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THE ARTIN RADICAL OF A NOETHERIAN RING

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Abstract

In this note we define the Artin radical of a Noetherian ring and describe some of its applications.

0. Introduction

In a Noetherian ring, the sum of all the Artinian right ideals is a two-sided ideal which has certain radical-like properties. More precisely, let R be a Noetherian ring and define the Artin radical A of R to be the sum of all the Artinian right ideals of R, then A is also the sum of all the Artinian left ideals of R and R/A has no non-zero Artinian one-sided ideals. In Section 1 we prove the basic properties of the Artin radical, including the useful fact that it is a complement both as a right ideal and also as a left ideal. The usefulness of the Artin radical in studying certain classes of Noetherian rings is shown in Section 2 where applications are made to Noetherian serial rings, and to injective one-sided ideals of Noetherian rings.

Throughout this paper all rings have identity element. "R is Noetherian (Artinian, serial, etc.)" means that R is both left and right Noetherian (Artinian, serial, etc.).

1. The Artin radical

The starting-point for this part of the theory of Noetherian rings is the following elegant result of T. H. Lenagan

THEOREM 1.1. (Lenagan (1975)) *Let R be a Noetherian ring and let I be an ideal of R which is Artinian as a right R-module, then I is also Artinian as a left R-module and the rings $R/r(I)$ and $R/l(I)$ are Artinian (where $r(I)$ and $l(I)$ denote the right and left annihilators of I).*

DEFINITION. *The Artin radical A, or more precisely $A(R)$, of a ring R is the sum of all the right ideals of R which are Artinian as right R-modules.*

We shall use Theorem 1.1 to show, among other things, that the Artin radical of a Noetherian ring is also the sum of all the Artinian left ideals. However, we first need the following result of S. M. Ginn and P. B. Moss which gives a way of proving that certain rings are Artinian.

THEOREM 1.2. (Ginn and Moss (1976)) *Let R be a Noetherian ring and suppose that the right socle E of R is essential as a right ideal of R, then R is Artinian.*

PROOF. Because E is essential as a right ideal, the left annihilator $l(E)$ of E is the right singular ideal of R, so that $l(E)$ is contained in the nilpotent radical N of R. By Theorem 1.1, $R/l(E)$ is Artinian. Therefore R/N is Artinian and the result now follows by a standard argument.

We can now list the basic properties of the Artin radical of a Noetherian ring.

THEOREM 1.3. *Let R be a Noetherian ring.* (i) *The Artin radical A of R is a two-sided ideal which is the unique largest Artinian right ideal of R and which is also the unique largest Artinian left ideal of R.*

(ii) *$A(R/A(R)) = 0$, i.e. R/A has no non-zero Artinian one-sided ideals.*

(ii) *A is a complement right ideal and a complement left ideal, i.e. A has no proper essential extensions in R either as a right R-module or as a left R-module.*

(iv) *A is a right annihilator and a left annihilator.*

PROOF. (i) Recall that A was defined to be the sum of all Artinian right ideals of R, i.e. of all the right ideals of R which are Artinian as right R-modules. Let S and T be Artinian right ideals of R. From the fact that $S + T$ is a homomorphic image of the external direct sum of S and T it follows that $S + T$ is Artinian. Also rS is a homomorphic image of S and so is Artinian for every element r of R. It now follows easily that A is a two-sided ideal and that A is the unique largest Artinian right ideal of R. Theorem 1.1 shows that the definition of A is right-left symmetric.

(ii) This follows immediately from the fact that an extension of an Artinian module by an Artinian module is Artinian.

(iii) Let I be a right ideal of R which contains an essential Artinian submodule (i.e. the socle of I is an essential submodule of I). Because RI is a finitely-generated left ideal of R, or because R satisfies the descending chain condition for right annihilators, there is a finite subset $x_1, \cdots, x_n$ of I such that

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$r(I) = r(x_1) \cap \cdots \cap r(x_n)$. This enables us to define a right R-module embedding of $R/r(I)$ into I^n by mapping $x + r(I)$ to $(x_1x, \cdots, x_nx)$. But I^n has an essential Artinian submodule. Hence the right socle of the Noetherian ring $R/r(I)$ is essential as a right ideal. Therefore $R/r(I)$ is an Artinian ring, by Theorem 1.2, so that I is Artinian. Thus an essential extension in R of an Artinian right ideal is itself Artinian, from which it follows that A is a complement as a right ideal and, by symmetry, as a left ideal. It is also a consequence of this argument that a right ideal I or R is Artinian if and only if $R/r(I)$ is an Artinian ring.

