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CARLEMAN ESTIMATE AND UNIQUE CONTINUATION PROPERTY FOR THE LINEAR STOCHASTIC KORTEWEG–DE VRIES EQUATION

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Abstract

In this paper, we obtain the well posedness of the linear stochastic Korteweg–de Vries equation by the Galerkin method, and then establish the Carleman estimate, leading to the unique continuation property (UCP) for the linear stochastic Korteweg–de Vries equation. This UCP cannot be obtained from the classical Holmgren uniqueness theorem.

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1. Introduction

The Korteweg–de Vries (KdV) equation was first derived by Korteweg and de Vries in 1895 as a model for the propagation of some surface water waves along a channel [4]. It has been intensively studied from various aspects of both mathematics and physics since the 1960s. It turns out that the equation is not only a good model for some water waves but also a very useful approximation model in nonlinear studies whenever one wishes to include and balance a weak nonlinearity and weak dispersive effects.

In recent years, a great deal of effort has been devoted to studying the controllability of stochastic partial differential equations (see, for instance, [1, 6, 7, 9, 10]). The Carleman estimates for the stochastic heat equation, wave equation and Schrödinger equation are complete, but nothing is known for the third-order stochastic dispersion equation. The main purpose of this paper is to establish the Carleman estimate for the following forward linear stochastic equation:

$$\begin{cases} dy + (y_x + y_{xxx})\,dt = f\,dt + g\,dw & \text{in } Q, \\ y(0,t) = 0 = y(1,t) & \text{in } (0,T), \\ y_x(1,t) = 0 & \text{in } (0,T), \\ y(x,0) = y_0(x) & \text{in } I, \end{cases} \tag{1.1}$$

where Q, T and I will be given later.

Using this Carleman estimate, we can obtain the unique continuation property (UCP) for the linear stochastic KdV equation. To the author's knowledge, this Carleman estimate is new: it is the first attempt for the linear stochastic KdV equation. The Carleman estimate with internal observation for the deterministic KdV equation was established in [2]; however, the method cannot be used for the stochastic KdV equation.

Throughout this paper, we make the following assumptions on the coefficients.

H1 Let $I = (0, 1)$, $T > 0$ and I_0 be a given nonempty open subset of I. We write Q and Q^{I_0} to stand for $(0, T) \times I$ and $(0, T) \times I_0$, respectively.

H2 Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete filtered probability space on which a one-dimensional standard Brownian motion $\{w(t)\}_{t\geq 0}$ is defined such that $\{\mathcal{F}_t\}_{t\geq 0}$ is the natural filtration generated by $w(\cdot)$, augmented by all the P-null sets in $\mathcal{F}$. Let H be a Banach space and let $C([0, T]; H)$ be the Banach space of all H-valued strongly continuous functions defined on $[0, T]$. We denote by $L^2_{\mathcal{F}}(0, T; H)$ the Banach space consisting of all H-valued $\{\mathcal{F}_t\}_{t\geq 0}$-adapted processes $X(\cdot)$ such that $E(|X(\cdot)|_{L^2(0,T;H)}) < \infty$, with the canonical norm; by $L^\infty_{\mathcal{F}}(0, T; H)$ the Banach space consisting of all H-valued $\{F_t\}_{t\geq 0}$-adapted bounded processes; and by $L^2_{\mathcal{F}}(\Omega; C([0, T]; H))$ the Banach space consisting of all H-valued $\{\mathcal{F}_t\}_{t\geq 0}$-adapted continuous processes $X(\cdot)$ such that $E(|X(\cdot)|^2_{C([0,T];H)}) < \infty$, with the canonical norm. We set $X_T = L^2_{\mathcal{F}}(\Omega; C([0, T]; H^3(I))) \cap L^2_{\mathcal{F}}(0, T; H^4(I))$.

