periodic of period one.

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Ultimately we would like to be able to characterize the nilpotent elements in the graded center. In that direction, a natural question to ask is whether there can be elements of the graded center that are nontrivially supported on more than one Σ-orbit. By this we mean: 'Do there exist elements in the graded center that vanish on all but a finite number of Σ-orbits, but which have a nonzero composition with some nonisomorphism?'. A main purpose of this paper is to show that the answer to that question is generally negative.

For the most part, we work in a Hom-finite, Krull–Schmidt, k-linear triangulated category C that is Calabi–Yau. For $F : C \to C$ an endofunctor, we define the support of a natural transformation $\alpha : \mathrm{Id}_C \to F$ to be the set of isomorphism classes of objects U with $\alpha_U : U \to F(U)$ not zero. We prove that the support of such an α is a single object if and only if F is the Serre functor and, for some object M, α_M is an almost vanishing morphism. Thus such natural transformations have the same form as those constructed by Linckelmann. We show that, under some reasonable assumptions on the Auslander–Reiten quiver, an element of the graded center that is supported nontrivially on more than one shift orbit in a component of the Auslander–Reiten quiver has the entire component in its support. Moreover, if C is the stable category of a group algebra of a p-group of wild representation type, then such an element ψ of the graded center has the property that there exists a map $\gamma : U \to M$ of indecomposable modules such that $\psi_M\gamma \neq 0$ and γ is not a composition of a finite number of irreducible morphisms. We show, by the example of a finite group with dihedral Sylow 2-subgroup, that this requirement does not hold if the group has tame representation type.

Throughout the paper we assume that k is an algebraically closed field and that C is a Hom-finite, Krull–Schmidt, k-linear triangulated category with shift Σ. For background on Auslander–Reiten theory, we refer to standard texts such as [1].

2. Linear functors on a triangulated category

We present some preliminaries on the category of linear functors defined on a triangulated category. Let $\mathrm{Fun}^{\mathrm{op}}\, C$ denote the category of contravariant k-linear functors from C to k-vector spaces. The first four results of this section are well known. While they are usually stated for functors on module categories, they hold for k-linear functors on k-linear categories (and even for additive functors on additive categories). In particular they hold for $\mathrm{Fun}^{\mathrm{op}}\, C$, ignoring the triangulated structure of C. There are proofs of these results in [1] stated in terms of functors on the module category of a ring, but the arguments there work for functors on an additive category without change. Our purpose is to point out that these results hold in the generality we consider here.

Recall our assumption that C is a Hom-finite, Krull–Schmidt, k-linear triangulated category with shift Σ.

Proposition 2.1. *The category* $\mathrm{Fun}^{\mathrm{op}}\, C$ *is an abelian category. Moreover, for each indecomposable object* M *the representable functor* $\mathrm{Hom}_C(-, M)$ *is indecomposable and projective, with endomorphism ring isomorphic to* $\mathrm{End}_C(M)$.

Proof. See [1, IV.6.2(a), IV.6.4(a) and A.2.9]. It is standard that $\mathrm{Fun}^{\mathrm{op}} C$ is an abelian category in which kernels, cokernels and exactness are determined by evaluation at the objects of C. The statements about representable functors are a consequence of the linear form of Yoneda's lemma, which in our usage says that any morphism from $\mathrm{Hom}_C(-, M)$ to $\mathrm{Hom}_C(-, N)$ is induced from a morphism from M to N. □

The simple functors $s^M \in \mathrm{Fun}^{\mathrm{op}} C$ are defined in [1, IV.6.7]. For each indecomposable object M of C,

$$s^M = \mathrm{Hom}_C(-, M)/\mathrm{Rad}_C(-, M),$$

where $\mathrm{Rad}_C(-, M)$ is the radical of $\mathrm{Hom}_C(-, M)$, the subfunctor whose value at an indecomposable object X is the set of nonisomorphisms from X to M. These functors have the description

$$s^M(N) = \begin{cases} k & \text{if } M \cong N, \\ 0 & \text{otherwise.} \end{cases}$$

The next result is also well known in the context of functors on module categories, and the proof given in the reference carries through *verbatim*.

Proposition 2.2. *The simple objects in* $\mathrm{Fun}^{\mathrm{op}} C$ *are all of the form* s^M *for some indecomposable object* M *in* C*. The relation* $M \leftrightarrow s^M$ *gives a one-to-one correspondence between isomorphism types of indecomposable objects in* C *and isomorphism types of simple objects in* $\mathrm{Fun}^{\mathrm{op}} C$.

In addition, the quotient map $\mathrm{Hom}_C(-, M) \to s^M$ *is a projective cover, having kernel* $\mathrm{Rad}_C(-, M)$*, which is the unique maximal subfunctor of* $\mathrm{Hom}_C(-, M)$.

Proof. See [1, IV.6.8]. □

We say that a functor F is finitely generated if there is an epimorphism $\mathrm{Hom}_C(-, M) \to F$ for some object M. We say that F is finitely presented if there is an exact sequence of functors $\mathrm{Hom}_C(-, M_1) \to \mathrm{Hom}_C(-, M_0) \to F \to 0$.

The next result is, again, usually only stated for functors on module categories, but it is also true for functors on additive or k-linear categories.

Proposition 2.3. *The representable functors* $\mathrm{Hom}_C(-, M)$*, where* M *is indecomposable, are a complete list of the indecomposable finitely generated projective functors in* $\mathrm{Fun}^{\mathrm{op}} C$.

Proof. See [1, IV.6.5]. □

We say that s^M is a *composition factor* of a functor F if there are subfunctors $F_0 \subset F_1$ of F so that $F_1/F_0 \cong s^M$.

Corollary 2.4. *Let* F *be a functor in* $\mathrm{Fun}^{\mathrm{op}} C$*. Then* F *has* s^M *as a composition factor if and only if* $F(M) \neq 0$*. Furthermore,* F *has finite composition length if and only if* F *is nonzero on only finitely many isomorphism classes of indecomposable objects of* C*, where its value is finite dimensional.*

PROOF. If F has s^M as a composition factor, then, since $s^M(M) \neq 0$, we must have $F(M) \neq 0$. Conversely, if $F(M) \neq 0$, then, by Yoneda's lemma, there is a nonzero morphism $\mathrm{Hom}_C(-, M) \to F$ showing that the unique simple quotient s^M of $\mathrm{Hom}_C(-, M)$ appears as a composition factor of F. The statement about finite composition length of F follows from the fact that each simple functor is nonzero on a single isomorphism class of indecomposable objects, where its value has dimension 1. □

We turn now to a result for triangulated categories that is not the same as for module categories. It is well known that finitely presented functors on a module category have projective dimension at most 2, because Hom is left exact on such a category [3, Proposition 4.2]. The situation for functors on a triangulated category is quite different.

PROPOSITION 2.5. *The only finitely presented functors of finite projective dimension in* $\mathrm{Fun}^{\mathrm{op}}\, C$ *are the representable functors. Equivalently, the only monomorphisms between representable functors are split.*

PROOF. Given a presentation $\mathrm{Hom}_C(-, M_1) \to \mathrm{Hom}_C(-, M_0) \to F \to 0$ of a functor F, the morphism between the representable functors comes from a morphism $\alpha : M_1 \to M_0$ in C, by Yoneda's lemma. Complete this morphism to a triangle and rotate it, to get a triangle

$$M_2 \xrightarrow{\beta} M_1 \xrightarrow{\alpha} M_0 \xrightarrow{\gamma} \Sigma M_2.$$

