

REGULAR ORBITS OF COPRIME LINEAR GROUPS IN LARGE CHARACTERISTIC

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Abstract

We prove that a finite coprime linear group G in characteristic $p \geq \frac{1}{2}(|G| - 1)$ has a regular orbit. This bound on p is best possible. We also give an application to blocks with abelian defect groups.

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1. Introduction

Linear groups, that is, subgroups G of $GL(V)$ where V is a vector space, play an important role in many branches of group representation theory. In the present note we are interested in the situation where V is an elementary abelian p -group and G is a p' -group. Then, by Maschke's theorem, the action of G on V is semisimple. A crucial fact used in the solution of the so-called $k(GV)$ -problem [16] is that G often has a *regular orbit* on V , that is, an orbit of size $|G|$. This is well known for abelian groups G and many other special cases have been handled in the literature (see, for example, [3, 4, 7, 10, 17]). It is also known that regular orbits exist if p is 'large', for example if $p > |G|$ (see [11, Lemma 2.2]). The main result of this paper establishes the existence of a regular orbit under the weaker condition $p \geq \frac{1}{2}(|G| - 1)$. The proof relies on a classification of finite groups with 'many' minimal subgroups given by Burness and Scott [2]. Conversely, for every odd prime p , we construct linear groups G without regular orbits such that $p = \frac{1}{2}|G| - 1$.

These results are motivated by Brauer's $k(B)$ -conjecture, which is still open even for blocks with abelian defect groups. In this situation, V is a defect group of a block B of a finite group and G is the corresponding inertial quotient. The number of irreducible characters in B is denoted by $k(B)$. By a result of Robinson [11], Brauer's $k(B)$ -conjecture, that is, $k(B) \leq |V|$, holds provided G has a regular orbit on V . In previous papers [13–15], we have applied some of the techniques from the $k(GV)$ -problem to

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this more general situation. Now in the present paper we will give a new application of our main theorem.

Recently there has been some interest in the base size of linear groups (or, more generally, permutation groups). Here a *base* in the situation above is a set of elements $b_1, \dots, b_n \in V$ such that $C_G(b_1) \cap \dots \cap C_G(b_n) = 1$. Our main theorem gives a sufficient condition for a linear group to have a base of size 1. In general, it was shown by Halasi and Podoski [6] that every coprime linear group has a base of size 2 (independent of p).

2. Main theorem

In the following we denote a cyclic group of order n by C_n . A dihedral group of order n is denoted by D_n and the symmetric and alternating groups of degree n are S_n and A_n , respectively. For a finite group G , we set $G^n := G \times \dots \times G$ (n copies). The exponent of G is defined by $\exp(G) := \min\{n \geq 1 : g^n = 1 \text{ for all } g \in G\}$. Note that the minimal subgroups of a finite group are precisely the subgroups of prime order.

PROPOSITION 2.1 [2, Theorem 1.1]. *If a nontrivial finite group G has more than $\frac{1}{2}|G| - 1$ minimal subgroups, then one of the following holds:*

- (i) $G \cong A \rtimes C_2$, where A is abelian and C_2 acts as inversion;
- (ii) $G \cong C_2^n \rtimes C_2$ for some $n \geq 2$;
- (iii) $G \cong C_2^{2n} \rtimes C_3$ for some $n \geq 1$ (Frobenius group);
- (iv) $G \cong D_8 * \dots * D_8 \times C_2^n$ for some $n \geq 0$ (central product);
- (v) $\exp(G) = 3$;
- (vi) $G \cong D_8^2 \times C_2^n$ for some $n \geq 0$;
- (vii) $G \cong S_3 \times D_8 \times C_2^n$ for some $n \geq 0$;
- (viii) $G \in \{S_3^2, S_4, A_5\}$.

Note that (i) in Proposition 2.1 includes all nontrivial elementary abelian 2-groups. In our main theorem we prove slightly more than what was promised in the introduction.

THEOREM 2.2. *Let G be a p' -automorphism group on a finite p -group P such that $|G| \leq 2p + 9$. Then one of the following holds:*

- (i) G has a regular orbit on P ;
- (ii) $G \cong D_{2p+2}$;
- (iii) $G \cong D_8 * C_4$ and $p = 5$.

