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COMPLETELY REGULAR MAPPINGS AND HOMOGENEOUS, APOSYNDETIC CONTINUA

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The purpose of this note is to prove an improved version of Jones' Aposyndetic Decomposition Theorem. Corollaries to the new theorem re-emphasize the importance of understanding aposyndetic, homogeneous continua.

The proof is a synthesis of results about homogeneous continua with results from an unexpected source: completely regular mappings. Completely regular mappings occur naturally and often in the study of homogeneous continua, which is a surprising and pleasing phenomenon, since these mappings were invented for quite another purpose [**1**]. The author believes that these maps are likely to provide even more new information about homogeneous continua.

A continuum is a compact, connected, nonvoid metric space. A curve is a one-dimensional continuum. A continuum M is homogeneous if for each pair of points p and q belonging to M, there exists a homeomorphism $h:M \rightarrow M$ such that $h(p) = q$.

A mapping $f:X \rightarrow Y$ of X onto Y is completely regular [**1**] if given $\epsilon > 0$ and $y \in Y$, there exists an open set V in Y containing y such that if $y' \in V$, then there is a homeomorphism h from $f^{-1}(y)$ to $f^{-1}(y')$ such that $d(x, h(x)) < \epsilon$. Each completely regular mapping is open.

We will need the following theorem, which should be well-known. A proof is included for completeness.

THEOREM 1. *If X is a curve and $H^1(X) = 0$, then X is hereditarily unicoherent.*

Proof. We first show that if Y is a subcontinuum of X, then $H^1(Y) = 0$. Consider the following part of the long exact sequence of the pair (X, Y):

$$\ldots \rightarrow H^1(X) \rightarrow H^1(Y) \rightarrow H^2(X, Y) \ldots$$

Since X is a curve, it follows that $H^2(X, Y) = 0$ and hence $H^1(Y) = 0$.

We now show that Y is unicoherent. Suppose $Y = Y_1 \cup Y_2$, where Y_1 and Y_2 are proper subcontinua of Y. Consider the following part of the

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reduced Mayer-Vietoris sequence of the triod $(Y; Y_1, Y_2)$:

$$\ldots \rightarrow \tilde{H}^0(Y_1) + \tilde{H}^0(Y_2) \rightarrow \tilde{H}^0(Y_1 \cap Y_2) \rightarrow H^1(Y) \rightarrow \ldots$$

It follows that $\tilde{H}^0(Y_1 \cap Y_2) = 0$ and so $Y_1 \cap Y_2$ is connected.

1. Jones' aposyndetic decomposition theorem. Let x and y be points of the continuum M. If M contains an open set G and a continuum H such that $x \in G \subset H \subset M - \{y\}$, then M is said to be *aposyndetic at x with respect to y*. If M is aposyndetic at each of its points with respect to every other point, then M is said to be *aposyndetic*.

The set L_x has as members the point x together with all points z of M such that M is not aposyndetic at z with respect to x. In the case that the homogeneous, decomposable continuum M is not aposyndetic, then the collection $\{L_x : x \in M\}$ yields the following decomposition of M [**8**].

THEOREM 2. (Jones' Aposyndetic Decomposition Theorem). *Suppose that M is a decomposable, homogeneous continuum. Then there exists a non-degenerate collection G of mutually exclusive continua filling up M such that*

(a) *the decomposition space N is a homogeneous, aposyndetic continuum,*

(b) *if x is a point of M, then L_x is an element of G,*

(c) *if g is an element of G and K is a subcontinuum of M that contains both a point of g and a point of M − g, then g is a subset of K,*

(d) *the associated quotient mapping $\pi: M \rightarrow N$ is a monotone, open map, and*

(e) *if g is an element of G, then g is a homogeneous continuum.*

We offer the following improvements of Theorem 2.

THEOREM 3. *If the continuum M in the Jones' Aposyndetic Decomposition Theorem is a curve, then conditions* (d) *and* (e) *can be strengthened as follows:*

(d) *the associated quotient map $\pi: M \rightarrow N$ is monotone and completely regular, and*

(e) *if g is an element of G, then g is a homogeneous, indecomposable, acyclic curve.*

Proof. Let $\epsilon > 0$ and let y belong to N. Let x belong to $\pi^{-1}(y)$. According to a corollary [**4**] of a theorem of Effros [**2**], x belongs to an open set W of M such that for every pair of points a and b in W, there exists a homeomorphism $h: M \rightarrow M$ such that $h(a) = b$ and h moves no point more than ϵ.

Let y' belong to the open set $\pi(W)$. Let x' belong to $\pi^{-1}(y') \cap W$. Let $h: M \rightarrow M$ be a homeomorphism such that $h(x) = x'$ and h moves no point more than ϵ. It follows that h maps $\pi^{-1}(y)$ onto $\pi^{-1}(y')$, for to do otherwise would violate the definition of the decomposition elements. Therefore, the map π is completely regular.

