EULER LINES IN INFINITE DIRECTED GRAPHS

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1. Introduction. While the contents of the author's doctoral thesis **(4)** have, owing to their lengthy nature, been published only in small part **(5**, §2; **6**; **7)**, the absence from the literature of graph theory of any characterization of infinite directed graphs with Euler lines seems to constitute a definite gap that prompts the publication in the present paper of some further material from **(4)**. The main results characterizing such directed graphs will be obtained in §§2 and 3. In §4, we shall indicate an alternative (and perhaps better) formulation of one of these results, some extensions obtained in **(4)**, and some comparisons between parallel results for undirected and directed graphs. A familiarity with the definitions and results of **(7)** will be assumed in §4, but not before.

The cardinal number of a set A will be denoted by $|A|$. A is *finite* or *infinite* according as $|A|$ is finite or infinite, and is *enumerable* if $|A| = \aleph_0$. The notation $\{a_1, \ldots, a_n\}$ will denote the set with *distinct* elements $a_1, \ldots, a_n$.

A *graph* G consists of two disjoint sets $V(G)$, $E(G)$ and a relationship associating with each $\lambda \in E(G)$ an unordered pair of distinct or coincident elements of $V(G)$ which λ is said to *join*. (We permit two elements of $V(G)$ to be joined by more than one element of $E(G)$.) The elements of $V(G)$ are *vertices* of G and the elements of $E(G)$ are its *edges*. The letter G will always denote a graph. G is *finite* (*infinite*, *enumerable*) if $V(G) \cup E(G)$ is finite (infinite, enumerable). An edge λ and vertex ξ are *incident* if λ joins ξ to itself or to another vertex. The *degree* $d(\xi)$ of a vertex ξ joined to itself by a edges and to other vertices by b edges is the cardinal number $2a + b$. A *subgraph* of G is a graph H such that $V(H) \subset V(G)$, $E(H) \subset E(G)$, and each edge of H joins the same vertices in H as in G. If H is a subgraph of G, we shall also say that H is *contained* in G and write $H \subset G$. A collection of subgraphs of G are *disjoint* (*edge-disjoint*) if no two of them have a common vertex (edge). The *union* (*intersection*) of a number of subgraphs of G is the subgraph whose vertex-set is the union (intersection) of the vertex-sets of the given subgraphs and whose edge-set is the union (intersection) of their edge-sets. Unions and intersections of subgraphs are denoted by the same symbolism as unions and intersections of sets. The union of a collection of edge-disjoint subgraphs will also be spoken of as their *edge-disjoint union*. The *empty* graph $\square$ has $V(\square) = E(\square) = \emptyset$. G is *connected* if it is non-empty and is not the union of two disjoint non-empty subgraphs. Any graph G is the union of a unique collection of disjoint connected subgraphs, called the *components* of G. The

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word *sequence* will in this paper denote either a finite sequence with at least one term or an infinite sequence of one of the types $a_1, a_2, \ldots$ or $\ldots, a_{-2}, a_{-1}$ or $\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots$. An infinite sequence will be called *right-infinite*, *left-infinite*, or *endless* according as it is of the first, second, or third of these types respectively: both left-infinite and right-infinite sequences will be termed *one-ended*. If s, t are sequences such that the last term of s is the same as the first term of t, st will denote the sequence obtained by writing out the sequence s with its last term omitted, followed by the sequence t. For instance, if s is 1, 3, 7, 4 and t is 4, 5, 3 then st is 1, 3, 7, 4, 5, 3. Extending this definition by associativity defines the "product" $s_1 s_2 \ldots s_n$ of n ($\geqslant 3$) sequences $s_1, \ldots, s_n$ such that the last term of s_{i-1} is the same as the first term of s_i for $i = 2, \ldots, n$. A *track* of G is a sequence s whose terms are alternately vertices and edges of G, starting and ending if at all with a vertex, such that each term of s that is an edge joins the terms immediately preceding and following it. (A sequence with just one term, that term being a vertex, counts as a track; but there is no such thing as a track with 0 terms.) A track in which no edge (vertex) of the graph appears more than once is a *path* (*way*). For instance, in Figure 1 the track

$$\xi, \lambda, \zeta, \theta, \sigma, \nu, \eta, \mu, \zeta, \psi, \tau$$

is a path but not a way, and the track

$$\xi, \lambda, \zeta, \psi, \tau, \phi, \sigma$$

is a way. The subgraph of G formed by the vertices and edges in a path p of G will be said to be *derived* from p and denoted by P. A subgraph of G is a

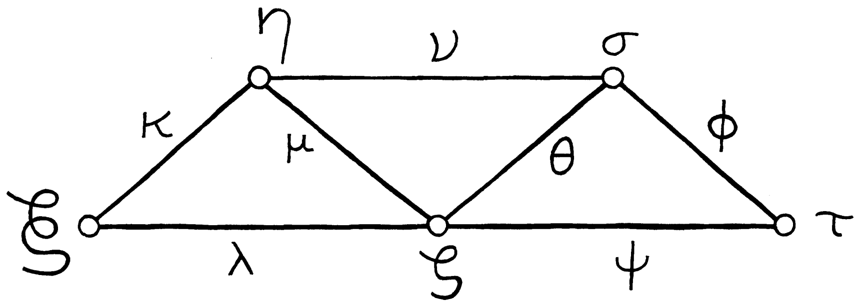

FIGURE 1.

pathoid (*one-ended pathoid*, *endless pathoid*) if it is derivable from a path (one-ended path, endless path); similarly, when any other type of path has been defined, the definition of the corresponding type of pathoid is immediate. The same subgraph can be both a one-ended and an endless pathoid since it may be derivable from more than one path. If X, Y are subsets of $V(G)$, $\bar{X}$ will denote $V(G) - X$, $X \circ Y$ will denote the set of those edges of G that join an element of X to an element of Y, and $X\delta$ will denote $X \circ \bar{X}$. For example,

$\{\xi, \eta\}\delta = \{\lambda, \mu, \nu\}$ in Figure 1. X is a *divisor* of G if $X\delta$ is finite. For example,

$$\{\xi_1, \eta_1, \xi_2, \eta_2, \xi_3, \eta_3, \ldots\}$$

is a divisor of the graph in Figure 2 but $\{\xi_2, \xi_4, \xi_6, \ldots\}$ is not. Whenever two or more graphs are under consideration and one of them is denoted by G, all graph-theoretic terminology and notation—such as "degree," $d(\xi)$, $X\delta$, $\bar{X}$—will relate to G unless the contrary is indicated. If it is necessary to relate such notations to some other graph H (such as a subgraph of G), this will be done by means of suffixes, e.g. $d_H(\xi)$, $X\delta_H$. If $\xi \in V(G)$, c_ξ will denote the integer-valued function with domain $V(G)$ which takes the value 1 on ξ and 0 on every other vertex of G. A *splitting* of G is a finite set $\{H_1, \ldots, H_l\}$ of disjoint infinite subgraphs of G such that G is the union of these and a finite subgraph: any finite subgraph H such that $G = H_1 \cup \ldots \cup H_l \cup H$ will be called a *completion* of the splitting. A splitting with l elements is an *l-splitting*. For any positive integer l, we call G *l-separable* if it possesses an l-splitting, and *l-coherent* if it does not possess an $(l + 1)$-splitting. (If G is l-coherent, it is clearly also l'-coherent for every $l' > l$.) The graph of Figure 2 is l-separable for $l = 1, 2, 3, 4$ but is 4-coherent. This graph has a 4-splitting

$$\{H_{\xi\lambda}, H_{\eta\mu}, H_{\zeta\nu}, H_{\tau\pi}\},$$

where the notation $H_{\chi\rho}$ means the subgraph with vertices $\chi_1, \chi_2, \chi_3, \ldots$ and edges $\rho_2, \rho_3, \rho_4, \ldots$. There are infinitely many possible completions of this splitting, one of them being the subgraph H with vertices $\omega, \xi_1, \xi_2, \eta_1, \zeta_1, \tau_1$ and edges $\lambda_1, \lambda_2, \mu_1, \nu_1, \pi_1$. A set such as $\{H_{\xi\lambda} \cup H_{\eta\mu}, H_{\zeta\nu}, H_{\tau\pi}\}$ would be a 3-splitting of this graph with completion H.