(iv) By Theorem 1.1, $R/r(A)$ is an Artinian ring. But $l(r(A))$ is a finitely-generated right $R/r(A)$-module. Therefore $l(r(A))$ is an Artinian right R-module. Combined with (i) this gives $A = l(r(A))$. Since A is also the unique largest Artinian left ideal of R it follows by symmetry that $A = r(l(A))$.

The Artin radical of a Noetherian ring can be the zero ideal, as in the ring of integers or in a simple Noetherian ring which is not Artinian, but we shall show by means of the applications in the next section that the Artin radical can be a useful tool in studying certain classes of Noetherian rings.

REMARK. An R-module K is said to be 1-critical if K is not Artinian and every proper factor module of K is Artinian. It is not hard to show that if M is a Noetherian module then the sum of all Artinian submodules of M is the intersection of all the submodules X such that M/X is an essential extension of a 1-critical module.

Thus the Artin radical of a Noetherian ring R is the intersection of all the right (left) ideals I such that R/I is an essential extension of a 1-critical module. However the following example shows that the Artin radical is not the intersection of all right ideals I such that R/I is 1-critical. Let S be the ring of integers localised at 2. Let M be the unique maximal ideal of S and let

$$R = \left\{ \begin{bmatrix} s & m \\ o & s \end{bmatrix}; s \in S, m \in M \right\}$$

with the usual operations then R is a commutative Noetherian ring with zero Artin radical but the nilpotent radical of R is the only ideal I of R such that R/I is 1-critical.

The definition of the Artin radical given above applies to any ring, but it appears that the theory works smoothly only for Noetherian rings. For example, let R be the ring of all matrices of the form

$$\begin{bmatrix} a & b \\ o & c \end{bmatrix}$$

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where a is an integer and b and c are rational numbers (see Small 1965 for details), then R is right but not left Noetherian, the right socle of R is essential as a right ideal but R is not right Artinian, and the sum of all the Artinian right ideals of R is an essential right ideal of R but R has no non-zero Artinian left ideals.

2. Applications

In this section we give three applications of the Artin radical in the theory of Noetherian rings, firstly to Noetherian rings with Artinian quotient rings, secondly to Noetherian serial rings, and thirdly to injective right ideals of Noetherian rings.

The first application, which we quote without proof, is the one made by Ginn and Moss (1976) to prove the following remarkable result:

Let R be a Noetherian ring which has an Artinian classical quotient ring, then the Artin radical of R is a direct summand of R.

The second application we give is to Noetherian serial rings. Using the terminology of Warfield (1975), a ring R is said to be right serial if R is the direct sum of (necessarily finitely many) right ideals $e_1R, \cdots, e_nR$ such that, for each i, the set of right R-submodules of e_iR is linearly ordered. We shall use the Artin radical to give a new proof of Warfield's decomposition theorem for Noetherian serial rings, but before doing that we prove that Noetherian right serial rings satisfy Jacobson's conjecture.

THEOREM 2.1. *Let R be a Noetherian right serial ring and let J be the Jacobson radical of R. Set $J' = \bigcap_{n=1}^{\infty} J^n$, then $J' = 0$.*

PROOF. Suppose that R is a counter-example to the theorem, i.e. that $J' \neq 0$. Because $J' \neq 0$ we have $J'J \neq J'$, by Nakayama's Lemma. Set $S = R/J'J$, then S is also a Noetherian right serial ring. Thus we could use S instead of R as a counter-example. We therefore assume without loss of generality, that $J'J = 0$. We can write $R = e_1R \oplus \cdots \oplus e_nR$ where the submodules of each e_iR are linearly ordered. Thus e_iJ is the unique maximal submodule of e_iR, so that each e_iR/e_iJ is a simple right R/J-module. Since $R/J \cong e_1R/e_1J \oplus \cdots \oplus e_nR/e_nJ$ it follows that R/J-is an Artinian ring. But $J'J = 0$. Therefore J' is a finitely generated right R/J-module. Hence $J' \subseteq A$ where A is the Artin radical of R.

Now we have $A = e_1A \oplus \cdots \oplus e_nA$. Hence for each i, e_iA is a direct summand of the complement right ideal A so that e_iA is also a complement right ideal of R. But e_iA is contained in the uniform right ideal e_iR. Therefore for each i we must have either $e_iA = 0$ or $e_iA = e_iR$. We shall now show that

$e_iJ' = 0$ for all i. This is trivial if $e_iA = 0$. On the other hand, if $e_iA = e_iR$ then e_iR is Artinian. So the chain $e_iR \supseteq e_iJ \supseteq e_iJ^2 \supseteq \cdots$ must stop. Nakayama's Lemma now gives $e_iJ^k = 0$ for some k. Hence $e_iJ' = 0$.