H3 $y_0 \in L^2(\Omega, \mathcal{F}_0, P; H^3(I) \cap H^1_0(I))$, $y_{0x}(1) = 0$, P-a.s., $f \in L^2_{\mathcal{F}}(0, T; H^2(I) \cap H^1_0(I)))$, $f_x(t, 1) = 0$, $g \in L^2_{\mathcal{F}}(0, T; H^3(I) \cap H^1_0(I)))$, $g_x(t, 1) = 0$.

H4 Let $\psi \in C^\infty(\bar{I})$ satisfy $\psi > 0$ in I, $\psi(0) = \psi(1) = 0$, $\|\psi\|_{C(\bar{I})} = 1$, $|\psi_x| > 0$ in $\bar{I}\backslash I_0$, $\psi_x(0) > 0$ and $\psi_x(1) < 0$. For any given positive constants λ and μ, we set $a(x, t) = (e^{\mu(\psi(x)+3)} - e^{5\mu})/t(T - t)$, $l = \lambda a(x, t)$, $\theta(x, t) = e^l$ and $\varphi(x, t) = e^{\mu(\psi(x)+3)}/t(T - t)$, for all $(x, t) \in Q$.

One of the main results in this paper is the following theorem.

THEOREM 1.1. *Let assumptions H1–H4 be satisfied. There are two positive constants λ_0 and C such that, for all $\lambda \geq \lambda_0$,*

$$\begin{aligned} &E\int_Q (\lambda\varphi\theta^2 y_{xx}^2 + \lambda^3\varphi^3\theta^2 y_x^2 + \lambda^5\varphi^5\theta^2 y^2)\,dx\,dt \\ &\quad\leq C\Big(E\int_Q (\theta^2 f^2 + \lambda^3\varphi^3\theta^2 g^2 + \lambda\varphi\theta^2 g_x^2)\,dx\,dt \\ &\qquad + E\int_{Q^{I_0}} (\lambda\varphi\theta^2 y_{xx}^2 + \lambda^5\varphi^5\theta^2 y^2)\,dx\,dt\Big), \end{aligned} \tag{1.2}$$

where y is the solution of (1.1) corresponding to y_0, f, g.

REMARK 1.2. The definition of a solution to (1.1) can be found in Section 2.

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This paper is organised as follows. Section 2 is devoted to the well posedness of (1.1). In Section 3, the Carleman estimate for the forward stochastic KdV equation is established. The UCP for some stochastic dispersion equations is obtained in Section 4.

2. Well posedness

According to [3], (1.1) has a unique mild solution in $L^2_{\mathcal{F}}(\Omega; C([0,T]; L^2(I)))$, but the regularity of the solution is not enough to establish the Carleman estimate in Theorem 1.1. Thus, we must improve the regularity of the solution.

Let us explain what we mean by a solution of the linear stochastic KdV equation.

DEFINITION 2.1. We call $y \in X_T$ a solution of (1.1) if the following hold:

(i) $y(0) = y_0$ in I, P-a.s.;

(ii) for any $t \in [0,T]$ and any $\varphi \in L^2(I)$,

$$\int_I y(t,x)\varphi(x)\,dx - \int_I y_0\varphi(x)\,dx + \int_0^t (y_{xxx}(s) + y_x(s), \varphi)_{L^2(I)}\,ds$$
$$= \int_0^t (f(s),\varphi)_{L^2(I)}\,ds + \int_0^t (g(s),\varphi)_{L^2(I)}\,dw.$$

In the sequel, C stands for a generic positive constant whose value can be changed from line to line.

THEOREM 2.2. *Let assumptions H1–H3 be satisfied. Then (1.1) has a unique solution $y \in X_T$ satisfying $y_x(1,t) = 0$ for almost every $t \in [0,T]$ P-a.s. Moreover, the following inequality holds:*

$$\|y\|_{X_T} \le C(\|f\|_{L^2_{\mathcal{F}}(0,T;H^2(I))} + \|g\|_{L^2_{\mathcal{F}}(0,T;H^3(I))} + \|y_0\|_{L^2(\Omega,\mathcal{F}_0,P;H^3(I))}). \tag{2.1}$$

