CLOSABLE DERIVATIONS OF SIMPLE C*-ALGEBRAS

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1. Introduction. In this note we show that any derivation of a simple C*-algebra, whose range is not dense, is closable. We also derive a necessary and sufficient condition for a *-derivation of a C*-algebra, which is defined on the domain of a closed *-derivation, to be closed.

A linear mapping \( \delta \) from a dense *-subalgebra \( D(\delta) \) of a C*-algebra \( \mathcal{A} \) into \( \mathcal{A} \) is called a derivation if \( \delta(ab) = \delta(a)b + a\delta(b) \) (a, b \( \in D(\delta) \)). If in addition \( \delta(a^*) = \delta(a)^* \), then \( \delta \) is called a *-derivation. For a linear mapping \( \delta \) from a linear subspace \( D \) of a Banach space \( \mathcal{A} \) into \( \mathcal{A} \), we let \( \sigma(\delta) \) denote the set \( \{ b \in \mathcal{A} : \text{there is a sequence} \ (a_n) \ \text{in} \ D \ \text{with} \ a_n \to 0 \ \text{and} \ \delta(a_n) \to b \} \), and call it the separating space of \( \delta \). The hypothesis on \( \sigma(\delta) \) forces \( \sigma(\delta) \) to be a closed linear subspace of \( \mathcal{A} \), and \( \delta \) is closable if and only if \( \sigma(\delta) = \{0\} \) [4, p. 8].

We show that the separating space of a derivation of a C*-algebra is a closed two-sided ideal. Then we apply this result to prove the main result of this paper. In the paper \( R(\delta) \) denotes the range of the derivation \( \delta \); i.e. \( R(\delta) = \delta(D(\delta)) \). S. Sakai [3] has asked: when is the range of a closed *-derivation of a simple C*-algebra not dense? Our result implies the answer to a converse question, namely, if the range of a *-derivation \( \delta \) of a simple C*-algebra is not dense, then \( \delta \) is closable.

Let \( \delta \) and \( \delta_0 \) be derivations of a C*-algebra defined on the same domains \( D \), say; then \( \delta \) is called \( \delta_0 \)-bounded if there is a number \( M > 0 \) such that \( \|\delta(a)\| \leq M(\|a\| + \|\delta_0(a)\|) \) \( (a \in D) \). It follows from [1] that if \( \delta_0 \) is a closed *-derivation and \( \delta \) is a *-derivation with \( D(\delta) \supseteq D(\delta_0) \), then \( \delta \) is \( \delta_0 \)-bounded. Sakai conjectured that \( \delta \) should be closable [2]. An easy argument shows that if \( D(\delta) = D(\delta_0) \), then \( \delta \) is closed if and only if \( \delta_0 \) is \( \delta \)-bounded.

2. The results. It will now be shown that, under one restriction, a derivation of a simple C*-algebra admits a closed extension.

Theorem 1. Let \( \mathcal{A} \) be a simple C*-algebra and \( \delta \) be a derivation of \( \mathcal{A} \). Then \( \delta \) is closable if \( R(\delta) \) is not dense in \( \mathcal{A} \).

Proof. It suffices to show that the separating space \( \sigma(\delta) \) is \( \{0\} \). The separating space \( \sigma(\delta) \) is obviously a linear subspace of \( \mathcal{A} \). We show that it is a closed two-sided ideal in \( \mathcal{A} \). Suppose \( (b_n) \) is a sequence in \( \sigma(\delta) \) and \( b_n \to b \); then there is a sequence \( (c_n) \subseteq D(\delta) \) such that \( \|c_n\| < 1/n \) and \( \|\delta(c_n) - b_n\| < 1/n \); it therefore follows that \( c_n \to 0 \) and \( \delta(c_n) \to b \). We conclude that \( \sigma(\delta) \) is closed. Let \( c \in D(\delta) \) and \( b \in \sigma(\delta) \); then there exists a sequence \( (a_n) \) in \( D(\delta) \) such that \( a_n \to 0 \) and \( \delta(a_n) \to b \). Hence \( ca_n \to 0 \), \( a_n b \to 0 \), and

\[
\delta(ca_n) = \delta(c)a_n + c \delta(a_n) \to cb,
\]

\[
\delta(a_n c) = \delta(a_n)c + a_n \delta(c) \to bc.
\]

Thus \( cb, bc \in \sigma(\delta) \). Suppose now that \( b \in \sigma(\delta) \) and \( c \in \mathbb{A} \). The density of \( D(\delta) \) in \( \mathbb{A} \) implies that there is a sequence \( (c_n) \) in \( D(\delta) \) such that \( c_n \to c \) and hence \( c_nb, bc_n \in \sigma(\delta) \), \( cb, bc \in \sigma(\delta) \), since \( \sigma(\delta) \) is closed. Thus \( \sigma(\delta) \) is a closed two-sided ideal in \( \mathbb{A} \). It follows now that \( \sigma(\delta) = \{0\} \) or \( \mathbb{A} \). If \( \sigma(\delta) = \mathbb{A} \), then \( R(\delta) \) would be dense in \( \mathbb{A} \); however, by our assumption \( R(\delta) \) is not dense. This contradiction shows that \( \sigma(\delta) = \{0\} \) and therefore \( \delta \) is closable.