PROOF. Let G be a minimal counterexample. By a result of Hartley and Turull [8, Lemma 2.6.2], we may assume that P is elementary abelian. Then G is nonabelian (see, for example, [4, Corollary 4.I]). If \overline{G} is a proper quotient of G , then either $|\overline{G}| \leq \frac{1}{3}|G| \leq \frac{2}{3}p + 3 < 2p + 2$ or $p > 2$ and $|\overline{G}| = \frac{1}{2}|G| \leq p + \frac{9}{2} < 2p + 2$. In particular, every proper quotient of G has a regular orbit. By [4, Lemmas 2.I and 3.I], it suffices to assume that P is an absolutely irreducible G -module over a finite field with $q = p^f$

elements. In particular, the centre $Z(G)$ of G is cyclic. Let \mathcal{M} be the set of minimal subgroups of G . Then for every $x \in P$ there exists $M \in \mathcal{M}$ such that $M \leq C_G(x)$. Hence,

$$P = \bigcup_{M \in \mathcal{M}} C_P(M). \tag{2.1}$$

Since G acts faithfully, $C_P(M) < P$ for every $M \in \mathcal{M}$. This shows that $m := |\mathcal{M}| \geq 2$ and

$$|P| < \sum_{M \in \mathcal{M}} |C_P(M)| \leq m \frac{|P|}{q}. \tag{2.2}$$

Therefore, $\frac{1}{2}(|G| - 9) \leq p \leq q \leq m - 1$. Let $|G| = 2p + k$ with $k \leq 9$ (here k may be negative). We discuss four cases.

Case 1. $k \in \{3, 5, 7, 9\}$. Here G has odd order. Since two distinct minimal subgroups of G intersect trivially, it follows that $m \leq \frac{1}{2}(|G| - 1)$. Equality can only hold if G is a 3-group. Then $p = 2$ according to [4, Theorem 1.II] and we easily get a contradiction. Hence, $m \leq \frac{1}{2}(|G| - 3)$. If we have equality this time, G consists of elements of order 3 and just one subgroup of order 5. But then G is nilpotent and must contain elements of composite order as well. This shows that $m \leq \frac{1}{2}(|G| - 5)$. Since the number of subgroups of prime order r is always congruent to 1 modulo r (Frobenius' extension of Sylow's theorem), equality in this case leads to $G \cong C_7 \rtimes C_3$. Of course, this is impossible and we conclude that $\frac{1}{2}(|G| - k) \leq p \leq m - 1 \leq \frac{1}{2}(|G| - 9)$ and $k = 9$. However, it is not hard to see that there is no group with the desired number of minimal subgroups.

Case 2. $k \in \{4, 8\}$. Now G has even order and $p > 2$. Hence, $|G|$ is not divisible by 4. It is well known that in this situation G has a normal 2-complement N . As usual, the number of minimal subgroups of G inside N is at most $\frac{1}{2}(|N| - 1) < \frac{1}{4}|G|$. Let S be a Sylow 2-subgroup of G . Then the number of minimal subgroups of G outside N is $|G : N_G(S)|$. If $S < N_G(S)$, then $m < \frac{1}{4}|G| + \frac{1}{6}|G| = \frac{5}{12}|G|$. Since $\frac{1}{2}|G| - 4 \leq p \leq m - 1$, we see that $|G| < 36$. These cases have been handled in [12, Proposition 14.9]. Let $S = N_G(S)$. Then S acts as inversion on N and N must be abelian. Ito's theorem implies that $\dim P = 2$ (see [9, Theorem 6.15]). Since $N \leq G' \leq \text{SL}(2, q)$, N is cyclic and so $G \in \{D_{2p+4}, D_{2p+8}\}$. Both cases are excluded by [12, Proposition 14.8].

Case 3. $k = 6$. As before, $\frac{1}{2}|G| - 3 = p \leq m - 1$. Suppose first that $m = \frac{1}{2}|G| - 2$. Let $\mathcal{M} = \{M_1, \dots, M_{p+1}\}$ and $|C_P(M_i)| = p^{d_i}$ with $d_1 \geq \dots \geq d_{p+1}$. Setting $|P| = p^n$, we refine (2.2) to

$$|P| \leq \sum_{i=1}^{p+1} |C_P(M_i)| - \sum_{i=2}^{p+1} |C_P(M_1) \cap C_P(M_i)| \leq p^{d_1} + (1 - p^{d_1-n}) \sum_{i=2}^{p+1} p^{d_i} \leq p^n. \tag{2.3}$$

This implies that $d_1 = \dots = d_{p+1} = n - 1$. In particular, every M_i acts faithfully on $[M_i, P] \cong C_p$ and we deduce that $|M_i| \mid p - 1$. Hence, $|M_i|$ divides $|G| - 2(p - 1) = 8$ and G is a 2-group. But then we may choose $M_i \leq Z(G)$ and obtain the contradiction