PROOF. By the Banach fixed point theorem, energy estimates and the standard extension argument, we know that it is sufficient to prove Theorem 2.2 for the system

$$\begin{cases} dy + y_{xxx}\,dt = f\,dt + g\,dw & \text{in } Q,\\ y(0,t) = 0 = y(1,t) & \text{in } (0,T),\\ y_x(1,t) = 0 & \text{in } (0,T),\\ y(x,0) = y_0(x) & \text{in } I. \end{cases} \tag{2.2}$$

From [5], we know that the eigenvalue problem

$$\begin{cases} \Lambda\varphi = \lambda\varphi \quad \text{in } I,\\ \varphi(0) = \varphi_{xxx}(0) = \varphi_{xxxx}(0) = 0,\\ \varphi(1) = \varphi_x(1) = \varphi_{xxx}(1) = 0 \end{cases}$$

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has solutions $\{\varphi_k\}_{k=1}^{\infty}$ which are a basis in $H^6(I)$ orthonormal in $L^2(I)$, and $\{\lambda_k\}_{k=1}^{\infty}$ are eigenvalues corresponding to eigenfunctions $\{\varphi_k\}_{k=1}^{\infty}$, where $\Lambda\varphi = -\partial_x^3((1+x)\partial_x^3\varphi)$.

Let us construct approximate solutions to (2.2) in the form

$$y^n = \sum_{k=1}^{n} c_k^n(t)\varphi_k,$$

where the unknown functions c_k^n are solutions to the Cauchy problem for the stochastic differential equations

$$\begin{aligned} dc_k^n + (y_{xxx}^n, \varphi_k)_{L^2(I)}\,dt &= f_k\,dt + g_k\,dw, \quad k = 1,2,3,\dots,n, \\ c_k(0) &= (y_0, \varphi_k)_{L^2(I)}, \end{aligned} \tag{2.3}$$

where $f_k = (f, \varphi_k)_{L^2(I)}$, $g_k = (g, \varphi_k)_{L^2(I)}$. Due to the classical theory of stochastic differential equations, we know that there is a pathwise-unique solution c_k^n adapted to $\{\mathcal{F}_t\}_{t\ge 0}$ such that $c_k^n \in C([0,T])$ for almost all $\omega \in \Omega$.

By Ito's rule,

$$d(c_k^n)^2 + 2(y_{xxx}^n, c_k^n\varphi_k)_{L^2(I)}\,dt = 2c_k^n f_k\,dt + 2c_k^n g_k\,dw + g_k^2\,dt \tag{2.4}$$

for all $t \in [0,T]$, for almost all $\omega \in \Omega$. Multiplying both sides of (2.4) by λ_k and taking sums from 1 to n about k yields

$$\begin{aligned} &d(y^n, \Lambda y^n)_{L^2(I)} + 2(y_{xxx}^n, \Lambda y^n)_{L^2(I)}\,dt \\ &\quad = 2(f^n, \Lambda y^n)_{L^2(I)}\,dt + 2(g^n, \Lambda y^n)_{L^2(I)}\,dw + (g^n, \Lambda g^n)_{L^2(I)}\,dt, \end{aligned}$$

where $f^n = \sum_{k=1}^n f_k\varphi_k$, $g^n = \sum_{k=1}^n g_k\varphi_k$. Integrating the above equality from 0 to t,

$$\begin{aligned} &(y^n, \Lambda y^n)_{L^2(I)}(t) + 2\int_0^t (y_{xxx}^n, \Lambda y^n)_{L^2(I)}\,ds \\ &\quad = (y^n, \Lambda y^n)_{L^2(I)}(0) + 2\int_0^t (f^n, \Lambda y^n)_{L^2(I)}\,ds \\ &\qquad + 2\int_0^t (g^n, \Lambda y^n)_{L^2(I)}\,dw + \int_0^t (g^n, \Lambda g^n)_{L^2(I)}\,ds. \end{aligned}$$

