MATRICES OVER ORTHOMODULAR LATTICES

by J. H. BEVIS

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In this paper elementary properties are established for matrices whose coordinates are elements of a lattice L. In particular, many of the results of Luce **[4]** are extended to the case where L is an *orthomodular* lattice, a lattice with an orthocomplementation denoted by $'$ in which $a \leqq b \Rightarrow a \vee (a' \wedge b) = b$. Originally, these were called orthocomplemented weakly modular lattices, Foulis **[2]**. In Theorem 1 a characterization is given of the nucleus with respect to matrix multiplication, which is in general nonassociative. Matrices with $A^{-1} = \text{transpose}(A)$ are characterized in Lemma 8. Theorems 3 and 4 respectively, give partial characterizations of zero divisors and inverses.

1. $\mathscr{A}_{mn}(L)$ and $\mathscr{A}_n(L)$ will denote, respectively, the set of all $m \times n$ matrices and the set of all $n \times n$ matrices whose coordinates are elements of L. The argument L will be suppressed when it is not needed. Except for L, capital letters denote matrices. A_{ij} will denote the (i, j)th element of the matrix A. For matrices of suitable size $A \vee B$, $A \wedge B$, and, if L has a complementation, A' are defined coordinatewise. $A \leqq B \Leftrightarrow A_{ij} \leqq B_{ij}$ for all i, j. For conformal matrices define the product by $(AB)_{ij} = \bigvee_k (A_{ik} \wedge B_{kj})$. We assume that L has a least element o and a greatest element 1. Define 0, I, and E to be the matrices with $0_{ij} = o$, $I_{ij} = 1$, $E_{ij} = o$ for $i \neq j$, and $E_{ii} = 1$, the sizes of these and other matrices being determined by the context, if not otherwise restricted. $\mathscr{A}_{mn}(L)$ is lattice isomorphic to the direct product of L with itself mn times; hence, if L is orthomodular, then so is $\mathscr{A}_{mn}(L)$. The proof of the first lemma is elementary and is therefore omitted.

LEMMA 1. *For matrices of appropriate size,*

(i) $BA \vee CA \leqq (B \vee C)A$, $AB \vee AC \leqq A(B \vee C)$, $A(B \wedge C) \leqq AB \wedge AC$, $(B \wedge C)A \leqq BA \wedge CA$,
(ii) $B \leqq C \Rightarrow AB \leqq AC$ *and* $BA \leqq CA$,
(iii) $0 \wedge A = 0A = A0 = 0$, $EA = AE = A$, $A \leqq AI \leqq I$, $A \leqq IA \leqq I$.

In an orthocomplemented lattice we say that *a commutes with b* and write $a\mathscr{C}b$ if $(a \vee b') \wedge b = a \wedge b$. The *centre* of L is defined as $\mathscr{C}(L) = \{a \in L \mid a\mathscr{C}b \text{ for all } b \in L\}$. Many of the results of Foulis [3] concerning the relation $\mathscr{C}$ will be used without making specific references. In particular, great use will be made of the Foulis–Holland theorem which states that, if L is an orthomodular lattice and two of the three relations $a\mathscr{C}b$, $a\mathscr{C}c$, $b\mathscr{C}c$ hold, then $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$ and $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$. If L is orthocomplemented, then so is $\mathscr{A}_{mn}(L)$, and in this case we have $A\mathscr{C}B \Leftrightarrow A_{ij}\mathscr{C}B_{ij}$ for all i, j and

$$\mathscr{C}(\mathscr{A}_{mn}) = \{A \in \mathscr{A}_{mn} \mid A_{ij} \in \mathscr{C}(L)\}.$$

These concepts should not be confused with matrix multiplication commuting or the multiplicative centre of $\mathscr{A}_n$. For the remainder of this section we assume that L is orthomodular.

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LEMMA 2. *Given A in $\mathscr{A}_{mn}$, $A \in \mathscr{C}(\mathscr{A}_{mn})$ if and only if any one (and thus all) of the following hold for all B, C in $\mathscr{A}_{pq}$, where p or q is chosen to make the multiplication conformal*:

$$BA \vee CA = (B \vee C)A,\ BC \vee AC = (B \vee A)C,\ AB \vee AC = A(B \vee C),\ BA \vee BC = B(A \vee C).$$

Proof. If $A \in \mathscr{C}(\mathscr{A}_{mn})$ one may easily check the identities. Conversely, choose B to have o entries except for $B_{ii} = b$ and C to have o entries except for $C_{ii} = b'$. Then $BA \vee CA = (B \vee C)A$ implies that $(b \wedge A_{ij}) \vee (b' \wedge A_{ij}) = A_{ij}$. Thus $A_{ij} \in \mathscr{C}(L)$ for all i and j. The other cases are similar.