We define a *directed graph* or *digraph* to be a graph G such that each edge of G is an ordered triple whose second and third components are the vertices joined by the edge—i.e., an edge (ϵ, ξ, η) joins the vertices ξ and η (Figure 3). If G is a digraph and $\lambda = (\epsilon, \xi, \eta) \in E(G)$, we call ξ the *tail* of λ and η its *head* and write $\xi = \lambda t$, $\eta = \lambda h$. We also say that λ is "*oriented in the direction from λt to λh.*" By defining a digraph to be a special type of graph, we ensure that all definitions relating to graphs apply immediately to digraphs. (Obviously, however, a concept defined for graphs in general will not, when applied to digraphs, take any particular account of the directions of orientation of their edges.) A track s in a digraph is a *ditrack* if each term of s that is an edge is immediately preceded in s by its tail and (therefore) immediately followed by its head. A *dipath* (*diway*) is a ditrack that is a path (way). In Figure 3, the sequence $\xi, \lambda, \eta, \mu, \tau, \nu, \zeta$ is a diway, whereas $\xi, \lambda, \eta, \pi, \zeta, \nu, \tau$ is a way but not a diway. An *Euler dipath* of G is a dipath p in G such that $P = G$. For instance, the digraphs in Figures 4 and 5 have Euler dipaths in which the edges occur in the order indicated by the numbering. If X, Y are subsets of $V(G)$, $X \rhd Y$ will denote $\{\lambda \in E(G) | \lambda t \in X, \lambda h \in Y\}$. X is *out-biased* if $X \rhd \bar{X}$ is infinite and $\bar{X} \rhd X$ is finite. G is *biased* if $V(G)$ has an out-biased subset. The digraph of Figure 6, which is biased, may be contrasted

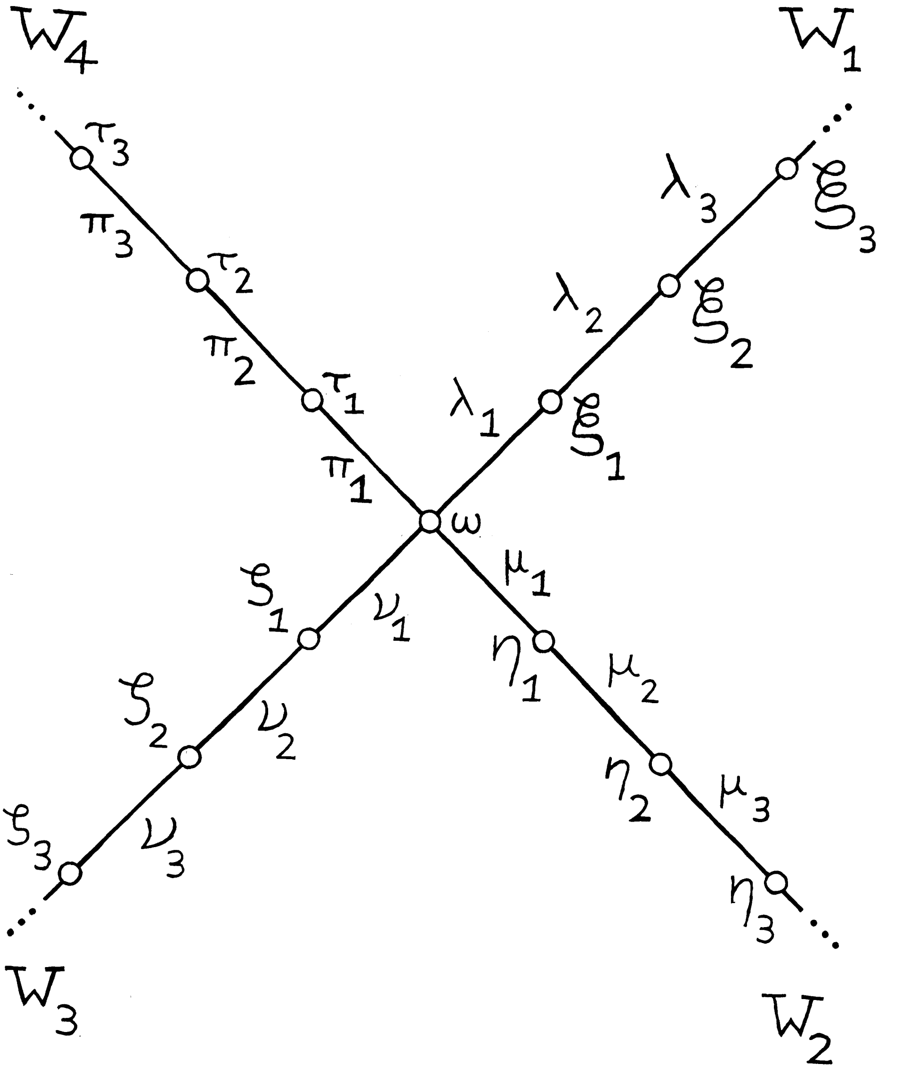

FIGURE 2.

with the unbiased digraph of Figure 7. The *flux* $F(X)$ of (or "out of") a divisor X of G is $|X \rhd \bar{X}| - |\bar{X} \rhd X|$. An *exit* (*entry*) of a vertex ξ is an edge with tail (head) ξ; the cardinal number of the set of exits (entries) of ξ will be denoted by $x(\xi)$ $(e(\xi))$. G is *solenoidal* if $x(\xi) = e(\xi)$ for every $\xi \in V(G)$ (Figures 8, 9, 11). If $\xi \in V(G)$ and $x(\zeta) = e(\zeta) + c_\xi(\zeta)$ for every $\zeta \in V(G)$, we shall call G *ξ-solenoidal* (Figure 5). If ξ, η are (not necessarily distinct) vertices of G and $x(\zeta) + c_\eta(\zeta) = e(\zeta) + c_\xi(\zeta)$ for every $\zeta \in V(G)$, we shall call G *$\xi\eta^{-1}$-solenoidal* (Figure 4).

The main theorems of this paper are as follows:

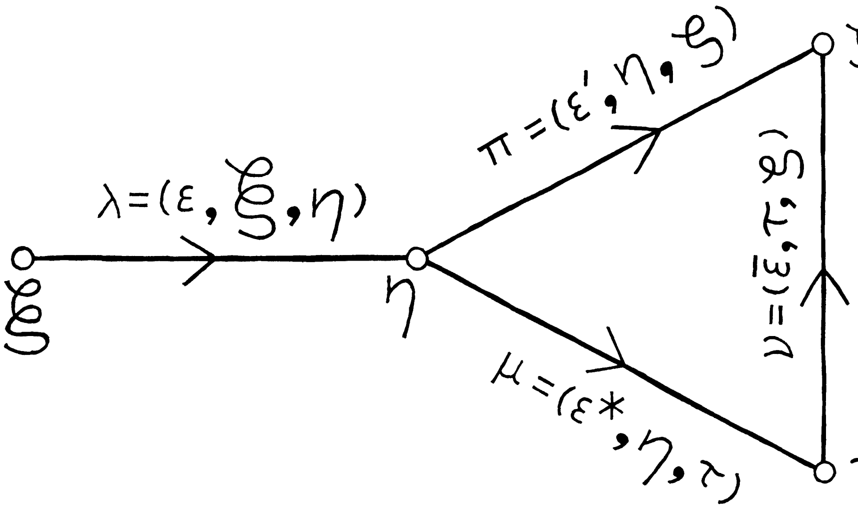

FIGURE 3.

THEOREM 1. *Let ξ be a vertex of a digraph G. Then G has a right-infinite Euler dipath with first term ξ if and only if G is enumerable, connected, ξ-solenoidal,* 1-*coherent, and unbiased.*

THEOREM 2. *Necessary and sufficient conditions for a digraph to have an endless Euler dipath are that it be enumerable, connected, solenoidal, unbiased, and* 2-*coherent and that, if* 2-*separable, it possess a divisor with flux* 1.

Necessary and sufficient conditions for a digraph G to have a left-infinite Euler dipath with a prescribed last term are, of course, deducible from Theorem 1

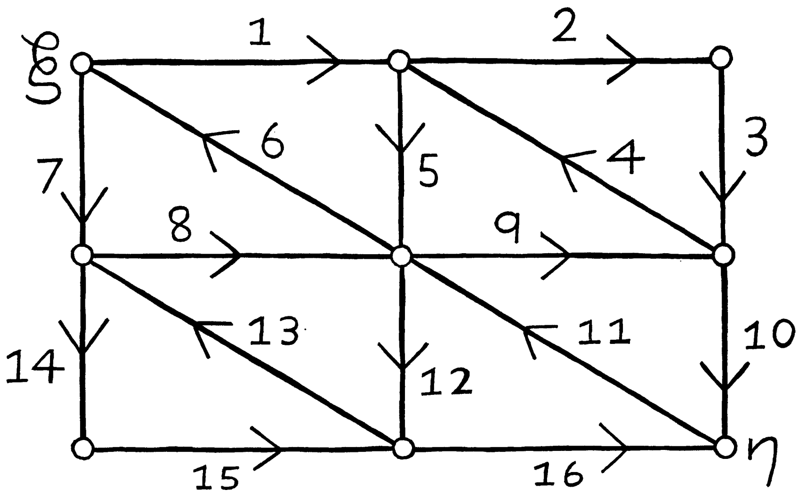

FIGURE 4.

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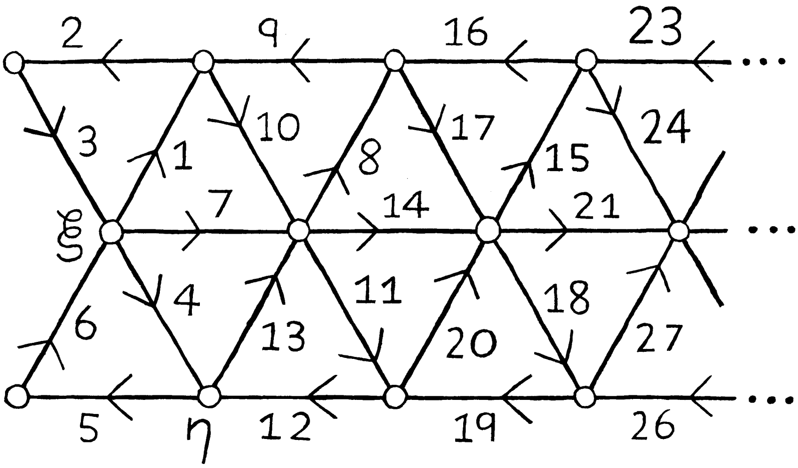

FIGURE 5.