We get a long exact sequence of representable functors that has F as one of its (co)kernels, giving a long exact sequence

$$\begin{array}{ccccc} \cdots & \longrightarrow & \mathrm{Hom}_C(-, \Sigma^{-1} M_1) & \longrightarrow & \mathrm{Hom}_C(-, \Sigma^{-1} M_0) \\ \longrightarrow \mathrm{Hom}_C(-, M_2) & \longrightarrow & \mathrm{Hom}_C(-, M_1) & \longrightarrow & \mathrm{Hom}_C(-, M_0) \\ \longrightarrow \quad F & \longrightarrow & 0, & & \end{array}$$

that is a projective resolution of F. By a standard result in homological algebra, F has finite projective dimension if and only if at some stage the kernel in this resolution is projective. Note that every kernel is finitely generated, because it is the image of the next term in the resolution. Thus, if F has finite projective dimension, then one of the morphisms $\mathrm{Hom}_C(-, U) \to \mathrm{Hom}_C(-, V)$ in the sequence factors as a surjection onto a projective functor followed by an injection from the projective functor as follows:

$$\mathrm{Hom}_C(-, U) \to \mathrm{Hom}_C(-, X) \to \mathrm{Hom}_C(-, V)$$

corresponding to morphisms $U \xrightarrow{\phi} X \xrightarrow{\theta} V$ in C. Here we are using the fact that projective functors are representable, by Proposition 2.3. Because the first of these maps of functors is surjective, the identity 1_X is an image of a map $X \to U$ after composition with ϕ, so that ϕ is a split epimorphism. The injectivity of the second map of functors is exactly the definition that θ is a monomorphism. In a triangulated category all monomorphisms are split, so that θ is a split monomorphism. It follows from this that there are decompositions $U \cong U_1 \oplus X$ and $V \cong X \oplus V_1$ so that the morphism $U \to V$ is the identity on X and 0 on U_1. Transferring back to the original triangle, we see that it is the sum of two triangles, one of the form $\Sigma^n X \xrightarrow{1} \Sigma^n X \to 0 \to \Sigma^{n+1} X$ for some n, and the other with zero as one of its three morphisms. The first of these produces contractible summands of the resolution of F, and the second produces a resolution that is split everywhere, showing that F is projective because the final map $\mathrm{Hom}_C(-, M_0) \to F$ must split.

The equivalence with the statement that monomorphisms between representable functors are split is immediate. If there is a nonsplit such monomorphism, then its cokernel has projective dimension 1 and is not projective. On the other hand, any nonprojective functor of finite projective dimension gives rise to a functor of projective dimension 1 (appearing at the end of the finite projective resolution), and this is presented by a nonsplit monomorphism of projectives. □

We characterize the existence of Auslander–Reiten triangles in terms of finite presentability of the corresponding simple functors. The result is familiar for functors on module categories, but less so for functors on triangulated categories.

Proposition 2.6. *Let C be a Hom-finite, Krull–Schmidt triangulated category and let M be an indecomposable object in C. The simple functor s^M is finitely presented if and only if there is an Auslander–Reiten triangle $U \to V \to M \to \Sigma U$. When there exists such an Auslander–Reiten triangle, the map of representable functors $\mathrm{Hom}_C(-, M) \to \mathrm{Hom}_C(-, \Sigma U)$ has s^M as its image.*

Proof. If there is such an Auslander–Reiten triangle, the long exact sequence

$$\cdots \to \mathrm{Hom}_C(-, U) \to \mathrm{Hom}_C(-, V) \to \mathrm{Hom}_C(-, M) \to \mathrm{Hom}_C(-, \Sigma U) \to \cdots$$

provides the start of a resolution

$$\mathrm{Hom}_C(-, V) \to \mathrm{Hom}_C(-, M) \to s^M \to 0$$

because the lifting property of the Auslander–Reiten triangle coupled with the fact that it is not split translates to the statement that the cokernel of $\mathrm{Hom}_C(-, V) \to \mathrm{Hom}_C(-, M)$ is s^M, which is also the image of $\mathrm{Hom}_C(-, M)$ in $\mathrm{Hom}_C(-, \Sigma U)$. This shows that s^M is finitely presented.

Conversely, if s^M is finitely presented by a three-term exact sequence of this form, then the morphism $\mathrm{Hom}_C(-, V) \to \mathrm{Hom}_C(-, M)$ comes from a morphism $V \to M$ in C that we may extend to a triangle $U \to V \to M \to \Sigma U$. This triangle satisfies the Auslander–Reiten lifting property at M, and the morphism $M \to \Sigma U$ is not zero since $V \to M$ is not a split epimorphism. The triangle gives rise to a long exact sequence of representable functors of the kind at the start of this proof. If the kernel of $\mathrm{Hom}_C(-, V) \to \mathrm{Hom}_C(-, M)$ has a nonzero projective direct summand, then such a summand is finitely presented and hence representable. By Proposition 2.5, it splits off from $\mathrm{Hom}_C(-, V)$, as well as from $\mathrm{Hom}_C(-, U)$. Thus we can remove such a summand and may assume that $\mathrm{Hom}_C(-, V)$ is a projective cover of $\mathrm{Rad}_C(-, M)$. In this case the

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morphism $V \to M$ is minimal right almost split, in the terminology of [1]. It is proven by Happel [11, page 36] (in the dual case of a minimal left almost split morphism) that the third term U in the triangle is indecomposable and hence the triangle is an Auslander–Reiten triangle. □

Given a Hom-finite, Krull–Schmidt triangulated category C over k, a *Serre functor* on C is a self-equivalence $\mathcal{S} : C \to C$ for which there are bifunctorial isomorphisms

$$D\,\mathrm{Hom}_C(X, Y) \cong \mathrm{Hom}_C(Y, \mathcal{S}(X)) \quad \text{for all } X, Y \in C.$$

Here $D(U) = \mathrm{Hom}_k(U, k)$ is the vector space duality. It was shown in [14] that C has a Serre functor $\mathcal{S}$ if and only if C has Auslander–Reiten triangles, and that the Auslander–Reiten triangles have the form

$$\Sigma^{-1}\mathcal{S}(U) \xrightarrow{\alpha} V \xrightarrow{\beta} U \xrightarrow{\gamma} \mathcal{S}(U)$$

with Auslander–Reiten translate $\tau = \Sigma^{-1}\mathcal{S}$.

We now point out that the presence of a Serre functor on C makes $\mathrm{Fun}^{\mathrm{op}}\,C$ into a self-injective category. We will use, particularly, the fact that representable functors for indecomposable objects have simple socles.

Proposition 2.7. *Let C be a Hom-finite, Krull–Schmidt triangulated category with Serre functor $\mathcal{S}$. Then each representable functor* $\mathrm{Hom}_C(-, M)$ *is injective (as well as projective), with simple socle $s^{\mathcal{S}^{-1}(M)}$.*

Proof. For each object X,

$$\mathrm{Hom}_C(X, M) \cong D\,\mathrm{Hom}_C(M, \mathcal{S}(X)) \cong D\,\mathrm{Hom}_C(\mathcal{S}^{-1}(M), X).$$

Because $\mathrm{Hom}_C(\mathcal{S}^{-1}(M), -)$ is a projective covariant functor on C, it follows that

$$D\,\mathrm{Hom}_C(\mathcal{S}^{-1}(M), -) \cong \mathrm{Hom}_C(-, M)$$

is an injective contravariant functor on C, as well as being projective. Now, if

$$\Sigma^{-1}M \to E \to \mathcal{S}^{-1}(M) \to M$$

is an Auslander–Reiten triangle, the image of

$$\mathrm{Hom}_C(-, \mathcal{S}^{-1}(M)) \to \mathrm{Hom}_C(-, M)$$

is the simple functor $s^{\mathcal{S}^{-1}(M)}$, by Proposition 2.6. This is the socle. □

We identify composition factors of functors in $\mathrm{Fun}^{\mathrm{op}}\,C$ in the spirit of [4].

Corollary 2.8. *Assume that C has a Serre functor $\mathcal{S}$. Let U and M be indecomposable objects of C. The following are equivalent:*

(1) *the functor s^U is a composition factor of* $\mathrm{Hom}_C(-, M)$*;*
(2) *there is a nonzero morphism $U \to M$;*

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(3) *there is a nonzero morphism* $\mathcal{S}^{-1}(M) \to U$*;*

(4) *the functor* $s^{\mathcal{S}^{-1}(M)}$ *is a composition factor of* $\mathrm{Hom}_{C}(-, U)$.

Proof. This is immediate from Corollary 2.4 and the definition of a Serre functor. □

Following Linckelmann [12] (who attributed the terminology to Happel [11]), we say that the third morphism γ in an Auslander–Reiten triangle

$$X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X$$

is *almost vanishing*. Notice that the domain and codomain of γ are both indecomposable in this definition. An almost vanishing morphism determines the corresponding Auslander–Reiten triangle by completing it to a triangle and rotating to put it in the right position. Equally, an almost vanishing morphism exists with domain Z (or codomain X) if and only if there is an Auslander–Reiten triangle with Z on the right (or X on the left—since these properties are preserved by Σ).