COROLLARY. *Matrix multiplication is distributive over joins in $\mathscr{A}_n(L)$ if and only if L is Boolean.*

Proof. This follows from the lemma, since $\mathscr{C}(L) = L \Leftrightarrow L$ is Boolean.

A collection $S = \{s_\alpha \mid \alpha \in \mathscr{I}$, an indexing set$\}$ with each s_α in L, is said to have property $\mathscr{D}$ on $\mathscr{M}$, if whenever $x \in \mathscr{M}$, $x \wedge \bigvee_\alpha x_\alpha = \bigvee_\alpha (x \wedge x_\alpha)$, where $x_\alpha \in L(o, s_\alpha) = \{s \in L \mid s \leqq s_\alpha\}$. Note that, if $s_\alpha = o$ for all but possibly one α or if $L(o, s_\alpha) \subset \mathscr{C}(L)$ for all α and either $\mathscr{I}$ is finite or L is complete, then S has property $\mathscr{D}$ on L. If S has property $\mathscr{D}$ on L then $L(o, s_\alpha \wedge s_\beta)$ is distributive for each pair α, β of distinct indices. We say that a matrix A in $\mathscr{A}_{mn}$ is right $\wedge$*-distributive* if $(B \wedge C)A = BA \wedge CA$ for all $B, C \in \mathscr{A}_{pm}$. For conformal matrices (A, B, C) is an *associative triple* if $A(BC) = (AB)C$. By elementary calculations, (A, B, C) is an associative triple if the entries in two of the three matrices are in $\mathscr{C}(L)$.

T. S. Blyth [1] has characterized $\wedge$-distributive matrices over a Boolean lattice. Lemma 3 and its corollary are generalizations of one of his results.

LEMMA 3. *Given A in $\mathscr{A}_{mn}(L)$, for each $j = 1, \ldots, n$, define*

$$\mathscr{M}_j = \{\bigvee_i x_i \mid x_i \in L(o, A_{ij}),\ i = 1, \ldots, n\}.$$

A is right $\wedge$-distributive if and only if $A_{ij} \wedge A_{kj} = o$ for all i, j, k with $i \neq k$, and for each j the jth column of A satisfies $\mathscr{D}$ on $\mathscr{M}_j$.

Proof. For sufficiency,

$$(BA \wedge CA)_{ij} = [\bigvee_k (B_{ik} \wedge A_{kj})] \wedge [\bigvee_h (C_{ih} \wedge A_{hj})] \tag{1}$$

$$= \bigvee_k [B_{ik} \wedge A_{kj} \wedge \bigvee_h (C_{ih} \wedge A_{hj})] \tag{2}$$

$$= \bigvee_k \bigvee_h (B_{ik} \wedge A_{kj} \wedge C_{ih} \wedge A_{hj}) = \bigvee_k (B_{ik} \wedge C_{ik} \wedge A_{kj}) = [(B \wedge C)A]_{ij}. \tag{3}$$

Conversely, set $B = E$ and $C = E'$; then $0 = (E \wedge E')A = A \wedge E'A$ implies that $A_{ij} \wedge A_{kj} = o$ for all i, j, k with $i \neq k$. If B is chosen so that the join over k has only one term, we obtain (2) = (3) which implies that, for each j, the jth column of A satisfies $\mathscr{D}$ on $\bigcup_i L(o, A_{ij})$. With this we obtain (1) = (2) for any B in $\mathscr{A}_{pm}$ and the necessity follows.

COROLLARY. *If $B \leqq A$ and A is right $\wedge$-distributive, then B is right $\wedge$-distributive.*

Similar results are obtained for left $\wedge$-distributive matrices. For the rest of Section 1, take $n \geqq 2$.

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LEMMA 4. *Let* $A, B, C \in \mathscr{A}_n$.