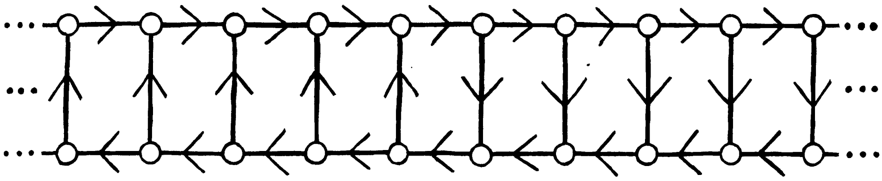

FIGURE 6.

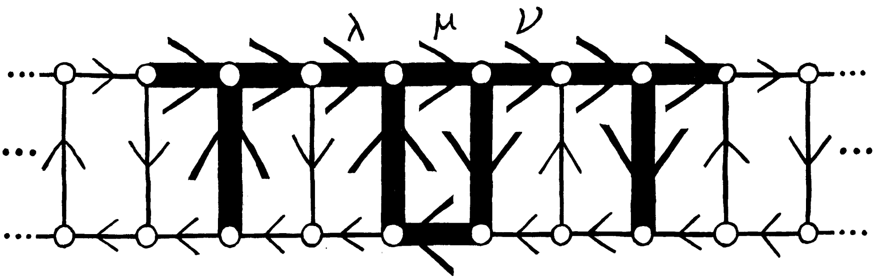

FIGURE 7.

by reversing the orientations of the edges of G (i.e., forming a new digraph with the same vertices in which each edge (ϵ, η, ζ) of G is replaced by an edge of the form (ϵ', ζ, η)).

Figure 5 depicts a digraph satisfying the conditions of Theorem 1, and an Euler dipath in this digraph with first term ξ is indicated by the numbering of the edges. The digraphs of Figures 8 and 9 satisfy the conditions of Theorem 2: the latter is 2-separable and the former is not. In Figure 9, the set of all vertices to the left of the broken line (drawn across the figure from top to bottom) is a divisor with flux 1. An endless Euler dipath in this digraph is given by adapting

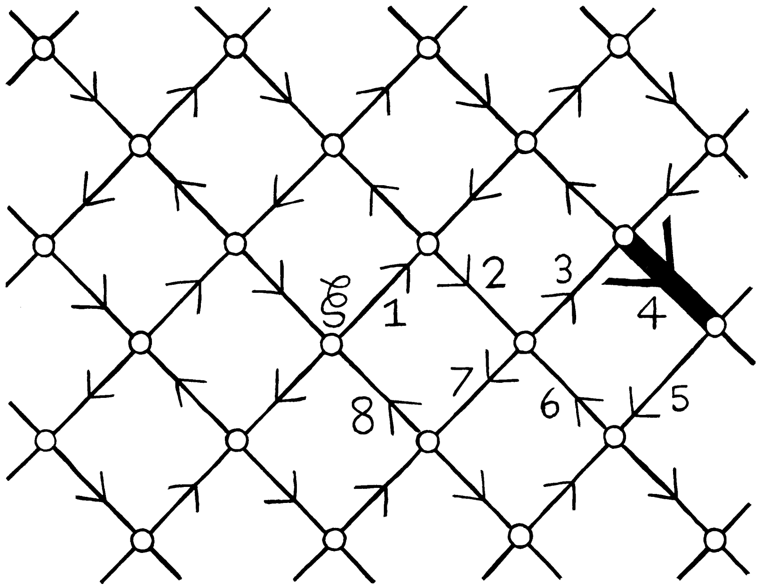

FIGURE 8.

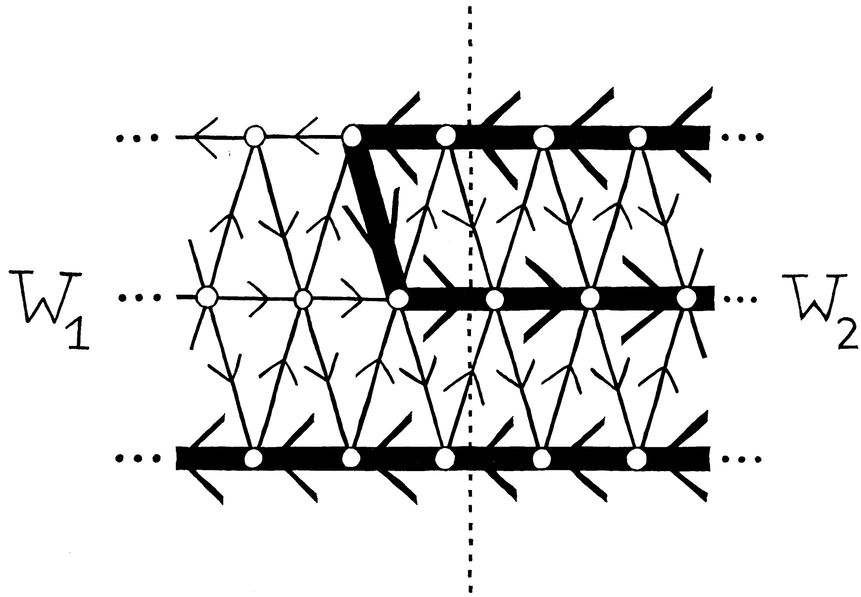

FIGURE 9.

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the construction indicated by the edge-numbering in Figure 5. The digraph of Figure 8 has an endless Euler dipath that can be described pictorially as follows. Suppose the vertices of the digraph to be the points (m, n) of the plane such that m, n are odd integers and $m \equiv n \pmod 4$. Then two edges incident with a vertex (m, n) will be successive edges in the Euler dipath if and only if either (i) $|m| \leqslant |n|$ and the two edges lie on the same side of the line $y = n$ or (ii) $|m| > |n|$ and the two edges lie on the same side of the line $x = m$.

2. Proof of Theorem 1. *Definitions.* The notation $\phi . A$ will stand for $\sum_{x \in A} \phi(x)$. If $L \subset E(G)$, $L\dagger$ will denote the subgraph of G formed by the elements of L and the vertices incident with them, and $G - L$ will denote the subgraph with vertex-set $V(G)$ and edge-set $E(G) - L$. If $X \subset V(G)$, X^* will denote the subgraph with vertex-set X and edge-set $X \circ X$. We shall call X *essential* if $X \circ V(G)$ is infinite and *inessential* if $X \circ V(G)$ is finite. If ξ is a vertex of finite degree in a digraph, $f(\xi)$ will denote $x(\xi) - e(\xi)$; we regard $f(\xi)$ as undefined for a vertex ξ of infinite degree. We note that, if X is an inessential set of vertices of G, the degrees of these vertices must be finite and only finitely many of them can differ from zero, so that, if G is a digraph, $f(\xi)$ is defined for every $\xi \in X$ and is non-zero for only finitely many such ξ. Consequently, $f . X$ is always well defined when X is an inessential set of vertices of a digraph. A *$\xi\eta$-ditrack* (*$\xi\eta$-dipath*, *$\xi\eta$-diway*) is a ditrack (dipath, diway) with first term ξ and last term η. A vertex η of a digraph is *accessible* from a vertex ξ if there exists a $\xi\eta$-ditrack in the digraph. For instance, in Figure 3 every vertex is accessible from ξ and no vertex except ζ itself is accessible from ζ, and in Figure 6 every vertex is accessible from every vertex.

LEMMA 1. *If k is a positive integer and G is the union of k one-ended pathoids, then G is k-coherent.*

Proof. Let the above pathoids be $P_1, \ldots, P_k$. Let $\{H_1, \ldots, H_l\}$ be an l-splitting of G; then this splitting has a completion H and, since H is finite, each P_i is the union of a one-ended pathoid Q_i having no edge in common with H and a finite pathoid R_i. Since the Q_i include no edges of H, each of them is contained in $H_1 \cup \ldots \cup H_l$ and therefore (being connected) in a single H_j. It follows that any H_j which contained no Q_i would be disjoint from $Q_1 \cup \ldots \cup Q_k$ and therefore contained in $R_1 \cup \ldots \cup R_k$, which is impossible since a finite graph cannot contain an infinite one. Therefore each H_j contains at least one Q_i and hence $l \leqslant k$.

LEMMA 2. *Any dipathoid is unbiased.*

Proof. If G is a dipathoid derivable from a dipath p and $X \subset V(G)$, it is clear that the elements of $X\delta$ form a subsequence of p whose terms belong alternately to $X \rhd \bar{X}$ and $\bar{X} \rhd X$; hence $X \rhd \bar{X}$ cannot be infinite if $\bar{X} \rhd X$ is finite.

If the digraph G of Theorem 1 has a right-infinite Euler dipath with first term ξ, it will obviously be enumerable, connected, and ξ-solenoidal, and will by Lemmas 1 and 2 be 1-coherent and unbiased; thus the necessity of the conditions in the theorem is established.