An example that motivates us is the stable module category **stmod**(A) of a symmetric algebra A (see [11]). In this situation the shift is Ω^{-1}, the Serre functor is Ω and Auslander–Reiten triangles are exactly the triangles obtained by completing the nonzero morphisms in an almost split sequence to a triangle. A morphism $\gamma : W \to U$ between indecomposable A-modules is almost vanishing if and only if, as an element of $\mathrm{Ext}^1_A(W, \Omega(U))$, it represents an almost split sequence of A-modules $0 \to \Omega(U) \to V \to W \to 0$.

Almost vanishing morphisms have been used in several places in the literature. They underlie the construction of natural transformations in the graded center in [12, 13]. They provide a construction of ghost maps showing that Freyd's generating hypothesis fails in general for the stable module category **stmod**(kG), when G is a finite group [8]. They were used by Happel [11] in constructing Auslander–Reiten triangles in bounded derived categories (where they exist). We present several characterizations of these morphisms. Most of these are well known, but conditions (2) and (3) may be less familiar.

Proposition 2.9. *Let* C *be a Hom-finite, Krull–Schmidt triangulated category with Serre functor* $\mathcal{S}$ *and let* $f : X \to Y$ *be a morphism between indecomposable objects in* C*. The following are equivalent:*

(1) *the map* f *is almost vanishing;*

(2) f *is nonzero and, for all objects* U*,* f *factors through every nonzero morphism* $U \to Y$*;*

(3) f *is nonzero and, for all objects* V*,* f *factors through every nonzero morphism* $X \to V$*;*

(4) *whenever* $g : U \to X$ *is not a split epimorphism in* C*, then* $fg = 0$*;*

(5) *whenever* $h : Y \to U$ *is not a split monomorphism, then* $hg = 0$*;*

(6) *the map* $\mathrm{Hom}_C(-, f) : \mathrm{Hom}_C(-, X) \to \mathrm{Hom}_C(-, Y)$ *factors through a simple functor.*

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Thus morphisms f satisfying any (and hence all) of the above conditions are determined up to scalar multiple. For such a morphism, $Y \cong \mathcal{S}(X)$.

The word 'split' is redundant in conditions (4) and (5) because all monomorphisms and epimorphisms in a triangulated category are split.

PROOF. We start by observing that the implication (1) $\Rightarrow$ (6) is part of Proposition 2.6. For the converse (6) $\Rightarrow$ (1), if (6) holds then the image of $\mathrm{Hom}_{\mathcal{C}}(-, f)$ must be the simple socle $s^{\mathcal{S}^{-1}Y}$ of $\mathrm{Hom}_{\mathcal{C}}(-, Y)$, by Proposition 2.7, and so $Y \cong \mathcal{S}(X)$, and f is determined up to a scalar multiple. We know that there exists an almost vanishing map $g : X \to \mathcal{S}X$, and it has the same property as f. Hence f is a scalar multiple of g, and f is almost vanishing.

(1) $\Rightarrow$ (2) Suppose that f is almost vanishing and let $U \to Y$ be a nonzero morphism. Then $X \cong \mathcal{S}^{-1}(Y)$ and $s^X \cong s^{\mathcal{S}^{-1}(Y)}$ is a composition factor of $\mathrm{Hom}_{\mathcal{C}}(-, U)$ by Proposition 2.8, and we have morphisms between projective functors $\mathrm{Hom}_{\mathcal{C}}(-, X) \to \mathrm{Hom}_{\mathcal{C}}(-, U) \to \mathrm{Hom}_{\mathcal{C}}(-, Y)$ with composite mapping to the simple socle of $\mathrm{Hom}_{\mathcal{C}}(-, Y)$. This means that the corresponding composite is almost vanishing and provides a factorization of f as in (2).

(2) $\Rightarrow$ (1) Suppose that f satisfies (2) and let $\phi : \mathcal{S}^{-1}(Y) \to Y$ be almost vanishing. There is a factorization of f as $X \xrightarrow{\gamma} \mathcal{S}^{-1}(Y) \xrightarrow{\phi} Y$. Since the image of $\mathrm{Hom}_{\mathcal{C}}(-, \phi)$ is the simple top $s^{\mathcal{S}^{-1}(Y)}$ and $f \neq 0$, $\mathrm{Hom}_{\mathcal{C}}(-, \gamma) : \mathrm{Hom}_{\mathcal{C}}(-, X) \to \mathrm{Hom}_{\mathcal{C}}(-, \mathcal{S}^{-1}(Y))$ maps onto the simple top and hence is surjective, by Nakayama's lemma. Therefore γ is an isomorphism, and f is almost vanishing.

The equivalence of (1) and (3) is similar.

That (1) implies (4) follows because g factors through W in the Auslander–Reiten triangle $\Sigma^{-1}Y \xrightarrow{\alpha} W \xrightarrow{\beta} X \xrightarrow{f} Y$ so that $fg = f\beta g'$ for some g', and this composite is zero because $f\beta = 0$.

To get that (4) implies (1), we complete f to a triangle and rotate to get a triangle $\Sigma^{-1}Y \xrightarrow{\alpha} W \xrightarrow{\beta} X \xrightarrow{f} Y$. This is an Auslander–Reiten triangle because $f \neq 0$, X and $\Sigma^{-1}Y$ are indecomposable, and condition (4) implies that any morphism $g : U \to X$ that is not a split epimorphism factors through W.

The equivalence (1) $\Leftrightarrow$ (5) is similar. □

In the next section we consider morphisms $f : X \to Y$ for which the image of the natural transformation of representable functors

$$\mathrm{Hom}_{\mathcal{C}}(-, f) : \mathrm{Hom}_{\mathcal{C}}(-, X) \to \mathrm{Hom}_{\mathcal{C}}(-, Y)$$

has finite composition length. To prepare for this, we present some results that identify the occurrence of composition factors.

PROPOSITION 2.10. *Assume that $\mathcal{C}$ has a Serre functor $\mathcal{S}$. Let $f : X \to Y$ be a morphism between indecomposable objects and let V be an indecomposable object. The following are equivalent.*

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(1) *The functor s^V is a composition factor of the image of*

$$\mathrm{Hom}_C(-,X) \xrightarrow{\mathrm{Hom}(-,f)} \mathrm{Hom}_C(-,Y).$$

(2) *There is a morphism $V \xrightarrow{\gamma} X$ so that $f\gamma \neq 0$.*

(3) *There are morphisms $\mathcal{S}^{-1}(Y) \xrightarrow{\xi} V \xrightarrow{\gamma} X$ so that $f\gamma\xi$ is almost vanishing.*

Proof. Because $\mathrm{Hom}(-,V)$ is projective and has unique simple quotient s^V, we see that s^V is a composition factor of the image of $\mathrm{Hom}(-,f)$ if and only if there is a morphism $\mathrm{Hom}(-,\gamma) : \mathrm{Hom}_C(-,V) \to \mathrm{Hom}_C(-,X)$ whose image is not in the kernel of $\mathrm{Hom}(-,f)$. By Yoneda's lemma, this happens if and only if there is a morphism $\gamma : V \to X$ so that $f\gamma \neq 0$. By Proposition 2.9, conditions (2) and (3) are equivalent. □

If $\alpha : F \to G$ is a natural transformation of functors defined on C, we say that the *support* of α is the set of isomorphism classes of indecomposable objects M for which $\alpha_M : F(M) \to G(M)$ is nonzero.

Corollary 2.11. *Assume that C has a Serre functor $\mathcal{S}$ and let $f : X \to Y$ be a morphism between indecomposable objects. The following are equivalent:*

(1) *the image of $\mathrm{Hom}_C(-,f)$ has finite composition length;*
(2) *there are only finitely many isomorphism classes of indecomposable modules V with a morphism $\gamma : V \to X$ so that $f\gamma \neq 0$;*
(3) *there are only finitely many isomorphism classes of indecomposable modules V such that there are morphisms $\mathcal{S}^{-1}(Y) \xrightarrow{\xi} V \xrightarrow{\gamma} X$ for which $f\gamma\xi$ is almost vanishing;*
(4) *the support of $\mathrm{Hom}_C(-,f)$ is finite;*
(5) *whenever $\phi : \mathcal{S}^{-1}(Y) \to X$ is such that $f\phi$ is almost vanishing, then ϕ can be expressed as a sum of composites of (finitely many) irreducible morphisms.*

Furthermore, if the image of $\mathrm{Hom}_C(-,f)$ has finite composition length, then its composition factors are the s^V for which there are morphisms $\mathcal{S}^{-1}(Y) \xrightarrow{\xi} V \xrightarrow{\gamma} X$, both of which are finite composites of irreducible morphisms, and so that $f\gamma\xi$ is almost vanishing.