(i) (A, B, C) *is an associative triple for all* B *and* C *if and only if* $A \in \mathscr{C}(\mathscr{A}_n)$ *and each row of* A *satisfies* $\mathscr{D}$ *on* L.

(ii) (B, C, A) *is an associative triple for all* B *and* C *if and only if* $A \in \mathscr{C}(\mathscr{A}_n)$ *and each column of* A *satisfies* $\mathscr{D}$ *on* L.

(iii) (B, A, C) *is an associative triple for all* B *and* C *if and only if each row and column of* A *satisfies* $\mathscr{D}$ *on* L.

Proof. For necessity in (i), let B have 1's in the jth row and o's elsewhere. By examining the (i,j)th element of the equation $(AB)C = A(BC)$, we obtain $\bigvee_h (C_{hk} \wedge A_{ij}) = A_{ij} \wedge \bigvee_h C_{hk}$. Thus C may be chosen to imply that $A_{ij} \in \mathscr{C}(L)$ for any i, j. Now, with A in $\mathscr{C}(\mathscr{A}_n)$, choose C to have o's except in the hth row, to obtain $\bigvee_k (A_{ik} \wedge B_{kh} \wedge C_{hj}) = C_{hj} \wedge \bigvee_k (A_{ik} \wedge B_{kh})$. This holds for all B_{kh} and C_{hj} if and only if the ith row of A satisfies $\mathscr{D}$ on L. For sufficiency, take the join with respect to h of both sides of the above equation. The other parts of the lemma are obtained in a similar manner.

The *nucleus* of $\mathscr{A}_n$ is $\{A \in \mathscr{A}_n \mid \text{any triple containing } A \text{ is associative}\}$. If one defines a *scalar meet* by $(x \wedge A)_{ij} = x \wedge A_{ij}$ for x in L, it may be shown by standard methods that $AB = BA$ for all B in $\mathscr{A}_n$ if and only if there is an a in L such that $A = a \wedge E$. The *multiplicative centre* of $\mathscr{A}_n$ is defined as the set of all A in the nucleus of $\mathscr{A}_n$ such that $AB = BA$ for all $B \in \mathscr{A}_n$. Since each row or column of $a \wedge E$ has only one nonzero element, the rows and columns of $a \wedge E$ satisfy $\mathscr{D}$ on L. We have obtained

THEOREM 1. *The nucleus of* $\mathscr{A}_n$ *is* $\{A \in \mathscr{C}(\mathscr{A}_n) \mid \text{rows and columns of } A \text{ satisfy } \mathscr{D} \text{ on } L\}$. *The multiplicative centre of* $\mathscr{A}_n$ *is* $\{a \wedge E \mid a \in \mathscr{C}(L)\}$.

COROLLARY. *Matrix multiplication is associative in* $\mathscr{A}_n(L)$ *if and only if* L *is Boolean.*

2. The *transpose* of a matrix A is defined to be A^t where $(A^t)_{ij} = A_{ji}$. It follows that $(A \vee B)^t = A^t \vee B^t$, $(A \wedge B)^t = A^t \wedge B^t$, $A^{tt} = A$, $(AB)^t = B^t A^t$, for A and B of suitable size, and, if L has a complementation, $(A^t)' = (A')^t$. If L is orthocomplemented, for square matrices we say that A is *symmetric* if $A' \perp A^t$ ($A \perp B \Leftrightarrow A \leqq B'$; in this case A and B are said to be *orthogonal*), and A is *skew-symmetric* if $A \perp A^t$.

LEMMA 5. *If* $A \in \mathscr{A}_n(L)$ *with* L *orthocomplemented, then* $A' \perp A^t \Leftrightarrow A = A^t$.

Proof. By taking the transpose of $A^t \leqq A$, we obtain $A \leqq A^t$. Hence

$$A' \perp A^t \Leftrightarrow A^t \leqq A \Leftrightarrow A = A^t.$$

THEOREM 2. *For* A *in* $\mathscr{A}_n(L)$ *with* L *orthomodular,* A *has an orthogonal decomposition into a symmetric and skew-symmetric matrix (i.e.,* $A = S \vee Q$ *with* S *symmetric,* Q *skew-symmetric, and* $S \perp Q$*) if and only if* $A \mathscr{C} A^t$. *If the decomposition exists, it is unique.*