LEMMA 3. *If G is 1-coherent and X is a divisor of G, then either X or $\bar{X}$ is inessential.*

Proof. If X is essential, $X \circ V(G)$ is infinite and therefore, since $X\delta$ is finite, $X \circ X$ is infinite and consequently X^* is infinite. Similarly $\bar{X}^*$ is infinite if $\bar{X}$ is essential. Hence, if X and $\bar{X}$ were both essential, $\{X^*, \bar{X}^*\}$ would be a 2-splitting of G with completion $(X\delta)\dagger$, and G would not be 1-coherent.

LEMMA 4. *If G is a digraph, then $F(X) = f.X$ for every inessential $X \subset V(G)$.*

Proof. Since an edge contributes 0, 1, or -1 to $f.X$ according as it belongs to $(X \circ X) \cup (\bar{X} \circ \bar{X})$, $X \rhd \bar{X}$, or $\bar{X} \rhd X$ respectively, we infer that $f.X = |X \rhd \bar{X}| - |\bar{X} \rhd X|$.

If G is as shown in Figure 10 and X is the set of vertices inside the broken contour, then

$$f.X = 4 + 2 - 1 - 2 = 3 \quad \text{and} \quad F(X) = 4 - 1 = 3.$$

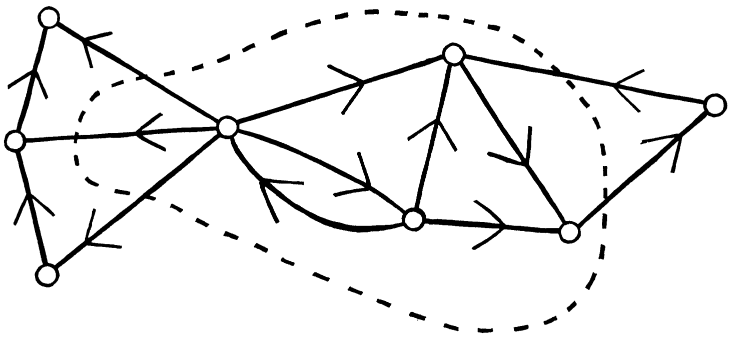

FIGURE 10.

COROLLARY 4A. *$f.\,V(K) = 0$ if K is a finite component of G.*

COROLLARY 4B. *If G is solenoidal and 1-coherent and X is a divisor of G, then $F(X) = 0$.*

Proof. By Lemma 3, either X or $\bar{X}$ is inessential. Since G is solenoidal, Lemma 4 gives $F(X) = f.X = 0$ in the former case and $-F(X) = f.\bar{X} = 0$ in the latter.

LEMMA 5. *A finite digraph has a closed Euler dipath if and only if it is connected and solenoidal.*

Proof. See (**3**, chap. 2; **1**, chap. 17; or **8**, chap. 3).

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LEMMA 6. *Let ξ, η be vertices of a finite digraph G. Then G is a $\xi\eta$-dipathoid if and only if it is connected and $\xi\eta^{-1}$-solenoidal.*

(For instance, the connected $\xi\eta^{-1}$-solenoidal digraph of Figure 4 is a $\xi\eta$-dipathoid derivable from the $\xi\eta$-dipath in which the edges of the digraph occur in the order indicated by the numbering.)

Proof. We may assume (although it does not really affect the argument) that $\xi \neq \eta$ since otherwise the assertion of Lemma 6 is equivalent to that of Lemma 5. It is obvious that a $\xi\eta$-dipathoid must be connected and $\xi\eta^{-1}$-solenoidal. Conversely, if G is connected and $\xi\eta^{-1}$-solenoidal, the addition of an edge λ with tail η and head ξ converts G into a connected solenoidal digraph that has, by Lemma 5, a closed Euler dipath p. If q, r are the portions of p preceding and following the term λ respectively, then rq is a $\xi\eta$-dipath from which G is derivable.

LEMMA 7. *Let $G = F \cup H$, where G is a digraph and F, H are subgraphs of G and F is finite. Then H is biased if and only if G is biased.*

The proof is left to the reader.

LEMMA 8. *If a vertex η is accessible from a vertex ξ in a digraph G, then there exists a $\xi\eta$-diway in G.*

Proof. A $\xi\eta$-ditrack with the minimum number of terms will clearly be a $\xi\eta$-diway.

LEMMA 9. *If $\xi \in V(G)$, $\lambda \in E(G)$, and G is a connected ξ-solenoidal 1-coherent unbiased digraph, then λ is a term of some finite dipath with first term ξ.*

Proof. Let A be the set of all vertices accessible from ξ. It is clear that $A \rhd \bar{A} = \emptyset$, and therefore, since G is unbiased, $\bar{A} \rhd A$ is finite. Therefore $A\delta$ is finite. Hence, by Lemma 3, either A or $\bar{A}$ is inessential. But, if A were inessential, the facts that G is ξ-solenoidal and $\xi \in A$ would imply that $f.A = 1$ and hence by Lemma 4 that $F(A) = 1$, which is impossible since $A \rhd \bar{A} = \emptyset$. Therefore $\bar{A}$ is inessential. Since G is ξ-solenoidal and $\xi \in A$, it follows that $f.\bar{A} = 0$. Therefore $F(\bar{A}) = 0$ by Lemma 4. Since $F(\bar{A}) = 0$ and $A \rhd \bar{A} = \emptyset$, it follows that $\bar{A} \rhd A$ must also be void and hence that $A\delta = \emptyset$, which, since G is connected and $\xi \in A$, implies that $A = V(G)$. Therefore, in particular, $\lambda t \in A$ and so by Lemma 8 there exists a $\xi(\lambda t)$-diway w in G. Since λ is clearly not a term of w, adding the terms λ, λh at the right-hand end of w gives the required dipath.

LEMMA 10. *Let $\xi \in V(G)$, $\lambda \in E(G)$, and G be an infinite connected ξ-solenoidal 1-coherent unbiased digraph. Then we can find a vertex η of G and a $\xi\eta$-dipath q such that $\lambda \in E(Q)$ and G is the edge-disjoint union of Q and an infinite connected η-solenoidal 1-coherent unbiased digraph.*

(For example, let G be the digraph of Figure 5, ξ be as shown in the figure, and λ be the edge numbered 10. Suppose that we take η to be as shown in the figure and q to be the dipath whose edges in the order of their appearance are 1, 10, 8, 9, 2, 3, 4. Then G is the edge-disjoint union of Q and the infinite connected η-solenoidal 1-coherent unbiased digraph $(E(G) - E(Q))\dagger$.)

Proof. By Lemma 9, we can find a vertex η of G and a $\xi\eta$-dipath p such that $\lambda \in E(P)$. Let $G - E(P) = H$, and let Q denote the union of P and the finite components of H. Since G is infinite and P finite, H is infinite. But H is not the union of two disjoint infinite subgraphs since two such subgraphs would constitute a 2-splitting of G with completion P. Hence H has a unique infinite component I and only finitely many finite components. It follows that Q is finite. Moreover, since G is connected, each component of H has a vertex in common with P and hence Q is connected. Since G is ξ-solenoidal and P is finite and (by Lemma 6) $\xi\eta^{-1}$-solenoidal, H is η-solenoidal. It follows, by Corollary 4A, that η cannot be in a finite component of H and hence that the finite components of H are solenoidal. From this and the fact that P is $\xi\eta^{-1}$-solenoidal, it follows that Q (which we have already shown to be finite and connected) is $\xi\eta^{-1}$-solenoidal and is therefore by Lemma 6 derivable from a $\xi\eta$-dipath q, say. Since $\lambda \in E(P) \subset E(Q)$ and $G = P \cup H = Q \cup I$ and Q is clearly edge-disjoint from I, it will suffice to check that I is infinite, connected, η-solenoidal, 1-coherent, and unbiased. But I is infinite and connected since it is the infinite component of H; and I is η-solenoidal since H is η-solenoidal and we have shown that η is not in a finite component of H. I is 1-coherent since, if it had a 2-splitting with completion C, this would also be a 2-splitting of G with completion $C \cup Q$. Finally, since Q is finite and $G = Q \cup I$ is unbiased, I is unbiased by Lemma 7.

To complete the proof of Theorem 1, we shall assume that G is enumerable, connected, ξ-solenoidal, 1-coherent, and unbiased and deduce the required conclusion. Since G is enumerable and connected, $E(G)$ is enumerable; let

$$\lambda_1, \lambda_2, \ldots \tag{1}$$

be an enumeration of $E(G)$. By Lemma 10, there exist a $\xi_1 \in V(G)$ and a $\xi\xi_1$-dipath q_1 in G such that $\lambda_1 \in E(Q_1)$ and G is the edge-disjoint union of Q_1 and an infinite connected ξ_1-solenoidal 1-coherent unbiased digraph I_1. If μ_2 is the first term of (1) in $E(I_1)$, then by Lemma 10 there exist a $\xi_2 \in V(I_1)$ and a $\xi_1\xi_2$-dipath q_2 in I_1 such that $\mu_2 \in E(Q_2)$ and I_1 is the edge-disjoint union of Q_2 and an infinite connected ξ_2-solenoidal 1-coherent unbiased digraph I_2. If μ_3 is the first term of (1) in $E(I_2)$, then by Lemma 10 there exist a $\xi_3 \in V(I_2)$ and a $\xi_2\xi_3$-dipath q_3 in I_2 such that $\mu_3 \in E(Q_3)$ and I_2 is the edge-disjoint union of Q_3 and an infinite connected ξ_3-solenoidal unbiased digraph I_3. By continuing this construction, we obtain (using an obvious extension of the "product" notation defined in §1) a dipath $q_1 q_2 q_3 \ldots$ that is clearly a right-infinite Euler dipath of G starting at ξ.