**Corollary 2.** Let \( \delta \) be a \(*\)-derivation of a simple \( C^* \)-algebra \( \mathbb{A} \). Then \( \delta \) is closable if one of the two sets \( \{a + \delta(a) : a \in D(\delta)\} \) is not dense in \( \mathbb{A} \).

**Proof.** This follows from the proof of the theorem above.

Let \( \delta_0 \) be a closed \(*\)-derivation of a \( C^* \)-algebra \( \mathbb{A} \) and let \( \delta \) be a \(*\)-derivation of \( \mathbb{A} \) with the domain \( D(\delta) = D(\delta_0) \). The next result gives a necessary and sufficient condition for \( \delta \) to be closed. By [1] \( \delta \) is \( \delta_0 \)-bounded. Suppose moreover that \( \delta_0 \) is \( \delta \)-bounded; then there exists two real numbers \( M, K > 0 \) such that for \( a \in D(\delta) = D(\delta_0) \) we have

\[
\|\delta(a)\| \leq M(\|a\| + \|\delta_0(a)\|),
\]

\[
\|\delta_0(a)\| \leq K(\|a\| + \|\delta(a)\|).
\]

If \( a_n \to a \, (a_n \in D(\delta)) \) and \( \delta(a_n) \to b \), then

\[
\|\delta_0(a_n) - \delta_0(a_m)\| = \|\delta_0(a_n - a_m)\| \leq K(\|a_n - a_m\| + \|\delta(a_n) - \delta(a_m)\|);
\]

thus \((\delta_0(a_n))\) is a Cauchy sequence in \( \mathbb{A} \) and hence is convergent. Thus \( a \in D(\delta_0) \) and \( \delta_0(a_n) \to \delta_0(a) \) and so \( \delta(a_n) \to \delta(a) \). This gives the following theorem.

**Theorem 3.** Let \( \delta_0 \) and \( \delta \) be \(*\)-derivations of a \( C^* \)-algebra \( \mathbb{A} \). Suppose \( \delta_0 \) is closed and \( D(\delta) = D(\delta_0) \). Then \( \delta \) is closed if and only if \( \delta_0 \) is \( \delta \)-bounded.

**3. Comments.** Let \( \mathbb{A} \) be a normed space. A subset \( \mathbb{A}_0 \) of \( \mathbb{A} \) is said to be a \( G_\delta \) set if there exists a countable family \( \{G_n\} \) of open sets such that \( \mathbb{A}_0 = \bigcap_{n=1}^\infty G_n \). For a closed linear mapping \( \delta \) of a normed space \( \mathbb{A} \) into \( \mathbb{A} \) with \( R(\delta) = \mathbb{A} \), the closedness condition of \( R(\delta) \) is equivalent to \( R(\delta) \) being a \( G_\delta \) set. In fact if \( R(\delta) \) is of second category, then \( R(\delta) = \mathbb{A} \). If we can show that the range \( R(\delta) \) of a closed \(*\)-derivation \( \delta \) of a simple \( C^* \)-algebra \( \mathbb{A} \) is not closed and is a \( G_\delta \) set, then it follows that \( \overline{R(\delta)} \subseteq \mathbb{A} \). The following problem poses itself: suppose \( \delta_0 \) is a closed \(*\)-derivation of a simple \( C^* \)-algebra \( \mathbb{A} \) and \( \overline{R(\delta_0)} \subseteq \mathbb{A} \). Let \( \delta \) be a \(*\)-derivation of \( \mathbb{A} \) with its domain \( D(\delta) = D(\delta_0) \). Then can we conclude that \( \overline{R(\delta)} \subseteq \mathbb{A} \) and thus \( \delta \) is closable?

Let \( \delta_0 \) be a closed \(*\)-derivation of a \( C^* \)-algebra \( \mathbb{A} \) and \( \delta \) be a \(*\)-derivation of \( \mathbb{A} \) with its domain \( D(\delta) = D(\delta_0) \); then \((\delta_0 - \delta)\) is \( \delta_0 \)-bounded. Hence there are two positive numbers \( N, M \) such that

\[
\|(\delta_0 - \delta)(a)\| \leq N\|a\| + M\|\delta_0(a)\|.
\]

An easy computation shows that \( \delta \) is closed if \( M < 1 \).
We note that the proof of Theorem 1 shows that the separating space of a derivation from a Banach algebra is a closed two-sided ideal. It also follows that if $T$ is a densely defined operator from a Banach algebra and $D(T)\sigma(T)\subseteq\sigma(T)$, $\sigma(T)D(T)\subseteq\sigma(T)$, then $\sigma(T)$ is a closed two-sided ideal.

REFERENCES


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