Proof. Suppose that S and Q exist such that $A = S \vee Q$, $S \perp Q$, $S' \perp S^t$ and $Q \perp Q^t$. Thus $A \wedge A^t = (S \vee Q) \wedge (S \vee Q^t) = S \vee (Q \wedge Q^t) = S$ and

$$A \wedge A^{t\prime} = (S \vee Q) \wedge (S' \wedge Q^{t\prime}) = Q \wedge S' \wedge Q^{t\prime} = Q.$$

Hence, if the decomposition exists, it is unique. Now $A = (A \wedge A^t) \vee (A \wedge A^{t\prime})$ implies that

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$A\mathscr{C}A^t$. If $A\mathscr{C}A^t$, then $A=(A\wedge A^t)\vee(A\wedge A^{t'})$, $A\wedge A^t\leqq A'\vee A^t=(A\wedge A^{t'})'$, $(A\wedge A^t)^t=A^t\wedge A$ and $A\wedge A^{t'}\leqq A\vee A^{t'}=(A^{t'}\wedge A)^{t'}$. Thus $S=A\wedge A^t$, and $Q=A\wedge A^{t'}$ is the required decomposition.

For A,B in $\mathscr{A}_n(L)$ with L orthocomplemented the following results are easily obtained. If A and B are skew-symmetric, then $A\wedge B$ is skew-symmetric. For A and B symmetric we obtain (i) $A\vee B$, $A\wedge B$, A^t and A' are symmetric, (ii) AB is symmetric if and only if $AB=BA$.

3. A is called *row* (*column*) *consistent* if $AI=I$ ($IA=I$), where the I's may not have the same size. The proofs of the lemmas of this section are elementary and are therefore omitted.

LEMMA 6. *The following are equivalent whenever A is of appropriate size*:
(i) *A is row* (*column*) *consistent.*
(ii) $\bigvee_k A_{ik}=1$ *for all* i ($\bigvee_k A_{kj}=1$ *for all* j).
(iii) $E\leqq AA^t$ ($E\leqq A^tA$).

LEMMA 7. *If AB is row* (*column*) *consistent, then A is row consistent* (*B is column consistent*).

COROLLARY. *If A has a left* (*right*) *inverse, then A is column* (*row*) *consistent and its left* (*right*) *inverse is row* (*column*) *consistent.*

Let L be the orthocomplemented modular lattice of subspaces of Euclidean 2-space. Let $A,B\in\mathscr{A}_n(L)$ be such that $A_{ij},B_{ij}\neq o,1$. If all of the elements of A and B are distinct, then A and B are both row and column consistent, but $AB=0$. Thus the converse of Lemma 7 that Luce [4] proved for L Boolean, does not obtain in general. For L Boolean, Rutherford [5] has shown that, for square matrices, A has a one sided inverse $\Leftrightarrow A$ has a two sided inverse $\Leftrightarrow A^{-1}=A^t$; examples similar to the one above show that, if $A\notin\mathscr{C}(\mathscr{A}_n)$, then A may have several one or two sided inverses. However the next lemma, which is due to Luce [4], holds for arbitrary lattices with o and 1.

LEMMA 8. $AA^t=E$ ($A^tA=E$) *if and only if A is row* (*column*) *consistent and* $A_{ik}\wedge A_{jk}=o$ ($A_{kj}\wedge A_{ki}=o$) *for all i,j,k with $i\neq j$.*

4. In this section we assume that L is orthomodular and that $B\in\mathscr{C}(\mathscr{A}_{mn})$. Conditions are given for finding a matrix X satisfying $XA\leqq B$ or $XA\geqq B$. The results can then be applied to the matrix equation $XA=B$. Dual statements are given for results concerning $AX\leqq B$, $AX\geqq B$, and $AX=B$.

LEMMA 9. (i) *If* $X\wedge(B'A^t)=0$, *then* $XA\leqq B$. *If* (X,A,B'^t) *is an associative triple, then* $XA\leqq B\Rightarrow X\wedge(B'A^t)=0$. (*Note that A in $\mathscr{A}_{pn}$ implies that $B'A^t$ and X are in $\mathscr{A}_{mp}$.*) (ii) *If* $X\wedge(A^tB')=0$, *then* $AX\leqq B$. *If* (B'^t,A,X) *is an associative triple, then* $AX\leqq B\Rightarrow X\wedge(A^tB')=0$. (*Note that A in $\mathscr{A}_{mp}$ implies that A^tB' and X are in $\mathscr{A}_{pn}$.*)