3. Proof of Theorem 2. Suppose, first, that G is a digraph possessing an endless Euler dipath p. Then G is clearly enumerable, connected, and solenoidal. It is 2-coherent by Lemma 1 and unbiased by Lemma 2. Finally, if G is 2-separable, it has a 2-splitting $\mathfrak{S}$ with completion C, say. Since C is finite, we can write $p = p_1 p_2 p_3$, where the p_i are dipaths and p_1, p_3 have no term in $E(C)$. Therefore each of P_1, P_3 is contained in the union of the members of $\mathfrak{S}$, and so each of them, being connected, is contained in a single member of $\mathfrak{S}$. If they were both contained in the same member of $\mathfrak{S}$, the other member of $\mathfrak{S}$ would be contained in P_2, which is impossible since a finite graph cannot contain an infinite one. Hence we can write $\mathfrak{S} = \{I, J\}$, where $P_1 \subset I$ and $P_3 \subset J$. Write $V(I) = X$. Since $X\delta \subset E(C)$, X is a divisor of G. Moreover, since $V(P_1) \subset V(I) = X$ and $V(P_3) \subset V(J) \subset \bar{X}$, the elements of $X\delta$ clearly form a subsequence of p in which elements of $X \rhd \bar{X}$ alternate with elements of $\bar{X} \rhd X$ and whose first and last terms are in $X \rhd \bar{X}$. Therefore

$$|X \rhd \bar{X}| - |\bar{X} \rhd X| = 1$$

and so X is a divisor of G with flux 1. We have thus proved the necessity of the conditions in Theorem 2.

The proof of sufficiency falls into two parts, one dealing with 1-coherent digraphs such as that of Figure 8 and the other dealing with 2-separable ones such as that of Figure 9. Lemmas 13 and 19 will complete the proof of the theorem for these two types of digraphs respectively.

LEMMA 11. *If G is a connected solenoidal 1-coherent unbiased digraph and $\xi \in V(G)$, $\lambda \in E(G)$, then λ is a term of some $\xi\xi$-dipath in G.*

Proof. Let $\lambda h = \eta$. Let A be the set of vertices accessible from ξ in G. Then $A \rhd \bar{A} = \emptyset$, and therefore, since G is unbiased, $A\delta$ is finite, and therefore, by Corollary 4B, $F(A) = 0$, which, since $A \rhd \bar{A} = \emptyset$, implies that $A\delta = \emptyset$ and hence (since $\xi \in A$ and G is connected) that $A = V(G)$. Thus $\lambda t \in A$ and so, by Lemma 8, we can form a $\xi\eta$-dipath p by adding the terms λ, η at the end of a $\xi(\lambda t)$-diway. Let B be the set of vertices accessible from η in $G - E(P)$. Then clearly $B \rhd \bar{B} \subset E(P)$, which is finite, and hence, since G is unbiased, $B\delta$ is finite, and so $F(B) = 0$ by Corollary 4B. Clearly $\eta \in B$, and therefore, if $\xi \notin B$, it would follow from the fact that P is $\xi\eta^{-1}$-solenoidal and Lemma 4 that

$$-1 = f_{P\cdot}(B \cap V(P)) = F_P(B \cap V(P)) \geqslant F(B) = 0,$$

where the inequality follows from the fact that $B \rhd \bar{B} \subset E(P)$. Hence $\xi \in B$. Therefore there is by Lemma 8 an $\eta\xi$-diway q in $G - E(P)$; and pq is the required dipath.

LEMMA 12. *Let $\xi \in V(G)$, $\lambda \in E(G)$, and G be an infinite connected solenoidal 1-coherent unbiased digraph. Then there exists a $\xi\xi$-dipath q such that $\lambda \in E(Q)$ and G is the edge-disjoint union of Q and an infinite connected solenoidal 1-coherent unbiased digraph.*

(For example, let G be the digraph of Figure 8, ξ be as shown, and λ be the thick edge in the figure. Suppose that we take q to be the dipath indicated by the numbered edges. Then G is the edge-disjoint union of Q and the infinite connected solenoidal 1-coherent unbiased digraph $(E(G) - E(Q))\dagger$.)

Proof. By Lemma 11, we can find a $\xi\xi$-dipath p such that $\lambda \in E(P)$. Let $G - E(P) = H$, and let Q denote the union of P and the finite components of H. Then Q is finite and connected by the argument used in the proof of Lemma 10. Since G is solenoidal and P is finite and solenoidal, H is solenoidal, and therefore Q is a union of edge-disjoint solenoidal subgraphs and so is solenoidal. Therefore, by Lemma 5, Q is derivable from a $\xi\xi$-dipath q, say. As in the proof of Lemma 10, we see that $\lambda \in E(Q)$ and that G is the edge-disjoint union of Q and the unique infinite component I of H. Moreover, I is solenoidal since H is solenoidal, and is infinite, connected, 1-coherent, and unbiased for the same reasons as before.

LEMMA 13. *Every enumerable connected solenoidal* 1-*coherent unbiased digraph has an endless Euler dipath.*

Proof. Let G be enumerable, connected, solenoidal, 1-coherent, and unbiased. Let $\lambda_1, \lambda_2, \ldots$ be an enumeration of $E(G)$. Select any $\xi_0 \in V(G)$. By Lemma 12, there exists a $\xi_0\,\xi_0$-dipath q_1 such that $\lambda_1 \in E(Q_1)$ and G is the edge-disjoint union of Q_1 and an infinite connected solenoidal 1-coherent unbiased digraph I_1. Since G is connected, we can select a $\xi_1 \in V(Q_1 \cap I_1)$. Then, if μ_2 is the first λ_i in $E(I_1)$, there exists by Lemma 12 a $\xi_1\,\xi_1$-dipath q_2 in I_1 such that $\mu_2 \in E(Q_2)$ and I_1 is the edge-disjoint union of Q_2 and an infinite connected solenoidal 1-coherent unbiased digraph I_2. Since I_1 is connected, we can select a $\xi_2 \in V(Q_2 \cap I_2)$. Then, if μ_3 is the first λ_i in $E(I_2)$, there exists by Lemma 12 a $\xi_2\,\xi_2$-dipath q_3 in I_2 such that $\mu_3 \in E(Q_3)$ and I_2 is the edge-disjoint union of Q_3 and an infinite connected solenoidal 1-coherent unbiased digraph I_3. Since I_2 is connected, we can select a $\xi_3 \in V(Q_3 \cap I_3)$; and so on. Clearly we can write $q_i = r_i\,s_i$ where r_i is a $\xi_{i-1}\,\xi_i$-dipath and s_i is a $\xi_i\,\xi_{i-1}$-dipath and r_i includes at least one edge if i is even and s_i includes at least one edge if i is odd. Then $\ldots s_3\,s_2\,s_1\,r_1\,r_2\,r_3 \ldots$ is an endless Euler dipath of G.

Definitions. A *source* (*sink*) of a finite digraph G is a vertex ξ such that $f(\xi) > 0$ ($f(\xi) < 0$), and the *strength* of this source or sink is $|f(\xi)|$. The sum of the strengths of the sources of G will be denoted by $\sigma(G)$. Taking $X = V(G)$ in Lemma 4 shows that $\sigma(G)$ is also the sum of the strengths of the sinks of G. In Figure 10, $\sigma(G) = 6$.

LEMMA 14. *If S_0 is a finite subgraph of an unbiased digraph G, then S_0 is contained in a finite subgraph S such that no component of $G - E(S)$ includes both a source and a sink of S.*

(For instance, if G is the digraph of Figure 7 and $S_0 = \{\lambda, \mu, \nu\}\dagger$, where λ, μ, ν are the three consecutive thick edges marked with these letters and lying along the top boundary of the figure, then one subgraph S satisfying the conditions of Lemma 14 is $L\dagger$, where L is the set of all thick edges in the figure.)

Proof. Select a finite subgraph T containing S_0 such that $\sigma(T)$ is as small as possible. If a source ξ of T were accessible in $G - E(T)$ from a sink η of T, there would by Lemma 8 be an $\eta\xi$-diway w in $G - E(T)$, in which case $T \cup W$ would clearly be a subgraph containing S_0 with $\sigma(T \cup W) < \sigma(T)$. Therefore no source of T is accessible in $G - E(T)$ from a sink of T and so, if A is the set of vertices accessible in $G - E(T)$ from at least one sink of T, all sinks of T belong to A and all its sources belong to $\bar{A}$. Clearly $A \rhd \bar{A} \subset E(T)$, which is finite, and therefore, since G is unbiased, $\bar{A} \rhd A$ is finite. Therefore $S = (\bar{A} \rhd A)\dagger \cup T$ is a finite subgraph of G containing S_0. Since clearly $f_S(\xi) \leqslant f_T(\xi)$ for every $\xi \in A$ (where $f_S(\xi)$, $f_T(\xi)$ may be interpreted as 0 if ξ is not in S or T respectively) and $f_S(\xi) \geqslant f_T(\xi)$ for every $\xi \in \bar{A}$, the fact that T has all its sinks in A and all its sources in $\bar{A}$ implies that S has the same property. But, since $A \rhd \bar{A} \subset E(T)$, it follows that $A\delta \subset E(S)$ and therefore a component of $G - E(S)$ cannot have one of its vertices in A and another in $\bar{A}$. Consequently, no component of $G - E(S)$ includes both a source and a sink of S.