Proof. The equivalence of the first four statements is immediate from Proposition 2.10. (1) $\Leftrightarrow$ (5) The image has finite composition length if and only if $\mathrm{Rad}^n_C(-,X)$ is contained in the kernel of $\mathrm{Hom}_C(-,f)$ for some n. Thus, assuming (1), no morphism $\gamma : Y \to X$ with $f\gamma \neq 0$ lies in $\mathrm{Rad}^n_C(Y,X)$, and such a morphism cannot be expressed as a sum of composites of n or more irreducible morphisms. Thus (5) holds. Conversely, assume that (5) holds. We may take an almost vanishing morphism $\phi : \mathcal{S}^{-1}(Y) \to X$ and express it as a sum of composites of irreducible morphisms, deducing that ϕ does not lie in $\mathrm{Rad}^n_C(\mathcal{S}^{-1}(Y),X)$ for some n. Since for each nonzero morphism $\gamma : Y \to X$ there is a factorization $\phi = \gamma\xi$ for some ξ, we deduce that every nonzero morphism γ lies outside $\mathrm{Rad}^n_C(Y,X)$. This shows that $\mathrm{Rad}^n_C(-,X)$ is contained in the kernel of $\mathrm{Hom}_C(-,f)$ and so (1) holds.

The composition factors are as claimed because firstly, by Proposition 2.10, the s^V for which $f\gamma\xi$ is as described are among the composition factors. The complete set of composition factor arises without the requirement that ξ and γ be finite composites of irreducible morphisms, but we see from (5) that they must be sums of composites of irreducible morphisms. The V that can arise from sums of composites of irreducible morphisms are the same as the V that arise from composites of irreducible morphisms. □

3. Elements of the graded center with finite support

The degree n elements of the graded center of a triangulated category C are the natural transformations $\mathrm{Id}_C \to \Sigma^n$ that commute with Σ up to $(-1)^n$. In the context of stable module categories **stmod**(A) for symmetric algebras A, Linckelmann [12] constructed certain elements of the graded center of **stmod**(A) of degree -1. In that situation the shift is given by $\Sigma = \Omega^{-1}$, the inverse of the Heller operator, and the Serre functor is $\mathcal{S} = \Omega$. His construction produced, for each finitely generated indecomposable nonprojective module U, a natural transformation $\zeta : \mathrm{Id}_C \to \Omega$ such that $\zeta_U : U \to \Omega(U)$ is almost vanishing (that is, represents an almost split sequence ending in U), and such that $\zeta(V) = 0$ for any finitely generated indecomposable nonprojective module V that is not isomorphic to $\Omega^n(U)$, for any integer n. Linckelmann and Stancu [13] then combined this construction with the existence of periodic modules of period one to produce elements of the graded center in degree 0. Their elements have support of size 1.

We assume throughout that C is a Hom-finite, Krull–Schmidt, k-linear triangulated category with Serre functor $\mathcal{S}$ and Auslander–Reiten translate $\tau M = \Sigma^{-1}\mathcal{S}(M)$. Our first goal is to show that the natural transformations of the kind constructed by Linckelmann are the only ones with small support.

Proposition 3.1. *Let $F : C \to C$ be a k-linear endofunctor and suppose that $\alpha : \mathrm{Id}_C \to F$ is a natural transformation with support consisting of a single indecomposable object M. Then $F(M) = \mathcal{S}(M)$ and $\alpha_M : M \to \mathcal{S}(M)$ is an almost vanishing morphism.*

Proof. Let α have support only on M. Consider the image of

$$\mathrm{Hom}_C(-, M) \xrightarrow{\mathrm{Hom}_C(-,\alpha_M)} \mathrm{Hom}_C(-, F(M))$$

as a subfunctor of $\mathrm{Hom}_C(-, F(M))$. If the image has only one composition factor (which must be s^M, the simple top of $\mathrm{Hom}_C(-, M)$), this composition factor must be the socle of $\mathrm{Hom}_C(-, F(M))$. Since the socle is $s^{\mathcal{S}^{-1}(F(M))}$, we deduce that $M = \mathcal{S}^{-1}(F(M))$, so $F(M) = \mathcal{S}(M)$, and that α_M is almost vanishing, by Proposition 2.9.

If the image has another composition factor s^X for some object X, this also appears as a composition factor of $\mathrm{Hom}_C(-, M)$. Hence, by projectivity of the representable functor $\mathrm{Hom}_C(-, X)$, there is a nonzero morphism $\mathrm{Hom}_C(-, X) \to \mathrm{Hom}_C(-, M)$ so that the composite

$$\mathrm{Hom}_C(-, X) \to \mathrm{Hom}_C(-, M) \xrightarrow{\mathrm{Hom}_C(-,\alpha_M)} \mathrm{Hom}_C(-, F(M))$$

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is nonzero. By Yoneda's lemma, this corresponds to a homomorphism $\phi : X \to M$ so that $\alpha_M \circ \phi \neq 0$. Since α is a natural transformation, it follows that $F(\phi) \circ \alpha_X \neq 0$, so that α has X in its support, contradicting our hypothesis. Hence the image has only one composition factor, a case we have already considered. □

We apply this to the situation considered by Linckelmann and Stancu [13], namely the stable module category $\mathbf{stmod}(kG)$ of a finite p-group G over an algebraically closed field of characteristic p. We show that the elements of the graded center that they constructed in degrees other than -1 are the only ones with support a single indecomposable module.

Corollary 3.2. *Let G be a finite group, k a field and r an integer. Let* $\phi : \mathrm{Id}_{\mathbf{stmod}(kG)} \to \Sigma^r$ *be a natural transformation with support a single indecomposable module* $\{M\}$*. Then* $\Sigma^r(M) \cong \Omega(M)$ *and* $\phi_M : M \to \Omega(M)$ *is an almost vanishing morphism. Thus, if* $r \neq -1$*, then* $M \cong \Omega^{r+1}(M)$ *is a periodic module and* ϕ *is, up to scalar multiple, one of the natural transformations constructed by Linckelmann and Stancu.*

Proof. In $\mathbf{stmod}(kG)$, we have $\mathcal{S} = \Omega = \Sigma^{-1}$. According to Proposition 3.1, $\Sigma^r M = \mathcal{S}(M)$ and $M \to \mathcal{S}(M)$ is almost vanishing. The remaining assertions are immediate. □

Because homogeneous elements of the graded center commute with Σ up to a sign, they can have support consisting of a single indecomposable object only if that object is periodic under Σ. A weaker condition is to allow the support to be a single Σ-orbit of indecomposable objects, which might not be periodic. We obtain the following result.

Proposition 3.3. *Let M be an indecomposable object of* C *for which there are no irreducible maps* $\Sigma^r M \to M$ *for any* $r \in \mathbb{Z}$*, and let* $F : C \to C$ *be a k-linear endofunctor. Suppose that* $\alpha : \mathrm{Id}_C \to F$ *is a natural transformation whose support is contained in* $\{\Sigma^r M \mid r \in \mathbb{Z}\}$*. Then* $F(M) = \mathcal{S}(M)$ *and, for each r, the map* $\alpha_{\Sigma^r M} : \Sigma^r M \to F(\Sigma^r M)$ *is almost vanishing. Thus* α *is one of the natural transformations constructed by Linckelmann in* [12].

The hypothesis that there are no irreducible maps $\Sigma^r M \to M$ for any $r \in \mathbb{Z}$ holds in many cases of interest. For example, it always holds if M belongs to an Auslander–Reiten quiver component of tree class A_∞. By [15], it can be seen to hold most of the time for $\mathbf{stmod}(kG)$ when G is a finite group.