$C_P(M_i) = 1$, since $C_P(M_i)$ is G -invariant and P is irreducible. This argument implies that $m \geq \frac{1}{2}|G| - 1$. If G is still a 2-group, p must be a Fermat prime or a Mersenne prime by [4, Theorem 2.II]. It follows easily that $p = 5$ (a Fermat prime) and $|G| = 16$. Here one can show with GAP [5] that $G \cong D_8 * C_4$ as given in the statement of our theorem. Now suppose that G is not a 2-group. If $m = \frac{1}{2}|G| - 1$, then, by (2.3), G contains just one minimal subgroup of odd (prime) order. Since the number of involutions is always odd, we conclude that $|G| \equiv 2 \pmod{4}$. But this gives the contradiction $p = 2$. Hence, we may assume that $m \geq \frac{1}{2}|G|$. At this point we refer to the next case.

Case 4. $m \geq \frac{1}{2}|G|$. First observe that this case includes the remaining possibility $k \leq 2$, since $\frac{1}{2}(|G| - k) \leq p \leq m - 1$. Moreover, G is given as in Proposition 2.1. Since $Z(G)$ is cyclic, some cases can be excluded immediately. If G has odd order, then $m \leq \frac{1}{2}(|G| - 1)$, which is impossible. Therefore, $|G|$ is even and $p > 2$.

Suppose first that $G \cong A \rtimes C_2$, where A is abelian and C_2 acts as inversion. Let M be a minimal subgroup of G lying inside A . Then $M \trianglelefteq G$ and $C_P(M)$ is G -invariant. It follows that $C_P(M) = 1$. Hence, in (2.1) we only need to consider the minimal subgroups outside A . This leads to $k \in \{2, 6\}$ in (2.2) after taking the last cases into account. By Ito’s theorem, we have $\dim P = 2$ and $O_2(A) \leq \text{SL}(2, q)$ is cyclic. Since also $Z(G)$ is cyclic, A contains at most one involution. This shows that A is cyclic and $G \in \{D_{2p+2}, D_{2p+6}\}$. The first case corresponds to the exception given in the statement of the theorem and the second case is excluded by [12, Proposition 14.8].

Next let $G \cong C_2^n \rtimes C_2$ with $n \geq 2$. Then $|Z(G)| = 2$ and [1, Lemma 1.1] gives $|G : G'| = 4$. A theorem of Taussky leads to $G \cong D_8$ (see [1, Proposition 1.6]). But then $p = 3$ and $G \cong D_{2p+2}$. Now let $G \cong C_2^{2n} \rtimes C_3$. Then Ito’s theorem implies that $\dim P \leq 3$ and the 2-rank of G is at most 3 (see, for example, [12, Proposition 7.13]). Consequently, $n = 1$, $G \cong A_4$ and $p = 5$. One can show that A_4 always has a regular orbit (see [12, Proposition 14.9]).

Now we discuss the extraspecial group $G \cong D_8 * \dots * D_8$ with $n \geq 2$ factors (note that there are no elementary abelian direct summands, because $Z(G)$ is cyclic). Then $p \geq \frac{1}{2}|G| - 3 \geq 13$ by the last cases. Let $z \in Z(G)$ be the unique central involution. Then $C_P(z) = 1$, since $C_P(z)$ is G -invariant. Hence, z inverts the elements of P . The noncentral involutions in G can be paired in the form $\{x, xz\}$. Then $P = [x, P] \times C_P(x)$ and $C_P(x) \subseteq [xz, P]$. This yields $|C_P(x)| |C_P(xz)| \leq |P|$ and

$$|C_P(x)| + |C_P(xz)| \leq |C_P(x)| + \frac{|P|}{|C_P(x)|} \leq \frac{|P|}{p} + p.$$

As in (2.2),

$$|P| < \frac{m - 1}{2} \left(\frac{|P|}{p} + p \right).$$

From [2, Table 1], it follows that $m \leq \frac{3}{4}|G|$. In the case $|P| \geq p^3$, we derive the contradiction

$$p < \frac{m}{2} \left(1 + \frac{1}{p} \right) \leq \frac{3}{8}|G| \frac{14}{13} = \frac{21}{52}|G| \leq \frac{1}{2}|G| - 3.$$

Therefore, $|P| = p^2$ and $\dim P = 2$. Then again G has 2-rank at most 3, which contradicts $n \geq 2$.