Lemma 15. *If S_0 is a finite subgraph of an unbiased connected digraph G, then G is the edge-disjoint union of a finite subgraph R containing S_0 and a subgraph I with no finite component such that no component of I includes both a source and a sink of R.*

(If, once again, G is the digraph of Figure 7 and $S_0 = \{\lambda, \mu, \nu\}\dagger$, we can satisfy the requirements of Lemma 15 by taking $R = L\dagger$, $I = (E(G) - L)\dagger$, where L is the set of all thick edges in the figure.)

Proof. Let S have the properties mentioned in the statement of Lemma 14. Let R be the union of S and the finite components of $G - E(S)$; and let I be the union of the infinite components of $G - E(S)$. Since G is connected, either $S = \square$ and G is the sole component of $G - E(S)$ or $S \neq \square$ and each component of $G - E(S)$ has a vertex in common with S. Hence $G - E(S)$ can have only finitely many components and consequently R is finite. A component I_0 of I will be an infinite component of $G - E(S)$ and so, if I_0 includes a vertex ξ of R, then ξ will be a vertex of S incident with precisely the same edges in S as in R. Consequently, if I_0 included both a source and a sink of R, it would include both a source and a sink of S, which is precluded by the fact that it is a component of $G - E(S)$. Thus no component of I includes both a source and a sink of R. The remaining required properties of I and R are obvious.

Lemma 16. *The elements of a splitting of an unbiased digraph are unbiased.*

Proof. Let I be an element of a splitting of an unbiased digraph G and H be a completion of this splitting. Then all edges incident with vertices of I are in

$I \cup H$. Suppose that $X \subset V(I)$ and that $X \rhd_I (V(I) - X)$ is infinite. Then $X \rhd \bar{X}$ is infinite since it contains $X \rhd_I (V(I) - X)$, and therefore $\bar{X} \rhd X$ is infinite since G is unbiased, and therefore $(V(I) - X) \rhd_I X$ must be infinite since it includes all elements of $\bar{X} \rhd X$ that are in $E(I)$, i.e. all elements of $\bar{X} \rhd X$ not in the finite set $E(H)$.

LEMMA 17. *Any member of an l-splitting of an l-coherent graph is* 1-*coherent.*

Proof. If I belonged to an l-splitting $\mathfrak{S}$ (with completion H) of an l-coherent graph G and if $\mathfrak{T}$ were a 2-splitting (with completion K) of I, then $\mathfrak{T} \cup (\mathfrak{S} - \{I\})$ would be an $(l + 1)$-splitting of G with completion $H \cup K$, which contradicts the l-coherence of G.

LEMMA 18. *If $G = F \cup H$, where F, H are subgraphs of G and F is finite and H is* 1-*coherent, then G is* 1-*coherent.*

Proof. If $\{I, J\}$ were a 2-splitting of G with completion C, then $\{I \cap H, J \cap H\}$ would be a 2-splitting of H with completion $C \cap H$.

LEMMA 19. *If G is an enumerable connected solenoidal* 2-*coherent unbiased digraph possessing a divisor with flux* 1, *then G has an endless Euler dipath.*

Proof. Let X be a divisor of G with flux 1. By Lemma 15, we can write $G = I \cup R$, where R is a finite subgraph containing $(X\delta)\dagger$, I has no finite component, no component of I includes both a source and a sink of R, and $E(I \cap R) = \emptyset$. Since G is solenoidal and $F(X) = 1$, $F(\bar{X}) = -1$, it follows from Lemma 4 that X, $\bar{X}$ are essential, and hence, since $X\delta$ is finite, that $X \circ X$, $\bar{X} \circ \bar{X}$ are both infinite. Therefore X^*, $\bar{X}^*$ are disjoint infinite subgraphs of G. Moreover, since $X\delta \subset E(R)$, which is disjoint from $E(I)$, it follows that $I \subset X^* \cup \bar{X}^*$ and so each component of I is contained in one of X^*, $\bar{X}^*$. Furthermore, since $X^* \subset G = I \cup R$ and X^* is infinite and R is finite, $X^* \cap I \neq \square$; and similarly $\bar{X}^* \cap I \neq \square$. It follows from these remarks that I has at least one component contained in each of X^*, $\bar{X}^*$. On the other hand, since the components of I constitute a splitting of G with completion R, I has at most two components. Therefore I has precisely two components M, N, say, where $M \subset X^*$, $N \subset \bar{X}^*$. Moreover,

$$f_R . V(R \cap N) = f_R . V(R \cap I \cap \bar{X}^*) = f_R . (\bar{X} \cap V(R) \cap V(I)) = f_R . (\bar{X} \cap V(R))$$

since the solenoidal character of $G = I \cup R$ implies that $f_R(\zeta) = 0$ for every $\zeta \in V(R) - V(I)$. But, by Lemma 4,

$$f_R . (\bar{X} \cap V(R)) = F_R(\bar{X} \cap V(R)) = F(\bar{X})$$

since $X\delta \subset E(R)$. Hence $f_R . V(R \cap N) = F(\bar{X}) = -1$ and so, since N does not include both a source and a sink of R, it follows that N includes a unique sink η of R and no source of R and that $f_R(\eta) = -1$. Similarly, M includes a unique source ξ of R and no sink of R and $f_R(\xi) = 1$. Moreover, R can have no source other than ξ or sink other than η since we have seen that $f_R(\zeta) = 0$ when $\zeta \in V(R) - V(I)$. Therefore R is $\xi\eta^{-1}$-solenoidal. Therefore, by Corollary 4A, ξ, η belong to the same component C of R and so (since R is $\xi\eta^{-1}$-solenoidal) C is $\xi\eta^{-1}$-solenoidal and the other components of R are solenoidal. By Lemma 6, $C = P$ for some $\xi\eta$-dipath p. Since G is connected, each component of R has a vertex in common with I; hence the set of components of R other than C can be divided into two disjoint subsets $\mathfrak{M}$, $\mathfrak{N}$ such that each member of $\mathfrak{M}$ has a vertex in common with M and each member of $\mathfrak{N}$ has a vertex in common with N. Let M' be the union of M and the members of $\mathfrak{M}$, and N' be similarly defined. Since G is solenoidal and R is finite and $\xi\eta^{-1}$-solenoidal, I is $\eta\xi^{-1}$-solenoidal and therefore N is η-solenoidal. Since the members of $\mathfrak{N}$ are solenoidal, it follows that N' is η-solenoidal. Since $\{M, N\}$ is a splitting of G with completion R, N is unbiased and 1-coherent by Lemmas 16 and 17, and therefore N' is unbiased and 1-coherent by Lemmas 7 and 18. Finally, N' is enumerable since $N \subset N' \subset G$, and is connected since N is connected and has a vertex in common with each member of $\mathfrak{N}$. Therefore, by Theorem 1, N' has a right-infinite Euler dipath n' with first term η. Similarly, M' has a left-infinite Euler dipath m' with last term ξ; and $m'pn'$ is an endless Euler dipath of G.

Lemmas 13 and 19 complete the proof of Theorem 2.

4. Further observations and results. The language and notation of (**7**) will be used in this section; but, where the words "path-sequence" and "path" were used in (**7**), we now replace them by "path" and "pathoid" respectively. An *Euler path* of G is a path p in G such that $P = G$. We shall call G *Eulerian* if it has no odd vertex (vertices of infinite degree being allowed). Figures 2, 8, 9, and 11 depict Eulerian graphs. If $\xi \in V(G)$ and $d(\xi)$ is odd or infinite and all other vertices of G have even or infinite degree, we call G *ξ-Eulerian*. Obviously an infinite graph is 1-coherent if and only if it is 1-limited (and, more generally, l-coherent if and only if it is l'-limited for some $l' \leqslant l$). Hence we see from Lemma 1 and (**7**, Lemma 11) that *a graph has a right-infinite Euler path with first term ξ if and only if it is enumerable, connected, ξ-Eulerian, and 1-coherent.* In effect, this exhibits the characterization due to Erdös, Grünwald, and Vázsonyi (**2**, p. 68; **8**, Theorem 3.2.1) of infinite graphs with one-ended Euler paths in a form designed for comparison with Theorem 1. We shall make a similar comparison of their theorem regarding endless Euler paths with Theorem 2 after some further preliminary observations.

We remark first that, since the proofs did not depend on G having no finite component or $E(G)$ being enumerable, (**7**, Corollary 2A) applies to inessential sets of vertices of Eulerian graphs in general and (**7**, Lemmas 5 and 6) and the related definitions of odd and even wings and of $p(G)$ apply to Eulerian limited graphs in general. We shall now establish an analogue, for solenoidal limited digraphs, of the parity of a wing of an Eulerian limited graph.