Proof. As in the proof of Proposition 3.1, consider the image of

$$\mathrm{Hom}_C(-, M) \xrightarrow{\mathrm{Hom}_C(-,\alpha_M)} \mathrm{Hom}_C(-, F(M)).$$

This has s^M as a composition factor, and if it has more composition factors than this it must have one of the form s^E where $E \to M$ is an irreducible morphism, since such simple functors form the second radical layer of the projective cover of s^M. This would mean that E does not have the form $\Sigma^r M$ and that α has E in its support, which is not possible. We conclude that the image is the simple functor s^M and, as before, $FM = \mathcal{S}(M)$ and α_M is an almost vanishing morphism. □

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We now consider elements of the graded center of C with support larger than a single Σ-orbit of objects. One way to construct such elements is to add two elements that have support on different shift orbits: the resulting natural transformation $\alpha : \mathrm{Id}_C \to \Sigma^r$ has the property that for every morphism $f : M \to N$ between indecomposable objects in different shift orbits we have $\alpha_N f = 0$. We consider α with $\alpha_N f \neq 0$ for some nonisomorphism f. This is equivalent to requiring that the support of the natural transformation $\mathrm{Hom}_C(-, \alpha_N)$ has size at least 2 for some N.

We recall that a triangulated category C is d-Calabi–Yau if Σ^d is a Serre functor. It follows from [14] that such a category has Auslander–Reiten triangles. In the next result we refer to the Auslander–Reiten quiver simply as the 'quiver'. We recall that the term *mesh* denotes a region of this quiver bounded by the objects that appear in the three left terms of an Auslander–Reiten triangle [6].

THEOREM 3.4. *Let C be a k-linear, Hom-finite, Krull–Schmidt triangulated category. Let $\alpha : \mathrm{Id}_C \to \Sigma^r$ be a natural transformation in the graded center of C. Fix an indecomposable object N of C. We suppose that:*

(1) *C is a d-Calabi–Yau category for some integer d;*
(2) *for all objects U in the quiver component of N, $\Sigma^{r-d}U$ and U lie in the same τ-orbit;*
(3) *every mesh in the quiver component of N has at most two middle terms; and*
(4) *for all objects U in the same quiver component as N, the support of the natural transformation $\mathrm{Hom}_C(-, \alpha_U) : \mathrm{Hom}_C(-, U) \to \mathrm{Hom}_C(-, \Sigma^r U)$ is finite and, for some U, it has size at least 2.*

Then the support of α contains the entire quiver component of N.

PROOF. Let U be an indecomposable object in the quiver component of N for which the support of $\mathrm{Hom}_C(-, \alpha_U)$ has size at least 2. Since $\mathrm{Hom}_C(-, \alpha_U)$ has finite composition length, by Proposition 2.9 and Corollary 2.11, there is a morphism $\phi : \Sigma^{r-d}U \to U$ that is a sum of finite composites of irreducible morphisms, such that $\alpha_U \phi$ is almost vanishing. Because the support of $\mathrm{Hom}_C(-, \alpha_U)$ has size at least 2, ϕ is not an isomorphism.

We claim that the composite of morphisms in any path in the Auslander–Reiten quiver from $\Sigma^{r-d}U$ to U also has the same property as ϕ, and in fact equals $\pm\phi$. This is because whenever we have a pair of consecutive irreducible morphisms in such a path of the form $\tau V \to W \to V$ the Auslander–Reiten triangle $\tau V \to E \to V \to \Sigma\tau V$ has middle term E with at most two indecomposable summands, one of which is W. If $E = W \oplus X$ for some X, we can replace the maps into and out of W by irreducible morphisms $\tau V \to X \to V$, because the composite $\tau V \to W \oplus X \to V$ is zero, so that the new irreducible morphisms have composite (-1) times the composite of the old. Repeating this operation allows us to move from any path from $\Sigma^{r-d}U$ to U to any other path, changing the composite by (-1) each time. Now ϕ must be a linear combination of composites along these paths, but since the composites are all the same up to sign,

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we deduce that ϕ could be taken to be the composite of the irreducible morphisms along any of the paths.

Since $\Sigma^{r-d}U$ and U lie in the same τ-orbit, there is a path in the quiver from $\Sigma^{r-d}U$ to U going through each member of the τ-orbit of U between these two objects. We deduce that for every irreducible morphism with codomain U the domain of that morphism lies in the support of α. This and the fact that α commutes (up to sign) with Σ, and hence with τ, imply that all objects in the component of N lie in the support of α. □

In the next section we present an example of a natural transformation satisfying the conditions of Theorem 3.4 in the context of the stable module category of a group with a dihedral Sylow 2-subgroup in characteristic 2. In general it is not always possible to find such examples, as we now see.

Corollary 3.5. *With the same hypotheses as in Theorem 3.4, suppose further that the quiver component containing N has type A_∞. Then no such natural transformation α can exist.*

Proof. Suppose that there were such a natural transformation α. Its support would have to contain the quiver component containing N and, for any choice of indecomposable object N_0 in this quiver component, the proof of Theorem 3.4 shows that there is an irreducible morphism $f : U \to N_0$ with $\alpha_{N_0} f \neq 0$. We may choose N_0 so that it lies on the rim of the quiver, as in the following diagram.

$$\begin{array}{ccccccccccccccc}
 & \vdots & & & & \vdots & & & & \vdots & & & & \vdots & \\
 & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \\
\cdots & & & M_2 & & & & N_2 & & & & O_2 & & & \cdots \\
 & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \\
 & L_1 & & & & M_1 & & & & N_1 & & & & O_1 & \\
 & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \\
 & & & L_0 & & & & M_0 & & & & N_0 & & &
\end{array}$$

There is no path of irreducible morphisms from $\Sigma^{r-d}N_0$ to N_0 with nonzero composite unless $r = d$. This is because $\Sigma^{r-d}N_0$ is also on the rim, and such a path has composite equal to that of a path that has two irreducible morphisms between consecutive objects on the rim, and the composition of these morphisms is zero. Such a path was necessary to the existence of α in the proof of Theorem 3.4, so this situation cannot occur. When $r = d$, the support of $\mathrm{Hom}_C(-, \alpha_{N_0})$ has size 1, because there is no finite chain of irreducible morphisms from N_0 to N_0 other than the empty chain at N_0. This shows that α_{N_0} is almost vanishing, so that $\alpha_{N_0} f = 0$, which is a contradiction. Hence no α can exist as in Theorem 3.4. □

Corollary 3.6. *Let $C =$ **stmod**(B) be the stable module category of a block with wild representation type of a group algebra kG. Let α be an element of the graded center of C.*

(1) *If α is supported on only finitely many τ-orbits, then α is a sum of elements that are supported on single τ-orbits, each of which is of the kind described in Proposition 3.3. Thus $\alpha_Y f = 0$ for every nonisomorphism $f : X \to Y$ between indecomposable objects.*

(2) *If there is any nonisomorphism $f : X \to Y$ between indecomposable objects so that $\alpha_Y f \neq 0$, then such an f can be found that is not a finite composite of irreducible morphisms. In this case α is not supported on only finitely many τ-orbits.*

Proof. We exploit the fact, using a theorem of Erdmann [10], that all quiver components of C have type A_∞ and satisfy conditions (1), (2) and (3) of Theorem 3.4.

To prove (1), if α were supported on only finitely many τ-orbits, then the support of $\mathrm{Hom}_C(-, \alpha_Y)$ would be finite for all indecomposable Y and, by Corollary 3.5, such α cannot exist unless this support has size 1 for every Y. This is equivalent to requiring that $\alpha_Y f = 0$ for every nonisomorphism $f : X \to Y$ between indecomposable objects, and that α is a sum of natural transformations supported on single τ-orbits.

With the hypothesis of (2), we must have that some $\mathrm{Hom}_C(-, \alpha_Y)$ has infinite support. Finding $f : X \to Y$ so that $\alpha_Y f$ is almost vanishing as in Proposition 2.9, we find by Corollary 2.11 that f is not a composite of irreducible morphisms. □

Remark 3.7. In the case of modules in a block of wild type in a group algebra, it seems likely that any map $f : X \to Y$ as above, that is not a composite of a finite number of irreducible maps, should factor through a module that is not in the quiver component of X, implying that α would have support on more than one quiver component. This is easily verified in some specific cases, but seems difficult to prove in general.