According to a theorem of [**11**, Theorem 1], for each element g of the decomposition, $H^1(g) = 0$. By Theorem 1, each element g of G is hereditarily unicoherent. Jones [**7**] (see also [**3**]) has shown that each homogeneous, hereditarily unicoherent continuum is indecomposable. This completes the proof.

Remark. If we only assume that M is a finite-dimensional continuum rather than a curve, then the map $\pi: M \rightarrow N$ is still completely regular, but we may only conclude about each element g of G that $H^n(g) = 0$, where $n = \dim(M)$.

Recall that a continuum M is λ*-connected* if each two of its points lie in a hereditarily decomposable subcontinuum of M.

COROLLARY 4. *Each* λ*-connected, homogeneous curve is aposyndetic.*

Piotr Minc once asked the author if it were possible to classify hereditarily decomposable, homogeneous continua. The next corollary is a first step.

COROLLARY 5. *Each hereditarily decomposable, homogeneous continuum is aposyndetic.*

The next corollary, actually a corollary to the original Jones' theorem, seems to have gone unnoticed.

COROLLARY 6. *Each arcwise-connected homogeneous continuum is aposyndetic.*

2. More completely regular maps related to homogeneous continua. The author [**9**] has recently constructed an uncountable collection of homogeneous continua called solenoids of pseudo-arcs. These continua are so termed because each admits a continuous decomposition into pseudo-arcs such that the resulting quotient space is a solenoid. It is easy to show that the quotient maps associated with these decompositions are completely regular. On the other hand, one can show that a "solenoid of solenoids" cannot exist, using the theorem of Wilson [**11**] and the fact that the quotient map of the decomposition would be completely regular.

The last theorem exploits a decomposition of a proper subcontinuum of a certain homogeneous curve.

THEOREM 7. *Suppose each proper subcontinuum of the homogeneous curve* M *is atriodic and unicoherent. Then each decomposable, proper subcontinuum of* M *that is not an arc contains an indecomposable, homogeneous, acyclic subcontinuum. Furthermore each decomposable, proper subcontinuum of* M *is acyclic.*

Proof. Let E be a decomposable, proper subcontinuum of M. Since E

is unicoherent and not a triod, E is irreducible between two points [**10**]. We follow arguments similar to those of [**4**, Theorem 1] and [**6**, Lemma 5] to find a monotone, open map $k:E \to [0, 1]$ such that for each $0 \leqq s < t \leqq 1$, (a) $k^{-1}(s)$ is homeomorphic to $k^{-1}(t)$, and (b) $k^{-1}(s)$ is homogeneous. An argument similar to the proof of Theorem 3 of this paper shows that the map k is completely regular, and that the continuum $k^{-1}(s)$ is acyclic and indecomposable. The last claim of the theorem is a consequence of the Vietoris-Begle Theorem.

The last corollary strengthens a theorem of [**5**].

COROLLARY 8. *Suppose each proper subcontinuum of the homogeneous continuum M is atriodic, unicoherent and decomposable. Then M is a solenoid.*

The following questions arise in trying to improve Jones' Aposyndetic Decomposition Theorem still further.

Question 1. Is each homogeneous, aposyndetic curve locally connected?

Question 2. Is each acyclic homogeneous curve tree-like? hereditarily indecomposable?

For the purpose of this paper, an affirmative answer to the last question would enable us to bypass Question 2.

Question 3. Suppose E is a curve and f is a monotone map of E onto the unit interval I. If $H^1(f^{-1}(t)) = 0$, for all t, must there exist a number s such that $f^{-1}(s)$ is tree-like?

REFERENCES

1. E. Dyer and M. E. Hamstrom, *Completely regular mappings*, Fund. Math. *45* (1958), 103–118.

2. E. G. Effros, *Transformation groups and C*-algebras*, Ann. of Math. *81* (1965), 38–55.

3. G. R. Gordh, Jr., *On homogeneous hereditarily unicoherent continua*, Proc. Amer. Math. Soc. *51* (1975), 198–202.

4. C. L. Hagopian, *Homogeneous plane continua*, Houston J. Math. *1* (1975), 35–41.

5. ——— *A characterization of solenoids*, preprint.

6. ——— *Indecomposable homogeneous plane continua are hereditarily indecomposable*, Trans. Amer. Math. Soc. *224* (1976), 339–350.

7. F. B. Jones, *Certain homogeneous unicoherent indecomposable continua*, Proc. Amer. Math. Soc. *2* (1951), 855–859.

8. ——— *On a certain type of homogeneous plane continuum*, Proc. Amer. Math. Soc. *6* (1955), 735–740.

9. J. T. Rogers, Jr., *Solenoids of pseudo-arcs*, Houston J. Math. *3* (1977), 531–537.

10. R. H. Sorgenfrey, *Concerning triodic continua*, Amer. J. Math. *66* (1944), 439–460.

11. D. C. Wilson, *Completely regular mappings and dimension*, Bull. Amer. Math. Soc. *76* (1970), 1057–1061.

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