LEMMA 20. *If W is a wing of a limited solenoidal digraph G, then $F(X)$ has the same value for every W-set X.*

Proof. Let X, Y be W-sets. Then by the proof of (**7**, Lemma 5), $X + Y$ is inessential, and therefore so are its subsets $X \cap \bar{Y}$, $\bar{X} \cap Y$. Therefore $F(X \cap \bar{Y}) = F(\bar{X} \cap Y) = 0$ by Lemma 4 and the fact that G is solenoidal. With the notation

$$X \cap Y = Z_1, \qquad X \cap \bar{Y} = Z_2, \qquad \bar{X} \cap Y = Z_3, \qquad \bar{X} \cap \bar{Y} = Z_4,$$
$$|Z_i \rhd Z_j| - |Z_j \rhd Z_i| = F_{ij}(= -F_{ji}),$$

we have

$$\begin{aligned} F(X) - F(Y) &= F_{13} + F_{14} + F_{23} + F_{24} - F_{12} - F_{32} - F_{14} - F_{34} \\ &= F_{21} + F_{23} + F_{24} - F_{31} - F_{32} - F_{34} = F(Z_2) - F(Z_3) = 0, \end{aligned}$$

and Lemma 20 is proved.

Definition. In the circumstances of Lemma 20, $f(W)$ will denote the value of $F(X)$ for every W-set X. (One can think of $f(W)$ as the "flux out of W." Figure 11 depicts a 3-limited solenoidal digraph with wings W_1, W_2, W_3 such that $f(W_1) = 1$, $f(W_2) = -3$, $f(W_3) = 2$.)

LEMMA 21. *If G is 2-limited and W_1, W_2 are its wings, a subset X of $V(G)$ is a W_1-set if and only if $\bar{X}$ is a W_2-set.*

(One can illustrate Lemma 21 by taking G to be the digraph of Figure 9 and X, $\bar{X}$ to be the sets of vertices to the left and right respectively of the broken line.)

Proof. If the symmetric difference $A + B$ of two sets A and B is finite, we shall write $A \sim B$; and the corresponding notation for subgraphs will be used as defined in (**7**). Let $\{H_1, H_2\}$ be a 2-splitting of G and let $H_1 \in W_1$, $H_2 \in W_2$. If X is a W_1-set, $X^* \sim H_1$ and therefore $X \sim V(H_1)$, $X \circ X \sim E(H_1)$ and hence

$$\bar{X} \sim V(G) - V(H_1) \sim V(H_2),$$

$$\bar{X} \circ \bar{X} \sim (\bar{X} \circ \bar{X}) \cup X\delta = E(G) - (X \circ X) \sim E(G) - E(H_1) \sim E(H_2),$$
and hence $\bar{X}^* \sim H_2 \in W_2$. Since $\bar{X}\delta = X\delta$, which is finite, it follows that $\bar{X}$ is a W_2-set. Similarly, if $\bar{X}$ is a W_2-set, then X is a W_1-set.

From Lemma 21, we see that the set of W_1-cinctures of G is the same as the set of W_2-cinctures; we shall call this set of cinctures the *neck* of G. It follows from this observation and definition that, if G is 2-limited and Eulerian, then both wings have the same parity (as one may also infer from (**7**, Lemma 6)) and all cinctures in the neck of G have this parity too; it is thus natural to call the neck of a 2-limited Eulerian graph *even* or *odd* according as it consists of

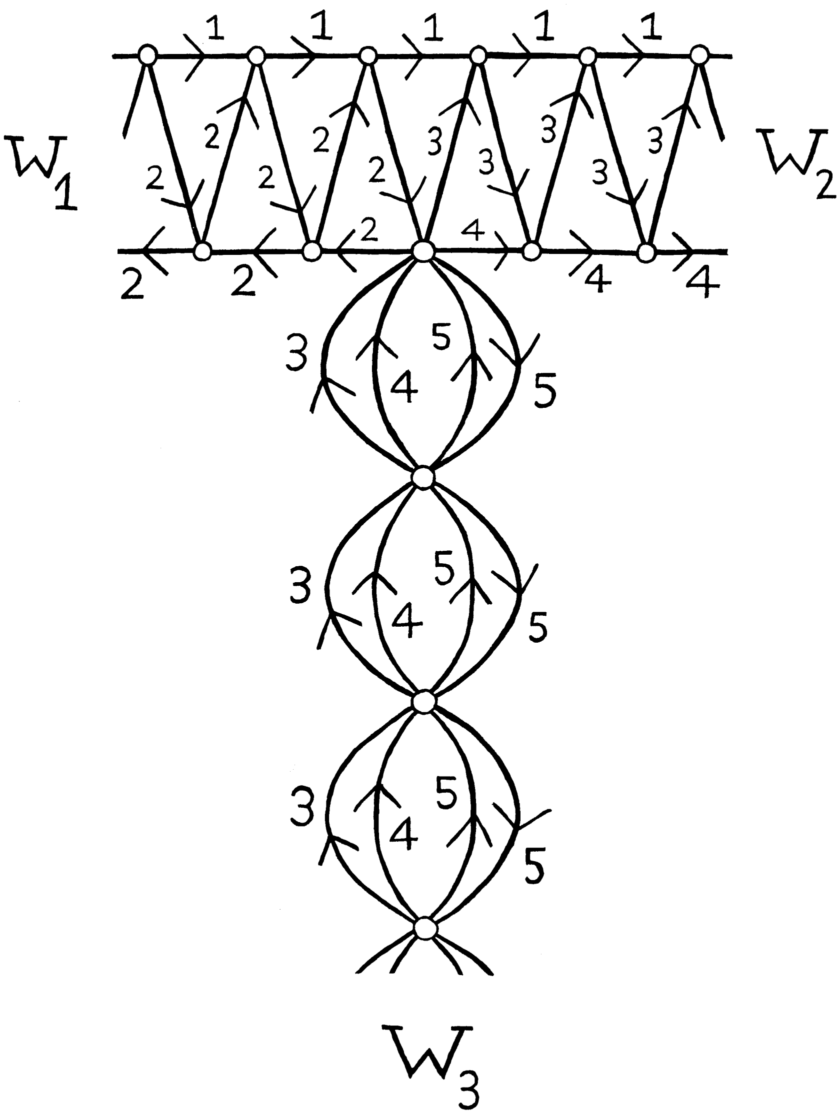

FIGURE 11.

even or odd cinctures respectively. It also follows from Lemma 21 that, if G is a 2-limited solenoidal digraph with wings W_1 and W_2, then

$$f(W_1) = F(X) = -F(\bar{X}) = -f(W_2)$$

where X is any W_1-set; we shall call the quantity $|f(W_1)| = |f(W_2)|$ the *flux through the neck* of G.

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LEMMA 22. *A* 2*-limited Eulerian graph has an odd cincture if and only if its neck is odd. A* 2*-limited solenoidal digraph has a divisor with flux* 1 *if and only if the flux through its neck is* 1.

Proof. Let X be a divisor of a 2-limited graph G with wings W_1, W_2. Then $X\delta$ is finite. Therefore X is inessential if X^* is finite, $\bar{X}$ is inessential if $\bar{X}^*$ is finite, and, if X^*, $\bar{X}^*$ are both infinite, they constitute a 2-splitting of G with completion $(X\delta)\dagger$, whence X^* must belong to W_1 or W_2 and so X must be a W_1-set or a W_2-set. Hence either X is inessential or $\bar{X}$ is inessential or X is a W_1-set or X is a W_2-set. If G is Eulerian, then by (**7**, Corollary 2A) (extended to apply to Eulerian graphs in general) $X\delta = \bar{X}\delta$ cannot be odd if X or $\bar{X}$ is inessential, and hence G has an odd cincture if and only if $X\delta$ is odd for some W_1-set or W_2-set X, which is equivalent to the neck of G being odd. If G is a solenoidal digraph, then by Lemma 4 $F(X) = F(\bar{X}) = 0$ if X or $\bar{X}$ is inessential and hence G will have a divisor with flux 1 if and only if there is a W_1-set or W_2-set X such that $F(X) = 1$, which is equivalent to the flux through the neck of G being 1.

If G has an endless Euler path, it is obviously enumerable, connected, and Eulerian, and by (**7**, Lemma 8) G is either 1-limited or 2-limited with odd neck. Conversely, if G is enumerable, connected, and Eulerian and is either 1-limited or 2-limited with odd neck, then by (**7**, Lemmas 10 and 13) G has an endless Euler path. From these remarks and Lemma 22, we see that *each of the following is a set of necessary and sufficient conditions for G to have an endless Euler path:*

(i) *G is enumerable, connected, Eulerian, and either* 1*-limited or* 2*-limited with odd neck,*

(ii) *G is enumerable, connected, Eulerian, and* 2*-coherent and, if* 2*-separable, possesses an odd cincture.*

Conditions (ii) provide the closer parallel with Theorem 2. It is a fairly easy exercise to prove directly the equivalence of (i) or (ii) to the conditions of Erdös, Grünwald, and Vázsonyi (**2**, p. 61; **8**, Theorem 3.2.2). Moreover, from Theorem 2 and Lemma 22, we see that *a digraph has an endless Euler dipath if and only if* (a) *it is enumerable, connected, solenoidal, and unbiased and* (b) *either it is* 1*-limited or it is* 2*-limited and the flux through its neck is* 1. This bears a resemblance to (i), and is possibly a slightly more natural, if slightly less simple-minded, formulation of Theorem 2.

