On the lines which intersect three given lines in space

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The following is an elementary discussion of certain propositions in solid geometry which are commonly left to a later stage.

Let ABCD be a quadrilateral formed of rods freely jointed, let points E, F, G, H be taken on AB, BC, CD, DA respectively and let EG, FH be joined by rods freely jointed to the former. Let us investigate whether the resulting framework is rigid.

It will be admitted that, when the given conditions enable us to find the lengths AC, BD, they fix the form of the framework. Let AE, EB, BF, FC, CG, GD, DH, HA, EG, FH be denoted by a, b, c, d, e, f, g, h, x, y respectively.

Then we have

$$b.\mathrm{CA}^2 + a.\mathrm{CB}^2 - (a+b)\,\mathrm{CE}^2 = ab(a+b),$$
$$f.\,\mathrm{EC}^2 + e.\mathrm{ED}^2 - (e+f)\,\mathrm{EG}^2 = ef(e+f),$$
$$b.\mathrm{DA}^2 + a.\mathrm{DB}^2 - (a+b)\mathrm{DE}^2 = ab(a+b)$$

and by elimination of CE, DE

$$fb.\mathrm{CA}^2 + ea.\mathrm{DB}^2 = (x^2+ab+ef)(a+b)(e+f) - af(c+d)^2 - be(g+h)^2 \quad \text{-} \quad (1)$$

for $\mathrm{BC} = c+d$, $\mathrm{DA} = g+h$.

In like manner, or by the interchanges ad, bc, eh, fg, xy, we have

$$cg.\mathrm{CA}^2 + dh.\mathrm{DB}^2 = (y^2+cd+gh)(c+d)(g+h) - dg(a+b)^2 - ch(e+f)^2 \quad \text{-} \quad (2)$$

The equations (1) and (2) determine CA^2 and DB^2 unless $aceg = bdfh$, that is, unless EFGH lie in one plane or in other words EG meets FH. Hence *the framework is rigid unless the rods* EG, FH *meet each other.*

If they meet then AC, BD are not determined by (1) and (2) but a relation must be satisfied by x, y, a, b...that is, the connexion FH suffices to keep the distance EG constant, and *vice versa.*

Further, if EG, FH meet in P, we have the sides of the quadrilateral ABFH met by a plane in E, C, P, D and therefore

$$\frac{\mathrm{HP}}{\mathrm{PF}} = \frac{\mathrm{HD}}{\mathrm{DA}} \cdot \frac{\mathrm{AE}}{\mathrm{EB}} \cdot \frac{\mathrm{BC}}{\mathrm{CF}} = \frac{g}{g+h} \cdot \frac{a}{b} \cdot \frac{c+d}{d}.$$

Hence P is a fixed point in FH and similarly in EG. Thus there would be no gain of rigidity by freely jointing the rods EG, FH to each other at P.

There would be no loss of rigidity if EG were taken away, nor on the other hand would there be any gain if another rod were added, connecting a point on AB with one on CD and *meeting FH*, or even any number of such rods, even if freely jointed with FH. The same is true of any number of rods from AD to BC, meeting EG.

Suppose then that we add E′G′ a rod meeting AB, FH, CD and F′H′ a rod meeting AD, EG, BC: the framework is not made stiff, and still less would E′G′ and F′H′ by themselves stiffen the quadrilateral ABCD. Hence E′G′ and F′H′ must meet each other. It follows then that *if three lines all meet three others, any line meeting the first triad and any line meeting the second triad must meet each other, and if any number of such lines are replaced by freely jointed rods the resulting framework is deformable.*

The first part of this theorem could be proved separately as follows, with the former notation.

Since EG meets FH, E, F, G, H lie in a plane and

$$AE . BF . CG . DH = EB . FC . GD . HA; \qquad - \quad - \quad - \quad - \qquad (3)$$

Since E′G′ meets FH, similarly

$$AE' . BF . CG' . DH = E'B . FC . G'D . HA; \qquad - \quad - \quad - \quad - \qquad (4)$$

Since EG meets F′H′,

$$AE . BF' . CG . DH' = EB . F'C . GD . H'A \qquad - \quad - \quad - \quad - \qquad (5)$$

Multiplying (4) and (5) and dividing by (3) we have

$$AE' . BF' . CG' . DH' = E'B . F'C . G'D . H'A$$

from which it follows that E′G′ meets F′H′.

We see also that the quadrilateral ABCD will not be stiffened by two rods, EG, E′G′ freely jointed with AB and CD if EG, E′G′, AD, BC all meet one line. It is easy to prove that this is the only case of failure.

The equation (1) still holds, and if AE′, E′B, CG′, G′D, E′G′ are denoted by a', b', e', f', x' respectively, we have also

$$\begin{aligned} f'b' . CA^2 + e'a' . DB^2 = (x'^2 + a'b' + e'f')\,(a' + b')\,(e' + f') \\ - a'f'\,(c+d)^2 - b'e'\,(g+h)^2 \qquad - \quad - \qquad (6) \end{aligned}$$

The equations (1) (6) fix CA, DB, so that the framework is stiff, unless $fb/f'b' = ea/e'a'$.

This condition expresses that the cross ratios AEE′B, DGG′C are equal and also shews that any line meeting AD, BC, EG, will meet E′G′,

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These results can be roughly verified by using hat-pins jointed together with small pieces of sheet india-rubber, as, for instance, from an old bicycle tire.

It has been assumed that the quadrilateral ABCD will not be stiffened by a single rod such as EG or FH. To see this, take ABCD with AB and AD fixed at any angle: then BD is fixed and the locus of G is a circle with BD as axis: this circle will meet a sphere, with centre E and radius not too small or too great, in two points. Thus EG can take any value, within certain limits, when the angle BAD has any value, or in other words fixing the length EG does not fix the angle BAD in general. There is an exception when EG has its maximum or minimum value.