A MOMENT PROBLEM

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1. Introduction

Let \( v \) be a discrete random variable taking on nonnegative integer values and set \( P\{v = k\} = P_k, \; k = 0, 1, \cdots \). Suppose that the binomial moments

\[
B_r = \mathbb{E}\left(\binom{v}{r}\right) = \sum_{k=r}^{\infty} \binom{k}{r} P_k, \quad r = 0, 1, \cdots,
\]

are finite. Frequently the problem arises under what conditions the probabilities \( P_k, \; k = 0, 1, \cdots \), can be determined uniquely by the sequence of moments \( B_r, \; r = 0, 1, \cdots \), and how it can be done.

In what follows we shall show that if \( \limsup_{r \to \infty} B_r < \infty \), then \( \{P_k\} \) can be determined uniquely by \( \{B_r\} \) and we shall give an explicit formula for \( P_k, \; k = 0, 1, \cdots \). If \( \limsup_{r \to \infty} B_r = \infty \), then, in general, \( \{P_k\} \) cannot be determined uniquely by \( \{B_r\} \).

2. An inversion formula

The probabilities \( P_k, \; k = 0, 1, \cdots \), can be determined in several possible ways, but formula (2) seems to be the most convenient one.

**Theorem.** Let \( v \) be a discrete random variable taking on nonnegative integer values and set \( P\{v = k\} = P_k, \; k = 0, 1, \cdots \). If the binomial moments \( B_r = \mathbb{E}(\binom{v}{r}), \; r = 0, 1, \cdots \), are finite and if \( \rho = \limsup_{r \to \infty} B_r < \infty \), then

\[
P_k = \sum_{r-k \geq 0} \binom{k}{r+k} \sum_{j=k}^{\infty} (-1)^{j-k} \binom{r}{j-k} q^{r-j} B_j
\]

where \( q \) is nonnegative and greater than \((\rho^2-1)\). If, in particular, \( \rho = \limsup_{r \to \infty} B_r < 1 \), then we can always choose \( q = 0 \) and (2) reduces to

\[
P_k = \sum_{r=k}^{\infty} (-1)^{r-k} \binom{r}{k} B_r.
\]

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PROOF. The generating function

\[ P(z) = \sum_{j=0}^{\infty} P_j z^j \]

is uniformly convergent in the circle \(|z| < 1| and \(P(z)\) is regular if \(|z| < 1|.

Hence

\[ P^{(k)}(z) = \frac{d^k P(z)}{dz^k} = k! \sum_{j=0}^{\infty} \binom{j}{k} P_j z^{j-k} \]

for \(k = 0, 1, \cdots \) and \(|z| < 1|.

If \(z = 0\) in (5), then we get

\[ P_k = \frac{1}{k!} P^{(k)}(0). \]

Thus the problem of finding \(P_k\) can be reduced to finding \(P(z)\) in a neighborhood of \(z = 0|.

If \(B_r\) is finite, then

\[ B_r = \frac{1}{r!} \left( \frac{d^r P(z)}{dz^r} \right)_{z=1} \]

and for \(|z-1| < 1/r\)

\[ P(z) = \sum_{r=0}^{\infty} B_r (z-1)^r. \]

The right hand side of (8) is uniformly convergent in the circle \(|z-1| < 1/r|\) and \(P(z)\) is regular if \(|z-1| < 1/r|.

Hence

\[ P^{(k)}(z) = \frac{d^k P(z)}{dz^k} = k! \sum_{r=0}^{\infty} \binom{r}{k} B_r (z-1)^{r-k} \]

for \(k = 0, 1, \cdots\), and \(|z-1| < 1/r|.

If \(\rho < 1\), and we put \(z = 0\) in (9), then by (6) we get (3). We note that (3) is an oscillating series which is convergent if and only if \(\lim_{r \to \infty} r^kB_r = 0|.

If \(\rho < \infty|, then (9) is a regular function of \(z) in the circle \(|z-1| < 1/r|.

By analytical continuation we can extend the definition of (9) to the domain \(|z| < 1| and in this domain (9) agrees with (5). Now we shall show that the definition of (9) can easily be extended to a neighborhood of \(z = 0| by using Euler's transformation of series. (Cf. Hardy [2], Chapter VIII.) Let \(q \geq 0\) and form the \(E_q\)-transform of (9),

\[ P_q^{(k)}(z) = k! \sum_{r-k}^{\infty} \frac{1}{(1+q)^{r+1}} \sum_{j=0}^{r} \binom{r}{j} \binom{r-j}{k} q^{r-j}(z-1)^{j-k} B_j. \]