It is easy to deduce that

$$\begin{aligned} (y^n, \Lambda y^n)_{L^2(I)} &= (1+x, |y_{xxx}^n|^2)_{L^2(I)}, \\ (y_{xxx}^n, \Lambda y^n)_{L^2(I)} &= \tfrac{3}{2}\|y_{xxxx}^n(t)\|_{L^2(I)}^2 + |y_{xxxx}^n(1,t)|^2, \\ (f^n, \Lambda y^n)_{L^2(I)} &= -(f_{xx}^n, ((1+x)y_{xxx}^n)_x)_{L^2(I)} \\ &\le \varepsilon\|y_{xxxx}^n(t)\|_{L^2(I)}^2 + C(\varepsilon)(\|f_{xx}^n(t)\|_{L^2(I)}^2 + (1+x, |y_{xxx}^n|^2)_{L^2(I)}), \\ (g^n, \Lambda g^n)_{L^2(I)} &= (1+x, |g_{xxx}^n|^2)_{L^2(I)}. \end{aligned}$$

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By the Burkholder–Davis–Gundy inequality and the Cauchy inequality,

$$
\begin{aligned}
E\sup_{0\le s\le t}\Big|\int_0^s (g^n,\Lambda y^n)_{L^2(I)}\,dw\Big| &\le CE\Big(\int_0^t |(g^n,\Lambda y^n)_{L^2(I)}|^2\,ds\Big)^{1/2}\\
&= CE\Big(\int_0^t |(g^n_{xxx},(1+x)y^n_{xxx})_{L^2(I)}|^2\,ds\Big)^{1/2}\\
&\le CE\Big(\int_0^t \|(1+x)y^n_{xxx}\|^2_{L^2(I)}\cdot\|g^n_{xxx}\|^2_{L^2(I)}\,ds\Big)^{1/2}\\
&\le CE\Big(\sup_{0\le s\le t}\|(1+x)y^n_{xxx}(s)\|_{L^2(I)}\cdot\Big(\int_0^t \|g^n\|^2_{H^3(I)}\,ds\Big)^{1/2}\Big)\\
&\le \delta E\sup_{0\le s\le t}\|(1+x)y^n_{xxx}(s)\|_{L^2(I)}+CE\int_0^t\|g^n\|^2_{H^3(I)}\,ds.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&E\sup_{0\le s\le t}(1+x,|y^n_{xxx}(t)|^2)_{L^2(I)}+E\int_0^t\|y^n_{xxxx}(s)\|^2_{L^2(I)}\,ds\\
&\quad\le C\Big(E\int_0^t(1+x,|y^n_{xxx}(s)|^2)_{L^2(I)}\,ds+E\int_0^t\|f^n(s)\|^2_{H^2(I)}\,ds\\
&\qquad+E\int_0^t\|g^n(s)\|^2_{H^3(I)}\,ds+E(1+x,|y^n_{xxx}(0)|^2)_{L^2(I)}\Big)\\
&\quad\le C\Big(E\int_0^t\sup_{0\le\tau\le s}(1+x,|y^n_{xxx}(\tau)|^2)_{L^2(I)}\,ds+E\int_0^t\|f^n(s)\|^2_{H^2(I)}\,ds\\
&\qquad+E\int_0^t\|g^n(s)\|^2_{H^3(I)}\,ds+E(1+x,|y^n_{xxx}(0)|^2)_{L^2(I)}\Big).
\end{aligned}
$$

According to Gronwall's inequality, for any $t\in[0,T]$,

$$
\begin{aligned}
&E\sup_{0\le s\le t}(1+x,|y^n_{xxx}(t)|^2)_{L^2(I)}+E\int_0^t\|y^n_{xxxx}(s)\|^2_{L^2(I)}ds\\
&\quad\le C\Big(E\int_0^T\|f^n(t)\|^2_{H^2(I)}dt+E\int_0^T\|g^n(t)\|^2_{H^3(I)}\,dt+E\|y^n(0)\|_{H^3(I)}\Big),
\end{aligned}
$$

namely,

$$
\|y^n\|^2_{X_T}\le C(\|f^n\|^2_{L^2_{\mathcal{F}}(0,T;H^2(I))}+\|g^n\|^2_{L^2_{\mathcal{F}}(0,T;H^3(I))}+\|y^n(0)\|^2_{L^2(\Omega,\mathcal{F}_0,P;H^3(I))}). \tag{2.5}
$$