Proof. By examining the (i,i)th element of the matrix product, one notes that

$$XY\leqq E'\Leftrightarrow X\wedge Y^t=0\Leftrightarrow YX\leqq E',\quad\text{for}\quad X,Y^t\quad\text{in}\quad\mathscr{A}_{mp}.$$

Now

$$X\wedge(B'A^t)=0\Rightarrow X(AB'^t)\leqq E'\Rightarrow X_{ik}\wedge A_{kj}\wedge B'_{ij}=o\quad\text{for all}\quad i,j,k.$$

Since $B \in \mathscr{C}(\mathscr{A}_{mn})$, we obtain $X_{ik} \wedge A_{kj} \leqq B_{ij}$ and thus $(XA)_{ij} \leqq B_{ij}$. Conversely, $XA \leqq B$ implies that $o = X_{ik} \wedge A_{kj} \wedge B'_{ij} = \bigvee_k (X_{ik} \wedge A_{kj} \wedge B'_{ij}) = (XA)_{ij} \wedge (B^t)'_{ji}$, the last step being accomplished since $B_{ij} \in \mathscr{C}(L)$. Taking the join over j, we obtain $E' \geqq (XA)B'^t = X(AB'^t)$. The result follows from the remark at the beginning of the proof.

THEOREM 3. *If A is not row (column) consistent, then A is a right (left) divisor of zero. If $A \in \mathscr{C}(\mathscr{A}_{pn})$ $(A \in \mathscr{C}(\mathscr{A}_{mp}))$ or if $A \in \mathscr{A}_{p1}$ $(A \in \mathscr{A}_{1p})$, then A is a right (left) divisor of zero if and only if A is not row (column) consistent.*

Proof. Set $B = 0$ in $\mathscr{A}_{mn}$ and $B'^t = I$ in $\mathscr{A}_{nm}$. If A is not row consistent, then $AI < I$ in $\mathscr{A}_{pm}$. Thus $0 < X \leqq (AI)'^t \Rightarrow X \wedge (AI)^t = 0 \Rightarrow XA = 0 \Rightarrow A$ is a right zero divisor. Now $\bigvee_h \bigvee_k (X_{ik} \wedge A_{kh}) = \bigvee_k (X_{ik} \wedge \bigvee_h A_{kh})$ and (X, A, I) is an associative triple, if $n = 1$ or if $A \in \mathscr{C}(\mathscr{A}_{pn})$. Hence in either case $XA = 0$ implies that $X \wedge (AI)^t = 0$. But, if A is row consistent, $(AI)^t = I$ in $\mathscr{A}_{mp}$ and $X = X \wedge I = 0$.

LEMMA 10. *If there is a matrix C such that $C \leqq A$ and $B \leqq IC$ $(B \leqq CI)$, then any X such that $X \geqq BC^t$ $(X \geqq C^t B)$ is a solution of $XA \geqq B$ $(AX \geqq B)$.*

Proof. If $A, C \in \mathscr{A}_{pn}$, then $X, I \in \mathscr{A}_{mp}$. With $j, k = 1, \ldots, n$,

$$[(BC^t)A]_{ij} = \bigvee_h [A_{hj} \wedge \bigvee_k (B_{ik} \wedge C_{hk})] \geqq \bigvee_h (A_{hj} \wedge B_{ij} \wedge C_{hj}) = B_{ij} \wedge (IC)_{ij} = B_{ij}.$$

COROLLARY. *If $X \geqq IC^t$ $(X \geqq C^t I)$, where $C \leqq A$ and C is column (row) consistent, then $XA = I$ $(AX = I)$.*

Proof. Set $B = I$ in Lemma 10.

LEMMA 11. *$XA \geqq E$ $(AX \geqq E)$ has a solution if and only if A is column (row) consistent.*

Proof. The result follows from Lemmas 6 and 7.

THEOREM 4. *For square matrices, if $C^t \leqq X \leqq (E'A^t)'$ $(C^t \leqq X \leqq (A^t E')')$, where $C \leqq A$ and C is column (row) consistent, then X is a left (right) inverse of A.*

Proof. The result is obtained by letting $B = E$ in Lemmas 9 and 10.

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VIRGINIA POLYTECHNIC INSTITUTE
BLACKSBURG, VIRGINIA

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