4. An example: groups with dihedral Sylow 2-subgroups

In this section we show, under certain circumstances, that there exist natural transformations in the graded center of the stable module category that are supported on only a single component of the Auslander–Reiten quiver, and which are not sums of the natural transformations constructed by Linckelmann in [12]. Furthermore, our natural transformations satisfy the finiteness condition of Theorem 3.4.

We assume throughout that k is an algebraically closed field of characteristic 2 and that G is a finite group with a dihedral Sylow 2-subgroup having order at least 8. The methods we will use apply in this generality: if we were to assume that G is a dihedral 2-group the string and band module methods of [7] would become available to us, resulting in some easier arguments.

The group algebra kG has tame representation type, and a primary fact in the example is that the Auslander–Reiten quiver component that contains the trivial module has tree class A_∞^∞ [15] and consists entirely of endotrivial modules (see [2]). By definition, a kG-module M is endotrivial provided $\mathrm{Hom}_k(M, M) \cong k \oplus P$, where P is a projective kG-module. We note that a kG-module is endotrivial if and only if its restriction to every elementary abelian p-subgroup is endotrivial and that the tensor product of two endotrivial modules is again an endotrivial module (see [9]).

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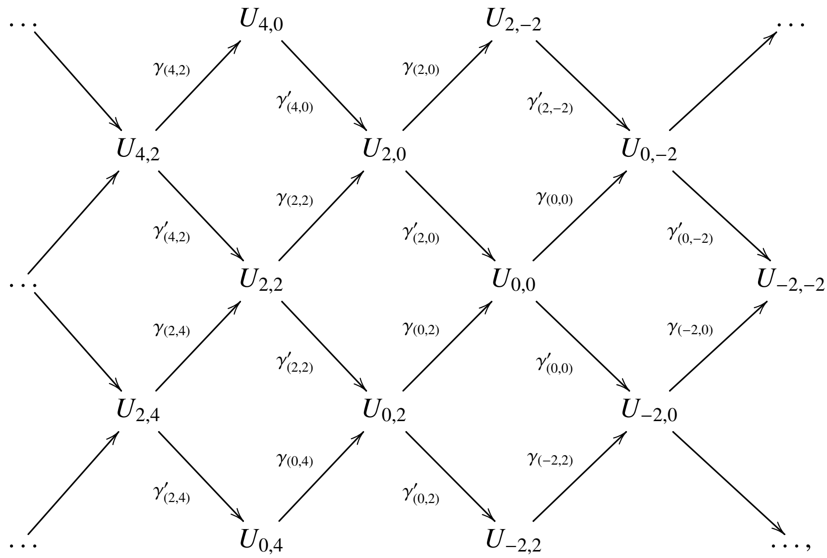
Suppose that S is a Sylow 2-subgroup of G and that E_1 and E_2 are representatives of the two conjugacy classes of elementary abelian subgroups of order 4 in S. The Auslander–Reiten quiver containing the trivial kG-module has the form



where $U_{0,0} \cong k$, $U_{i,i} \cong \Omega^i(k)$ and $U_{i,j}$ is an endotrivial module with the property that $U_{i,j}\downarrow_{E_1} \cong \Omega^i(k_{E_1})$ and $U_{i,j}\downarrow_{E_2} \cong \Omega^j(k_{E_2})$ (see [2, 15]). The modules $U_{0,j}$ where $j \geq 0$ is even, are characterized by the property that there is a chain of irreducible morphisms $U_{0,j} \to W_1 \to \cdots \to W_r \to U_{0,0}$ that are all isomorphisms in $\mathbf{stmod}(kE_1)$ after restriction to E_1. The modules $U_{j,0}$ where $j \geq 0$ are similarly characterized with respect to restriction to E_2. The almost split sequence ending in the trivial module has the form

$$0 \longrightarrow \Omega^2(k) \longrightarrow U_{2,0} \oplus U_{0,2} \longrightarrow k \longrightarrow 0.$$

Note that all modules in the component of the trivial module have odd dimension since they are endotrivial. If M is an indecomposable kG-module of odd dimension, then, by [2], the almost split sequence ending in M is (modulo projective summands)

$$0 \longrightarrow \Omega^2(k) \otimes M \longrightarrow U_{2,0} \otimes M \oplus U_{0,2} \otimes M \longrightarrow M \longrightarrow 0.$$

From this property and the above characterizations we see by induction that $U_{0,2n} \cong (U_{0,2})^n$ and $U_{2n,0} \cong (U_{2,0})^n$ in $\mathbf{stmod}(kG)$. This allows us to express every module $U_{i,j}$ in the form $\Omega^{2m}((U_{0,2})^n)$ or $\Omega^{2m}((U_{2,0})^n)$ in $\mathbf{stmod}(kG)$, for some m and n. For example, if $i \geq j$, then $U_{i,j} \cong \Omega^i(U_{2,0}^{(i-j)/2})$. Using this description, the fact that Ω commutes with $\otimes$ and the relation $U_{0,2} \otimes U_{2,0} \cong U_{2,2} \cong \Omega^2(k)$ in $\mathbf{stmod}(kG)$, we deduce that $U_{i,j} \otimes U_{s,t} \cong U_{i+s,j+t} \oplus P$ for some projective module P, for all i, j, s, t.

Lemma 4.1. *For any n, there is an exact sequence having the form*

$$\mathcal{E}_n : \qquad 0 \longrightarrow \Omega^{2n}(k) \xrightarrow{(\alpha,\beta)} U_{2n,0} \oplus U_{0,2n} \xrightarrow{\begin{pmatrix}\gamma\\ \delta\end{pmatrix}} k \longrightarrow 0\,,$$

where $\alpha = \gamma_{(2n,2)} \ldots \gamma_{(2n,2n-2)}\gamma_{(2n,2n)}$, $\beta = \gamma'_{(2,2n)} \cdots \gamma'_{(2n-2,2n)}\gamma'_{(2n,2n)}$, *etc. That is, each map is the obvious composition of irreducible maps in the Auslander–Reiten quiver.*

Proof. The modules in the sequence are positioned in the Auslander–Reiten quiver as the vertices of a diamond. By an argument similar to the one used to prove Theorem 3.4, we see that the two composites of irreducible morphisms from $\Omega^{2n}(k)$ to k, obtained by going round the two sides of the diamond, are equal of opposite sign. This shows that the composite of the two middle morphisms in the sequence is zero. We will show that $U_{2n,0} \oplus U_{0,2n} \to k$ is surjective and that the left-hand side of the sequence is the kernel of this surjection. In what follows, we write Rad^n_{kG} for the nth radical of Hom_{kG}.

We use functorial methods to establish these properties. This may seem complicated, but the generality of the methods avoids the question of extending the string and band module description of [7] from dihedral 2-groups to groups with dihedral Sylow 2-subgroups. The approach depends on the shape of the quiver, and seems to have potential applications in other situations with a quiver of this shape. When G is a dihedral 2-group, a shorter argument can be given.

We claim that for any module M in a stable component of the Auslander–Reiten quiver of kG-modules of type A^{∞}_{∞}, the composition factors of the functor $\mathrm{Hom}_{kG}(-, M)/\mathrm{Rad}^{\infty}_{kG}(-, M)$ are the s^V for which there is a path of irreducible morphisms from V to M. More specifically, the composition factors of $\mathrm{Rad}^n_{kG}(-, M)/\mathrm{Rad}^{n+1}_{kG}(-, M)$ are the s^V for which there is a path of n irreducible morphisms V to M, each s^V taken with multiplicity 1. This may be proved by considering the projective resolutions of simple functors, such as

$$0 \to \mathrm{Hom}_{kG}(-, \tau M) \to \mathrm{Hom}_{kG}(-, L_1) \oplus \mathrm{Hom}_{kG}(-, L_2) \\ \to \mathrm{Hom}_{kG}(-, M) \to s^M \to 0,$$

where $0 \to \tau M \to L_1 \oplus L_2 \to M \to 0$ is an almost split sequence. For each $n \geq 2$, this restricts to an exact sequence

$$0 \to \mathrm{Rad}^{n-2}_{kG}(-, \tau M) \to \mathrm{Rad}^{n-1}_{kG}(-, L_1) \oplus \mathrm{Rad}^{n-1}_{kG}(-, L_2) \\ \to \mathrm{Rad}^n_{kG}(-, M) \to 0,$$

since the morphisms are obtained by composition with an irreducible morphism. Hence we obtain for each $n \geq 1$ an exact sequence

$$0 \to \mathrm{Rad}^{n-2}_{kG}(-, \tau M)/\mathrm{Rad}^{n-1}_{kG}(-, \tau M) \\ \to \mathrm{Rad}^{n-1}_{kG}(-, L_1)/\mathrm{Rad}^n_{kG}(-, L_1) \oplus \mathrm{Rad}^{n-1}_{kG}(-, L_2)/\mathrm{Rad}^n_{kG}(-, L_2) \\ \to \mathrm{Rad}^{n-1}_{kG}(-, M)/\mathrm{Rad}^n_{kG}(-, M) \to 0,$$