For \(|z-1| < 1/r| we have \(P_q^{(k)}(z) = P^{(k)}(z)\) given by (9) because Euler's transformation is consistent. Now by using a theorem of Knopp [6] we can
establish the domain \( R_q \) in which (10) is convergent and represents a regular function of \( z \). Suppose that \( P_q^{(k)}(z) \) is analytically extended along every ray of origin \( z = 1 \) until we reach the first singular point (if any) of \( P_q^{(k)}(z) \) on the ray. Denote by \( \Gamma \) the set of all singular points obtained in this way. Then \( R_q \) can be represented as the set of points common to all the circles 
\[ |z-\gamma| < (1+q)|\gamma-1| \text{ for } \gamma \in \Gamma. \]
Evidently \( |\gamma-1| \geq 1/p \) and \( |\gamma| \geq 1 \) for all \( \gamma \in \Gamma \) and there exists a \( \gamma \in \Gamma \) such that \( |\gamma-1| = 1/p \). Hence it follows that \( R_q \) always contains the point \( z = 0 \) if \( q > (p^2-1) \) and \( R_q \) never contains \( z = 0 \) if \( q \leq (p-1)/2 \). For example, if \( |\gamma| \geq (1+\rho)/p \) for every \( \gamma \in \Gamma \), then we can choose \( q \) as any nonnegative number greater than \((p-1)/2 \), however, if \( \Gamma \) contains a \( \gamma \) for which \( |\gamma-1| = 1/p \) and \( |\gamma| = 1 \), then \( q \) must be chosen greater than \((p^2-1) \) in order that \( R_q \) contain \( z = 0 \). Accordingly if \( q > (p^2-1) \), then in some neighborhood of \( z = 0 \) we have \( P_q^{(k)}(z) = P^{(k)}(z) \) given by (5). Thus by (6) we have \( P_k = P_q^{(k)}(0)/k! \), \( k = 0, 1, \cdots \), which yields (2).

3. Examples
(i) Suppose that \( B_r = E\{\mathbb{C}\} = a^r/r! \), \( r = 0, 1, \cdots \), where \( a \) is a positive number. Then \( \lim_{r \to \infty} B_r^{1/r} = 0 \) and \( \rho = 0 \). By (3)
\[
P_k = \sum_{r=k}^{\infty} (-1)^{r-k} \binom{r}{k} \frac{a^r}{r!} = e^{-a} \frac{a^k}{k!}, \quad k = 0, 1, \cdots.
\]
(ii) Suppose that \( B_r = E\{\mathbb{C}\} = a^r, r = 0, 1, \cdots \), where \( a \) is a positive number. Then \( \lim_{r \to \infty} B_r^{1/r} = a \) and \( \rho = a \). If \( a < 1 \), then we can apply formula (3) to obtain
\[
P_k = \sum_{r=k}^{\infty} (-1)^{r-k} \binom{r}{k} a^r = \frac{a^k}{(1+a)^{k+1}}, \quad k = 0, 1, \cdots.
\]
If \( a < \infty \), then we obtain by (2) that
\[
P_k = \sum_{r=k}^{\infty} \binom{r}{k} \frac{a^k(q-a)^{r-k}}{(1+q)^{r+1}} = \frac{a^k}{(1+a)^{k+1}}, \quad k = 0, 1, \cdots,
\]
where \( q > (a-1)/2 \). The best choice is \( q = a \).
(iii) Let \( A_1, A_2, \cdots, A_n, \cdots \) be an infinite sequence of events. Denote by \( n \) the number of events occurring among \( A_1, A_2, \cdots, A_n, \cdots \). It can easily be seen that
\[
B_r = E\{\mathbb{C}\} = \sum_{1 \leq i_1 < i_2 < \cdots < i_r < \infty} P\{A_{i_1} A_{i_2} \cdots A_{i_r}\}.
\]
If \( \rho = \limsup_{r \to \infty} B_r^{1/r} < 1 \), then the probability that exactly \( k \) events
occur among \( A_1, A_2, \ldots, A_n, \ldots \) is given by (3). Formula (3) was found first by Jordan [3], [4], [5], for the case when \( A_{n+1} = A_{n+2} = \cdots = 0 \), the impossible event. (Cf. also [8].)

If \( \rho = \lim \sup_{r \to \infty} B_{r}^{(r)} < \infty \), then the probability that exactly \( k \) events occur among \( A_1, A_2, \ldots, A_n, \ldots \) is given by (2) with \( q > (\rho^2 - 1) \). In some particular cases we can choose \( q > (\rho-1)/2 \).

(iv) Consider the previous example. The probability that at least one event occurs among \( A_1, A_2, \ldots, A_n, \ldots \) is given by (2) with \( q > (p^2 - 1) \). In some particular cases we can choose \( q > (p-1)/2 \).

(15) \[
P\{A_1 + A_2 + \cdots + A_n + \cdots\} = \sum_{r=1}^{\infty} (-1)^{r-1} B_r.
\]
If \( p < \infty \), then by (2)

(16) \[
P\{A_1 + A_2 + \cdots + A_n + \cdots\} = 1 - \sum_{r=0}^{\infty} \frac{1}{(1+q)^{r+1}} \sum_{j=0}^{r} \binom{r}{j} q^{r-j} B_r,
\]
where \( q > (\rho^2 - 1) \). In some particular cases we can choose \( q > (\rho-1)/2 \).

Formula (15) was found first by Poincaré [7] for the case when \( A_{n+1} = A_{n+2} = \cdots = 0 \), the impossible event. Dvoretzky [1] proved that (15) holds if \( \lim_{r \to \infty} B_r = 0 \).

(v) It is interesting to mention also the following simple example. A balanced coin is tossed repeatedly. We say that event \( A_n \) occurs if head does not appear among the first \( n \) tossings. Denote by \( v \) the number of events occurring among \( A_1, A_2, \ldots, A_n, \ldots \). By (14) \( B_r = B\{r\} = 1 \) for \( r = 0, 1, \ldots \). In this case (3) is divergent, but by (2) with \( q > 0 \) we get that \( P_k = P\{v = k\} = 1/2^{k+1} \) for \( k = 0, 1, \ldots \), in agreement with a direct calculation.

References


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