Thus $e_iJ' = 0$ for all i and therefore $J' = 0$ which is the required contradiction.

REMARK. It follows from the above proof that if A is the Artin radical of the Noetherian right serial ring R, then $A = eR$ for some idempotent e. This is becasue for each i we have either $e_iA = 0$ or $e_iA = e_iR$.

EXAMPLE. The following ring R is Noetherian right serial but not left serial. Let S be the ring of integers localised at a non-zero prime ideal, let M be the unique maximal ideal of S, let $F = S/M$ and set

$$R = \begin{bmatrix} F & F \\ O & S \end{bmatrix}.$$

REMARK. As a consequence of Theorem 2.1 it can be shown that any Noetherian right serial ring is fully right bounded.

We next give a new proof of a theorem of R. B. Warfield.

THEOREM 2.2. (Warfield (1975)) *A Noetherian serial ring is a direct sum of Artinian serial rings and of prime Noetherian serial rings.*

PROOF. We suppose that R is an indecomposable Noetherian serial ring which is not Artinian; we must show that R is prime. As in the above remark, the Artin radical A or R is idempotently generated both as a left ideal and as a right ideal. Hence A is a direct summand of R. But R is indecomposable and $A \neq R$. Therefore $A = 0$. We can write $R = e_1R \oplus \cdots \oplus e_nR$ where the submodules of each e_iR are linearly ordered. The factor modules of the chain $e_iR \supseteq e_iJ \supseteq e_iJ^2 \supseteq \cdots$ are modules over the semi-simple Artinian ring R/J and it follows from the linear ordering that each e_iJ^k/e_iJ^{k+1} is either simple or zero. But $\bigcap_{k=1}^{\infty} e_iJ^k = 0$ by Theorem 2.1. Therefore $e_iR \supseteq e_iJ \supseteq e_iJ^2 \supseteq \cdots \supseteq 0$ are the only submodules of e_iR. Hence e_iR/U is Artinian for any non-zero submodule U of e_iR. Let K be an essential right ideal of R, then K contains a non-zero submodule of e_iR for each i, so that R/K is Artinian. In particular, $R/l(N)$ is Artinian where $l(N)$ is the left annihilator of the nilpotent radical N or R. Therefore N is an Artinian left R-module and so is 0 because $A = 0$. Hence R is semi-prime.

Next, let P be a minimal prime ideal of R, then P is an annihilator ideal. Hence P is a complement right ideal of R. As in the remark after Theorem 2.1 we see that P is generated as a right ideal by an idempotent element. By symmetry, P is also generated as a left ideal by an idempotent. Thus P is a direct summand of R. Therefore $P = 0$, as required.

Similar methods can be used to give information about the structure (in terms of triangular matrices) of Noetherian right serial rings.

The third and final application is to injective one-sided ideals of Noetherian rings.

THEOREM 2.3. *Let R be a Noetherian ring with an injective right ideal I, then I is Artinian.*

PROOF. Because I is a direct sum of indecomposable (and hence uniform) injective right ideals of R we may suppose that I is itself a uniform injective right ideal. Let U be a non-zero submodule of I such that $r(U)$ is maximal, then $r(U)$ is a prime ideal. Set $P = r(U)$ and $S = R/P$. As in the proof of (iii) of Theorem 1.3, there is a positive integer n such that S embeds (as a right R-module) in U^n. Because S is a Noetherian prime ring it has a classical quotient ring Q. But the embedding of S into U^n can be extended to a right R-module embedding of Q into I^n. Hence Q is finitely-generated as a right R-module and it follows that $Q = S$. Therefore U, and hence also I, is Artinian (cf. proof of Theorem 1.3 (iii)).

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References

S. M. Ginn and P. B. Moss (1976), 'A decomposition theorem for Noetherian orders in Artinian rings', to appear, *Bull. London Math. Soc.*

T. H. Lenagan (1975), 'Artinian ideals in Noetherian rings', *Proc. Amer. Math. Soc.* **(51)2**, 499–500.

L. W. Small (1965), 'An example in Noetherian rings', *Proc. Nat. Acad. Sci. USA* **54**, 1035–1036.

R. B. Warfield Jr. (1975), 'Serial rings and finitely presented modules', *J. of Algebra* **37**, 187–222.

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