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where we take $\mathrm{Rad}^{-1} = \mathrm{Rad}^0$. We also know that the composition factors of

$$\mathrm{Rad}_{kG}^{n-2}(-, \tau M)/\mathrm{Rad}_{kG}^{n-1}(-, \tau M)$$

are the composition factors of

$$\mathrm{Rad}_{kG}^{n-2}(-, M)/\mathrm{Rad}_{kG}^{n-1}(-, M)$$

with τ applied and that each indecomposable representable functor has a simple top. This provides a system of equations that allows us to compute the composition factors by recurrence: in a Grothendieck group,

$$\begin{aligned}
\mathrm{Rad}_{kG}^{0}(-, M)/\mathrm{Rad}_{kG}^{1}(-, M) &= s^M,\\
\mathrm{Rad}_{kG}^{1}(-, M)/\mathrm{Rad}_{kG}^{2}(-, M) &= \mathrm{Rad}_{kG}^{0}(-, L_1)/\mathrm{Rad}_{kG}^{1}(-, L_1) + \mathrm{Rad}_{kG}^{0}(-, L_2)/\mathrm{Rad}_{kG}^{1}(-, L_2)\\
&= s^{L_1} + s^{L_2},\\
\mathrm{Rad}_{kG}^{2}(-, M)/\mathrm{Rad}_{kG}^{3}(-, M) &= \mathrm{Rad}_{kG}^{1}(-, L_1)/\mathrm{Rad}_{kG}^{2}(-, L_1) + \mathrm{Rad}_{kG}^{1}(-, L_2)/\mathrm{Rad}_{kG}^{2}(-, L_2)\\
&\qquad - \mathrm{Rad}_{kG}^{0}(-, \tau M)/\mathrm{Rad}_{kG}^{1}(-, \tau M)\\
&= s^{L_{11}} + s^{\tau M} + s^{L_{22}} + s^{\tau M} - s^{\tau M}\\
&= s^{L_{11}} + s^{\tau M} + s^{L_{22}},
\end{aligned}$$

where $0 \to \tau L_i \to L_{ii} \oplus \tau M \to L_i \to 0$, $i = 1, 2$, are almost split sequences; and so on. We conclude that each irreducible morphism, such as $L_1 \to M$, induces a monomorphism

$$\mathrm{Hom}_{kG}(-, L_1)/\mathrm{Rad}_{kG}^{\infty}(-, L_1) \to \mathrm{Hom}_{kG}(-, M)/\mathrm{Rad}_{kG}^{\infty}(-, M).$$

Hence every composite of irreducible morphisms also induces such a monomorphism. By counting composition factors, we see that

$$\begin{aligned}
0 &\to \mathrm{Hom}_{kG}(-, \Omega^{2n}(k))/\mathrm{Rad}_{kG}^{\infty}(-, \Omega^{2n}(k))\\
&\to \mathrm{Hom}_{kG}(-, U_{2n,0} \oplus U_{0,2n})/\mathrm{Rad}_{kG}^{\infty}(-, U_{2n,0} \oplus U_{0,2n})\\
&\to \mathrm{Hom}_{kG}(-, k)/\mathrm{Rad}_{kG}^{\infty}(-, k)
\end{aligned}$$

is exact (and the last cokernel has composition factors inside the diamond we are considering in the quiver).

We may now deduce that the morphism $U_{2n,0} \oplus U_{0,2n} \to k$ is surjective, because it induces a nonzero map of representable functors and hence must be nonzero, to a module of dimension 1. Let K be the kernel of this morphism. Thus $0 \to K \to U_{2n,0} \oplus U_{0,2n} \to k \to 0$ is exact and our task is to show that K is $\Omega^{2n}(k)$. Then

$$0 \to \mathrm{Hom}_{kG}(-, K) \to \mathrm{Hom}_{kG}(-, U_{2n,0} \oplus U_{0,2n}) \to \mathrm{Hom}_{kG}(-, k)$$

is exact (by left exactness of Hom) and hence so is the similar sequence we get after factoring out Rad^{∞} from each term. Since the composite

$$\Omega^{2n}(k) \to U_{2n,0} \oplus U_{0,2n} \to k$$

is zero, we get a morphism $\Omega^{2n}(k) \to K$ (by the universal property of the kernel). This passes to a map

$$\mathrm{Hom}_{kG}(-, \Omega^{2n}(k))/\mathrm{Rad}^{\infty}_{kG}(-, \Omega^{2n}(k)) \to \mathrm{Hom}_{kG}(-, K)/\mathrm{Rad}^{\infty}_{kG}(-, K)$$

that is an isomorphism since both terms act as the kernel in the sequences of Rad^{∞} quotients. It follows from this that the irreducible morphisms to $\Omega^{2n}(k)$ and to (the summands of) K are the same, so that the summands of $\Omega^{2n}(k)$ and of K occupy the same positions in the Auslander–Reiten quiver. Thus K is indecomposable and the map $\Omega^{2n}(k) \to K$ is an isomorphism. We deduce that the sequence

$$0 \to \Omega^{2n}(k) \to U_{2n,0} \oplus U_{0,2n} \to k \to 0$$

is exact. □

We notice that the sequence $\mathcal{E}_n$ represents an element

$$\mu_n \in \mathrm{Ext}_{kG}(k, \Omega^{2n}(k)) \cong \underline{\mathrm{Hom}}_{kG}(k, \Omega^{2n-1}(k)) \cong \widehat{\mathrm{H}}^{1-2n}(G, k).$$

It is not necessary for our development, but perhaps interesting to note that, considered as an element in Tate cohomology $\widehat{\mathrm{H}}^{1-2n}(G, k)$, μ_n is perpendicular (under Tate duality) to the subspace of $\widehat{\mathrm{H}}^{2n-2}(G, k)$ spanned by the transfers from the proper 2-subgroups of the Sylow subgroup of G.

The important thing is that multiplication by μ_n induces a natural transformation from the identity functor to Ω^{2n-1} in the stable category $\mathbf{stmod}(kG)$. That is, we first choose a cocycle $\mu_n : k \to \Omega^{2n-1}(k)$ representing μ_n. The class of the cocycle as a map in the stable category is unique. Then for any M we have a composition map $\mu_{n,M}$ given by

$$M \xrightarrow{\cong} k \otimes M \xrightarrow{\mu_n \otimes 1} \Omega^{2n-1}(k) \otimes M \longrightarrow \Omega^{2n-1}(M)\ ,$$

where the first map sends m to $1 \otimes m$, and the last is the isomorphism in the stable category. This is well defined in the stable category and does not depend on the choice of a cocycle representing μ_n or the choice of a splitting $\Omega^{2n-1}(k) \otimes M \cong \Omega^{2n-1}(M) \oplus P$ for some projective module P. Thus we see that $\mu_{n,-}$ is an element of the graded center of the stable category of kG-modules.

Next we note the following relevant fact.

Proposition 4.2. *Suppose that $\phi : M \to N$ is a homomorphism of indecomposable kG-modules such that M and N do not lie in the same component of the Auslander–Reiten quiver. Then $\mu_{n,N}\phi = 0$, and $\Omega^{2n}(\phi)\mu_{n,M} = 0$ in the stable category* $\mathbf{stmod}(kG)$.