In the case $\exp(G) = 3$, G has odd order, which was already excluded. Finally, the cases $G \in \{S_3 \times D_8, S_3^2, S_4, A_5\}$ give $\frac{1}{2}|G| - 3 \leq p \leq m - 1 \leq \frac{1}{2}|G|$ (see [2, Table 1]) and $k = 2$. Again one can show that these groups have regular orbits (see [12, Proposition 14.9]). □

Theorem 2.2 does not extend to $|G| = 2p + 10$. For example, the semidihedral group SD_{16} of order 16 acts faithfully on C_3^2 without having regular orbits, since $16 > 9$. Similarly, `SmallGroup(24, 8)` is an automorphism group on C_7^2 without regular orbits. There is no doubt that one can classify these examples as well, but this becomes increasingly tedious and probably will not reveal new insights.

One can check that the exception $G = D_8 * C_4$ in Theorem 2.2 actually occurs. In the following, we show that the other exceptions occur for every odd prime p .

PROPOSITION 2.3. *Let p be an odd prime. Then there exists an automorphism group G on a finite p -group P such that $|G| = 2p + 2$ and G has no regular orbit on P .*

PROOF. Let P be a two-dimensional vector space over \mathbb{F}_p . We regard P as the additive group of the field \mathbb{F}_{p^2} . Let γ be a generator of $\mathbb{F}_{p^2}^\times$. Then $H := \langle \gamma^{p-1} \rangle$ has order $p + 1$ and acts by multiplication on P . Let F be the Frobenius automorphism of \mathbb{F}_{p^2} . Then $F(\gamma^{p-1}) = \gamma^{p^2-p} = \gamma^{1-p}$ and $G := H \rtimes \langle F \rangle \cong D_{2p+2}$ acts faithfully on P . It suffices to show that F fixes every orbit of H on P . Any nontrivial element of P has the form γ^i for some $i \in \mathbb{Z}$. Then $F(\gamma^i) = \gamma^{ip} = \gamma^i \gamma^{i(p-1)}$. Hence, γ^i and $F(\gamma^i)$ lie in the same orbit of H . □

For $p = 2$, the smallest p' -automorphism group without regular orbits is the semilinear group $C_7 \rtimes C_3$ acting on C_2^3 . It is obvious from the group order that in this situation there are no regular orbits.

3. Application

In the next theorem we consider blocks of finite groups with respect to an algebraically closed field of characteristic $p > 0$. We use the standard notation, which can be found for example in [12].

THEOREM 3.1. *Let B be a p -block of a finite group with abelian defect group D and inertial index $e \leq 6p + 5$. Then Brauer’s $k(B)$ -conjecture holds for B , that is, $k(B) \leq |D|$.*

PROOF. By [12, Theorem 14.13], we may assume that $e > 255$ and therefore $p \geq 43$. The inertial quotient $I(B)$ of B is a p' -group of order e and acts faithfully on D . In order to find a large orbit of $I(B)$ on D , we may assume by [8, Lemma 2.6.2] that D is elementary abelian of rank n . Then $I(B) \leq \text{GL}(n, p)$. Let $N := I(B) \cap \text{SL}(n, p) \trianglelefteq I(B)$. We use the arguments from the proof of Theorem 2.2 to show that N has a regular

orbit on D . Suppose by way of contradiction that N has no regular orbit. Let \mathcal{M} be the set of minimal subgroups of N . Then $D = \bigcup_{M \in \mathcal{M}} C_D(M)$. Since $N \leq \text{SL}(n, p)$, we have $[[D, M]] \geq p^2$ and $|D : C_D(M)| \geq p^2$ for $M \in \mathcal{M}$. Consequently,

$$|D| < \sum_{M \in \mathcal{M}} |C_D(M)| \leq |\mathcal{M}| \frac{|D|}{p^2}$$

and $p^2 < |\mathcal{M}| \leq e \leq 6p + 5$. This contradicts $p \geq 43$.

Hence, N has a regular orbit, that is, there exists $x \in D$ such that $C_N(x) = 1$. It follows that $|C_{I(B)}(x)| \leq |I(B)/N|$. Now if $|I(B)/N|$ is a prime or 1, then the claim follows from [14, Proposition 11]. In the remaining case there exists a normal subgroup $K \trianglelefteq I(B)$ such that $|I(B)/K|$ is 4 or an odd prime. We conclude that $|K| \leq e/3 < 2p + 2$. Theorem 2.2 implies that K has a regular orbit on D . Now the claim follows from [12, Lemma 14.5] and [14, Corollary 12]. \square

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