The flux through the neck of the digraph in Figure 9 is 1. If, however, we reverse the orientations of all the thick edges, this digraph remains solenoidal and 2-limited but the flux through its neck becomes 3. The reader will soon find by experiment that the digraph then no longer has an endless Euler dipath, the difficulty being, roughly speaking, that an endless dipath can at most carry a flux of 1 through the neck of the digraph.

We now summarize some further results of (**4**). A *diendless dipathoid* is a pathoid derivable from an endless dipath. We recall that, by (**5**, Theorem 2′), a digraph is decomposable into diendless dipathoids if and only if it is solenoidal

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and has no finite component. A wing W of a solenoidal limited digraph G is a *source-wing of strength* r if $f(W) = r > 0$, a *sink-wing* of *strength* r if $f(W) = -r < 0$, and *neutral* if $f(W) = 0$. An argument resembling the proof of (7, Lemma 6) shows that the sum (which we denote by $s(G)$) of the strengths of the source-wings of G is equal to the sum of the strengths of its sink-wings. The number of neutral wings of G will be denoted by $n(G)$. Now let G' denote an enumerable solenoidal (not necessarily limited) digraph with no finite component, and let us define $q(G')$ to mean $s(G') + n(G')$ if G' is both limited and unbiased and to mean $\aleph_0$ if G' is either unlimited or biased or both. Then a result of (4) states that *$q(G')$ is the minimum number of diendless dipathoids into which G' is decomposable.* This is the analogue for digraphs of the main result of (7), and its proof is not very much more difficult. A somewhat more elaborate argument shows that *G' is decomposable into α diendless dipathoids if and only if the cardinal number α lies between $q(G')$ and $\frac{1}{2}w(G')$ inclusive;* this is the analogue of the result stated without proof in (7, §4). For instance, the limited unbiased digraph of Figure 11 is decomposable into α diendless dipathoids if and only if $3 \leqslant \alpha \leqslant 5$, as the reader may easily verify by experiment.

If p is an endless path in a limited graph G, it is not hard to prove that there is a unique wing W of G such that every tail (in the sense of (7)) of p has an infinite number of edges in common with each element of W and a unique wing W' of G such that every head of p has an infinite number of edges in common with each element of W'; we call p a *WW'-path.* It can further be shown that every endless path from which P is derivable is either a WW'-path or a $W'W$-path. We may therefore unambiguously define P to be a *1-wing* pathoid with *end-wing* W if $W = W'$, and to be a *2-wing* pathoid with *end-wings* W, W' if $W \neq W'$. It can also be shown that the end-wings of a 2-wing diendless dipathoid Q in a digraph distinguish themselves as the *tail-wing* W and *head-wing* W' of Q by the property that every endless dipath from which Q is derivable is a WW'-path.

In Figure 2, let W_1, W_2, W_3, W_4 be the wings to which the subgraphs $H_{\xi\lambda}$, $H_{\eta\mu}$, $H_{\zeta\nu}$, $H_{\tau\pi}$ defined in §1 respectively belong; then

$$\ldots, \xi_3, \lambda_3, \xi_2, \lambda_2, \xi_1, \lambda_1, \omega, \mu_1, \eta_1, \mu_2, \eta_2, \mu_3, \eta_3, \ldots$$

is a $W_1 W_2$-path. In Figure 9, let X be the set of vertices to the left of the broken line and W_1, W_2 be the wings to which X^*, $\bar{X}^*$ respectively belong. In this figure, the thick edges, together with the vertices incident with them, constitute two diendless dipathoids; the upper dipathoid is a 1-wing dipathoid with end-wing W_2 and the lower one is a 2-wing dipathoid with tail-wing W_2 and head-wing W_1.

It is not hard to show that the number of 2-wing pathoids in a decomposition of a limited graph into endless pathoids is always finite. If ϕ is a cardinal-number-valued and ω is a non-negative integer-valued function on the set of wings of a limited graph G, a *(ϕ, ω)-decomposition* of G is a decomposition $\mathfrak{D}$

of G into endless pathoids such that each wing W is an end-wing of exactly $\phi(W)$ 1-wing and $\omega(W)$ 2-wing members of $\mathfrak{D}$. If ϕ is a cardinal-number-valued and ψ, χ are non-negative integer-valued functions on the set of wings of a limited digraph G, a *(ϕ, ψ, χ)-decomposition* of G is a decomposition $\mathfrak{D}$ of G into diendless dipathoids such that each wing W is the end-wing of exactly $\phi(W)$ 1-wing members of $\mathfrak{D}$ and is the tail-wing of exactly $\psi(W)$ and head-wing of exactly $\chi(W)$ 2-wing members of $\mathfrak{D}$. If L_i is the set of edges numbered i in Figure 11, then $\{L_i\dagger, \ldots, L_5\dagger\}$ is a (ϕ, ψ, χ)-decomposition of the digraph, where ϕ, ψ, χ are as given in the following table:

	W_1	W_2	W_3
ϕ	1	0	1
ψ	1	0	2
χ	0	3	0

A *$\xi\infty$-path* (*$\infty\xi$-path*) is a right-infinite (left-infinite) path with first (last) term ξ; and a corresponding definition can be given for dipaths. If u is a finite sequence $\xi_1, \ldots, \xi_m$ of (not necessarily distinct) vertices of G and ν is a cardinal number, a *$(u; \nu)$-decomposition* of G is a decomposition of G of the form $\{P_1, \ldots, P_m\} \cup \mathfrak{S}$, where P_i is a $\xi_i\,\infty$-pathoid and $\mathfrak{S}$ is a set of ν endless pathoids distinct from $P_1, \ldots, P_m$. If u, v are finite sequences $\xi_1, \ldots, \xi_m$ and $\eta_1, \ldots, \eta_n$ respectively of vertices of a digraph G, a *$(u, v; \nu)$-decomposition* of G is a decomposition of G of the form $\{P_1, \ldots, P_m, Q_1, \ldots, Q_n\} \cup \mathfrak{S}$, where P_i is a $\xi_i\,\infty$-dipathoid and Q_j is an $\infty\,\eta_j$-dipathoid and $\mathfrak{S}$ is a set of ν diendless dipathoids distinct from $P_1, \ldots, P_m, Q_1, \ldots, Q_n$. In these definitions (and in the statements of Problems (iii) and (vi) below), we permit the special cases in which u or v have 0 terms, thus waiving a rule made in §1 concerning our use of the word "sequence." In Figure 5, let u be the sequence ξ, η and v be the sequence with sole term η. Let L be the set of edges numbered $7n$, M be the set of edges numbered $7n + 5$, and N be the set of edges numbered $7n + 4$ and $7n + 6$, where n runs through the set of positive integers in each case, and let P be the set of all remaining edges. Then $\{L\dagger, M\dagger, N\dagger, P\dagger\}$ is a $(u, v; 1)$-decomposition of the digraph.

The following further problems were investigated in **(4)**.

(i) Let G be a limited graph and ϕ, ω be a pair of functions as described above. What are necessary and sufficient conditions for G to admit a (ϕ, ω)-decomposition?

(ii) If ν_1 is a cardinal number and n_2 is a non-negative integer, what are necessary and sufficient conditions for a limited graph to be decomposable into ν_1 1-wing and n_2 2-wing endless pathoids?

(iii) Let u be a finite sequence of vertices of a graph G and ν be a cardinal number. What are necessary and sufficient conditions for G to admit a $(u; \nu)$-decomposition?

(iv) Let G be a limited digraph and ϕ, ψ, χ be a triple of functions as described above. What are necessary and sufficient conditions for G to admit a (ϕ, ψ, χ)-decomposition?

(v) If ν_1 is a cardinal number and n_2 is a non-negative integer, what are necessary and sufficient conditions for a limited digraph to be decomposable into ν_1 1-wing and n_2 2-wing diendless dipathoids?

(vi) Let u, v be finite sequences of vertices of a digraph G and ν be a cardinal number. What are necessary and sufficient conditions for G to admit a $(u, v; \nu)$-decomposition?

Problems (i), (ii), (iii), and (vi) were solved completely in **(4)**. Problems (iv) and (v) were solved completely for unbiased digraphs, but not for biased ones. It appears that their complete solution for biased digraphs may possibly present a difficulty of a substantially higher order, and some possibility of a remote connection between this problem and the four-colour problem has even been detected. Since, however, it is easily deducible from Lemma 2 that only unbiased digraphs can be decomposed into a finite number of dipathoids, it follows that (iv) and (v) are completely solved for the cases in which ϕ is finite-valued and ν_1 is finite respectively. A slightly curious feature of the solutions of (ii) and (for unbiased digraphs) (v) is that they are substantially more complicated for disconnected than for connected graphs. It is true that the solutions for disconnected graphs are simply a matter of "adding the contributions of their components"; but this addition involves appreciable difficulties. It will be observed that the solution to (vi) may be regarded as containing both Theorems 1 and 2 as special cases; and similarly (iii) in a sense subsumes both the one-ended and endless Euler path problems for undirected graphs. A variant of (iii), in which it was supposed that G was limited and that it was specified how many of the ν endless pathoids were to be 1-wing and how many were to be 2-wing, was also solved in **(4)**, and the corresponding variant of (vi) solved for unbiased digraphs.

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