By the same argument, for $n\ge m\ge 1$,

$$
\begin{aligned}
\|y^n-y^m\|^2_{X_T}\le C(&\|f^n-f^m\|^2_{L^2_{\mathcal{F}}(0,T;H^2(I))}+\|g^n-g^m\|^2_{L^2_{\mathcal{F}}(0,T;H^3(I))}\\
&+\|y^n(0)-y^m(0)\|^2_{L^2(\Omega,\mathcal{F}_0,P;H^3(I))}).
\end{aligned}
$$

It follows that $\{y^n\}_{n=1}^\infty$ is a Cauchy sequence that converges strongly in X_T. Let y be the limit.

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From (2.3),

$$\int_I y^n(t,x)\varphi(x)\,dx - \int_I y^n(0,x)\varphi(x)\,dx + \int_0^t (y^n_{xxx}(s),\varphi)_{L^2(I)}\,ds \\ = \int_0^t (f^n(s),\varphi)_{L^2(I)}\,ds + \int_0^t (g^n(s),\varphi)_{L^2(I)}\,dw. \tag{2.6}$$

Passing to the limit as $n \to \infty$ in (2.5) and (2.6), we find that y is the solution of (2.2) and (2.1) holds.

The uniqueness can be obtained directly from (2.1).

This completes the proof of Theorem 2.2. □

3. Proof of Theorem 1.1

As in [8], it is enough to prove (1.2) for

$$dy + y_{xxx}\,dt = f\,dt + g\,dw. \tag{3.1}$$

In fact, assume that we have proved (1.2) for (3.1). Then

$$E\int_Q \theta^2|f - y_x|^2\,dx\,dt \le 2E\int_Q \theta^2|f|^2\,dx\,dt + 2E\int_Q \theta^2 y_x^2\,dx\,dt.$$

By choosing $\lambda > 0$ large, it is possible to absorb $2E\int_Q \theta^2 y_x^2\,dx\,dt$ with the left-hand side of (1.2), concluding that (1.2) also holds.

Set $u = \theta y$. A direct computation shows that

$$\theta(dy + y_{xxx}\,dt) = du + (A - l_t)u\,dt + Bu_x\,dt + Gu_{xx}\,dt + u_{xxx}\,dt,$$

where

$$A = -l_x^3 + 3l_x l_{xx} - l_{xxx}, \quad B = 3l_x^2 - 3l_{xx}, \quad G = -3l_x.$$

Set

$$\begin{aligned} P &= (Gu_x)_x + (A - \Phi)u, \\ P_1 &= ((Gu_x)_x + (A - l_t - \Phi)u)\,dt, \\ P_2 &= du + (u_{xxx} + (B - G_x)u_x + \Phi u)\,dt, \end{aligned}$$

where Φ will be given later. Then

$$\theta(dy + y_{xxx}\,dt) = P_1 + P_2.$$

Step 1. We shall prove the inequality

$$\begin{aligned} E\int_I\int_0^T (A-\Phi)(du)^2\,dx &- E\int_I\int_0^T G(du_x)^2\,dx + E\int_Q \theta^2 f^2\,dx\,dt \\ &\ge E\int_Q (\cdot)_{xxx}\,dt\,dx + E\int_Q (\cdot)_{xx}\,dt\,dx + E\int_Q (\cdot)_x\,dt\,dx \\ &\quad + E\int_Q u^2(\cdot)\,dt\,dx - E\int_Q l_t^2 u^2\,dt\,dx + E\int_Q u_x^2(\cdot)\,dt\,dx \\ &\quad + E\int_Q u_{xx}^2(\cdot)\,dt\,dx + E\int_I ((A-\Phi)u^2 - Gu_x^2)|_0^T\,dx, \end{aligned} \tag{3.2}$$