Proof. There is an isomorphism $\underline{\mathrm{Hom}}_{kG}(M, N) \cong \underline{\mathrm{Hom}}_{kG}(M \otimes N^*, k)$ that is natural in both M and N. Hence, letting $X = M \otimes N^*$, and $\theta : X \to K$ be the homomorphism corresponding to ϕ, it is only necessary to show that $\mu_n\theta = 0$ in $\underline{\mathrm{Hom}}_{kG}(X, \Omega^{2n-1}(k)) \cong$

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$\mathrm{Ext}_{kG}(X, \Omega^{2n}(k))$. That is, we need to prove that the map θ factors through the middle term of the sequence in the diagram:

$$\begin{array}{lcccccccc} & & & & & & X & & \\ & & & & & \swarrow^{\sigma} & \downarrow{\scriptstyle\theta} & & \\ \mathcal{E}_n : \quad 0 & \longrightarrow & \Omega^{2n}(k) & \longrightarrow & U_{2n,0} \oplus U_{0,2n} & \longrightarrow & k & \longrightarrow & 0. \end{array}$$

In other words, it must be shown that there exist two maps $\sigma_1 : X \to U_{2n,0}$ and $\sigma_2 : X \to U_{0,2n}$ such that

$$\theta = \sigma_1 \gamma'_{(2,0)} \cdots \gamma'_{(2n-2,0)} \gamma'_{(2n,0)} + \sigma_2 \gamma_{(0,2)} \cdots \gamma_{(0,2n-2)} \gamma_{(0,2n)}.$$

Observe that, by our hypotheses, no indecomposable direct summand Y of X is in the Auslander–Reiten component of the trivial module k. If it were otherwise, then Y would have odd dimension, implying that M and N would also have odd dimension [5]. Moreover, Y would be an endotrivial module, requiring that $Y \otimes N$ have only a single nonprojective summand. However, k is a direct summand of $N^* \otimes N$, and hence M must be the unique nonprojective direct summand of $Y \otimes N$. Recall from [2] that the Auslander–Reiten component of N consists of the nonprojective summands of $Y \otimes N$ for Y in the Auslander–Reiten component of k. Thus we would have that M is in the same Auslander–Reiten component as N, contradicting our hypotheses.

Because the row in the diagram

$$\begin{array}{ccccccccc} & & & & & & X & & \\ & & & & & \swarrow & \downarrow{\scriptstyle\theta} & & \\ 0 & \longrightarrow & \Omega^{2}(k) & \longrightarrow & U_{2,0} \oplus U_{0,2} & \longrightarrow & k & \longrightarrow & 0. \end{array}$$

is an almost split sequence, there are maps $\mu_1 : X \to U_{2,0}$ and $\mu_2 : X \to U_{0,2}$ such that $\theta = \gamma_{(2,0)}\mu_1 + \gamma_{(0,2)}\mu_2$. We can iterate this process. That is, in the next iteration, we write $\mu_1 = \gamma_{(4,0)}\nu_1 + \gamma_{(2,2)}\nu_2$ for $\nu_1 : X \to U_{4,0}$ and $\nu_2 : X \to U_{2,2}$ using the fact that $0 \to U_{4,2} \to U_{4,0} \oplus U_{2,2} \to U_{2,0} \to 0$ is an almost split sequence.

In this way, for some $m > n$, we write θ as a sum of maps of the form $\zeta\sigma$, where $\sigma : X \to U_{2m-2j,2j}$ and $\zeta : U_{2m-2j,2j} \to k$ is a composition of irreducible maps and $0 \le j \le m$. Next we note that $\gamma'_{(2i,2j-2)}\gamma_{(2i,2j)} = \gamma_{(2i-2,2j)}\gamma'_{(2i,2j)}$. Thus, since $m > n$, the map ζ factors either through $U_{2n,0}$ or through $U_{0,2n}$. It follows that θ factors through $U_{2n,0} \oplus U_{0,2n} \to k$, as asserted. This proves half of the proposition. The proof of the other half is dual to this one. □

Armed with this proposition, we can prove the main theorem for this example.

THEOREM 4.3. *Suppose that G is a finite group with a dihedral Sylow 2-subgroup of order at least 8, and that k is a field of characteristic 2. Suppose that $\mathcal{D}$ is a component of the stable Auslander–Reiten quiver of kG that contains a module of odd dimension. Then, for any $n > 0$, there exists a natural transformation $\psi : \mathrm{Id} \to \Omega^{2n-1}$ in the stable category* **stmod**(kG) *with the property that ψ is supported only on the set of modules in $\mathcal{D}$.*

Proof. Let M be a module in $\mathcal{D}$ having odd dimension. Recall that the collection of indecomposable modules in $\mathcal{D}$ coincides with the collection of nonprojective direct summands of modules of the form $M \otimes U_{2i,2j}$ for i and j in $\mathbb{Z}$ [2]. Hence every indecomposable module in $\mathcal{D}$ has odd dimension.

Now define the natural transformation ψ by the following rule. For M an indecomposable kG-module, let

$$\psi_M = \begin{cases} \mu_{n,M} & \text{if } M \text{ is in } \mathcal{D}, \\ 0 & \text{otherwise.} \end{cases}$$

To prove the theorem we must show that, given a homomorphism $\varphi : M \to N$, for M and N indecomposable modules, the diagram

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & N \\ \downarrow{\scriptstyle \psi_M} & & \downarrow{\scriptstyle \psi_N} \\ \Omega^{2n-1}(M) & \xrightarrow{\Omega^{2n-1}(\varphi)} & \Omega^{2n-1}(N) \end{array}$$

commutes. This is clear from the definitions if either both M and N are in $\mathcal{D}$ or both are not in $\mathcal{D}$. If one of M and N is in $\mathcal{D}$ and the other is not, then we need only appeal to Proposition 4.2. □

We now show that the natural transformation just constructed satisfies the conditions of Theorem 3.4, thereby showing that the circumstances of this theorem can actually arise in a nontrivial way.

Proposition 4.4. *The natural transformation $\psi : \mathrm{Id} \to \Omega^{2n-1}$ just constructed satisfies the conditions of Theorem 3.4. Moreover, if $f : V \to M$ is a map of indecomposable modules such that $\psi_M f \neq 0$, then f factors as a sum of composites of finitely many irreducible maps.*

Proof. We know when $C = \mathbf{stmod}(kG)$ that $\tau = \Omega^2$, $\Sigma = \Omega^{-1}$ and $\mathcal{S} = \Omega$. Thus $\mathbf{stmod}(kG)$ is a (-1)-Calabi–Yau category. The fact that for each indecomposable U, the only objects of the form $\Sigma^t U$ in the component of U lie in the same τ-orbit as U, as well as the fact that each mesh has at most two middle terms, follows from [7, 15].

We show that for all indecomposable modules M, the natural transformation $\mathrm{Hom}_C(-, \psi_M)$ has finite support. When M is not in $\mathcal{D}$ this is clear, so we suppose that M lies in $\mathcal{D}$. The construction of $\psi_M = \mu_{n,M} : M \to \Omega^{2n-1}(M)$ is that it is the third homomorphism in a triangle in $\mathbf{stmod}(kG)$ of the form

$$\Omega^{2n}(M) \to (U_{2n,0} \oplus U_{0,2n}) \otimes M \to M \to \Omega^{2n-1}(M)$$

corresponding to a short exact sequence of kG-modules

$$0 \to \Omega^{2n}(M) \to (U_{2n,0} \oplus U_{0,2n}) \otimes M \to M \to 0.$$

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The argument of Lemma 4.1 showed that the sequence of functors

$$0 \to \mathrm{Hom}_{kG}(-, \Omega^{2n}(M)) \\ \to \mathrm{Hom}_{kG}(-, (U_{2n,0} \oplus U_{0,2n}) \otimes M) \to \mathrm{Hom}_{kG}(-, M)$$

is exact, and the final cokernel has composition factors lying in the diamond of the Auslander–Reiten quiver determined by the modules $M, \Omega^{2n}(M), U_{n,0} \otimes M$ and $U_{0,n} \otimes M$, including the right-hand edge of this diamond, but not the left-hand edge or the modules $U_{n,0} \otimes M$ and $U_{0,n} \otimes M$. This cokernel is the image of $\mathrm{Hom}_{\mathbf{stmod}(kG)}(-, \psi_M)$, so that condition (1) of Corollary 2.11 is satisfied. This shows that $\mathrm{Hom}_{\mathcal{C}}(-, \psi_M)$ has finite support.

The final statement follows from part (5) of Corollary 2.11 and condition (2) of Proposition 2.9. If $\psi_M f \neq 0$, then $\psi_M f \xi$ is almost vanishing for some ξ. Thus $f\xi$ is a sum of composites of irreducible morphisms and hence so is f. □

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