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where

$$
\begin{aligned}
(\cdot)_{xxx} &= ((A-\Phi)u^2)_{xxx},\\
(\cdot)_{xx} &= (G\Phi u^2 + G_x u_x^2 - 3(A-\Phi)_x u^2)_{xx},\\
(\cdot)_{x} &= (-2(G\Phi)_x u^2 + G_x\Phi u^2 + (A-\Phi)(B-G_x)u^2 + G(B-G_x)u_x^2 + Gu_{xx}^2\\
&\qquad + 3(A-\Phi)_{xx}u^2 + 2\int_0^T Gu_x du - 3(A-\Phi)u_x^2 - 2G_{xx}u_x^2)_x,\\
u^2(\cdot) &= u^2(-(G_x\Phi)_x + (G\Phi)_{xx} - (A-\Phi)_{xxx} + 2(A-\Phi)\Phi - (A-\Phi)_t\\
&\qquad - ((A-\Phi)(B-G_x))_x),\\
u_x^2(\cdot) &= u_x^2(G_{xxx} - (G(B-G_x))_x + 3(A-\Phi)_x + 2G_x(B-G_x)_x - 2G\Phi + G_t),\\
u_{xx}^2(\cdot) &= u_{xx}^2(-3G_x).
\end{aligned}
$$

Indeed,

$$
\begin{aligned}
2\int_0^T PP_2 &= \int_0^T (\cdot)_{xxx}\,dt + \int_0^T (\cdot)_{xx}\,dt + \int_0^T (\cdot)_x\,dt\\
&\qquad + \int_0^T u^2(\cdot)\,dt + \int_0^T u_x^2(\cdot)\,dt + \int_0^T u_{xx}^2(\cdot)\,dt\\
&\qquad + (A-\Phi)u^2|_0^T - Gu_x^2|_0^T + \int_0^T G(du_x)^2 - \int_0^T (A-\Phi)(du)^2.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&E\int_Q \theta^2 f^2\,dx\,dt + E\int_Q P^2\,dx\,dt\\
&\quad\geq 2E\int_Q P\theta f\,dx\,dt\\
&\quad= 2E\int_Q P\theta(f\,dt + g\,dw)\,dx\\
&\quad= 2E\int_Q P\theta(dy + y_{xxx}\,dt)\,dx\\
&\quad= 2E\int_Q P(P_1+P_2)\,dx\\
&\quad= 2E\int_Q P(P - l_t u)\,dx\,dt + 2E\int_Q PP_2\,dx\\
&\quad\geq E\int_Q P^2\,dx\,dt + E\int_Q (\cdot)_{xxx}\,dx\,dt + E\int_Q (\cdot)_{xx}\,dx\,dt + E\int_Q (\cdot)_x\,dx\,dt\\
&\qquad + E\int_Q u^2(\cdot)\,dx\,dt - E\int_Q l_t^2 u^2\,dx\,dt + E\int_Q u_x^2(\cdot)\,dx\,dt + E\int_Q u_{xx}^2(\cdot)\,dx\,dt\\
&\qquad + E\int_Q G(du_x)^2\,dx - E\int_Q (A-\Phi)(du)^2\,dx + E\int_I ((A-\Phi)u^2 - Gu_x^2)|_0^T\,dx.
\end{aligned}
$$

This implies (3.2).

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Step 2. We shall prove the estimate

$$
\begin{aligned}
E\int_Q (\lambda\varphi\theta^2 y_{xx}^2 + \lambda^3\varphi^3\theta^2 y_x^2 + \lambda^5\varphi^5\theta^2 y^2)\,dx\,dt \\
\le C\Big(E\int_Q (\theta^2 f^2 + \lambda^3\varphi^3\theta^2 g^2 + \lambda\varphi\theta^2 g_x^2)\,dx\,dt \\
+ E\int_{Q^{I_0}} (\lambda\varphi\theta^2 y_{xx}^2 + \lambda^3\varphi^3\theta^2 y_x^2 + \lambda^5\varphi^5\theta^2 y^2)\,dx\,dt\Big).
\end{aligned}
\tag{3.3}
$$

We shall prove (3.3) by further estimates for each term in (3.2). Indeed, we take $\Phi = l_x l_{xx}$ in (3.2) and $\mu \ge \mu_0$, where $\mu_0 \ge 1$ will be fixed later. By the definition of a, φ, ψ and μ, it is obvious that

$$
\begin{gathered}
|a_x| \le C(\psi)\mu\varphi, \quad |a_{xx}| \le C(\psi)\mu^2\varphi, \quad |a_{xxx}| \le C(\psi)\mu^3\varphi, \\
|a_{xxxx}| \le C(\psi)\mu^4\varphi, \quad |a_t| \le CT\varphi^2, \quad |a_{xt}| \le C(\psi)\mu T\varphi^2
\end{gathered}
$$

and $\varphi \le (T^2/4)\varphi^2$.

For the term $u^2(\cdot)$ in (3.3), if we choose $\lambda \ge \mu C(\psi)(T+T^2)$ with $C(\psi)$ large enough, then

$$
\begin{gathered}
|-(G_x\Phi)_x + (G\Phi)_{xx} + (\Phi(B-G_x))_x - (A-\Phi)_{xxx} - (A-\Phi)_t| \le C(\psi)\lambda^5\mu^5\varphi^5, \\
2(A-\Phi)\Phi - (A(B-G_x))_x = 13\lambda^5\mu^6\varphi^5\psi_x^6 + D_1,
\end{gathered}
$$

where $|D_1| \le C(\psi)\lambda^5\mu^5\varphi^5$. Thus,

$$
u^2(\cdot) = 13\lambda^5\mu^6\varphi^5\psi_x^6 u^2 + E_0 u^2,
$$

where $|E_0| \le C(\psi)\lambda^5\mu^5\varphi^5$. Using the same method,

$$
u_x^2(\cdot) = 6\lambda^3\mu^4\varphi^3\psi_x^4 u_x^2 + E_1 u_x^2, \quad u_{xx}^2(\cdot) = 9\lambda\mu^2\varphi\psi_x^2 u_{xx}^2 + E_2 u_{xx}^2,
$$

where

$$
|E_1| \le C(\psi)\lambda^3\mu^3\varphi^3, \quad |E_2| \le C(\psi)\lambda\mu\varphi.
$$

We now estimate the term $E\int_Q((\cdot)_{xxx} + (\cdot)_{xx} + (\cdot)_x)\,dx\,dt$ in (3.2):

$$
\begin{aligned}
E\int_Q ((\cdot)_{xxx} + (\cdot)_{xx} + (\cdot)_x)\,dx\,dt \\
= E\int_0^T ((G(B-G_x) - G_{xx} - (A-\Phi))u_x^2 + 2G_x u_x u_{xx} + G u_{xx}^2)|_0^1\,dt \\
\triangleq V(1) - V(0).
\end{aligned}
$$

By the definition of Φ, for any $\varepsilon > 0$, if we choose $\lambda \ge \mu C(\varepsilon,\psi)(T+T^2)$ with $C(\varepsilon,\psi)$ large enough, then

$$
|2G_x(t,0)u_x u_{xx}(t,0)| \le \varepsilon^{-1}\lambda\mu\varphi(t,0)u_x^2(t,0) + \varepsilon\lambda\mu\varphi(t,0)u_{xx}^2(t,0).
$$

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Note that $\psi_x(1) < 0$, $\psi_x(0) > 0$. If we choose ε sufficiently small and $\lambda \geq \mu C(\varepsilon, \psi) \times (T + T^2)$, then there exist positive constants N, K such that

$$\begin{aligned} V(1) &= E\int_0^T ((G(B - G_x) - G_{xx} - (A - \Phi))u_x^2 + 2G_x u_x u_{xx} + G u_{xx}^2)(t, 1)\, dt \\ &= E\int_0^T (-3)\lambda\mu\varphi(t, 1)\psi_x(t, 1)u_{xx}^2(t, 1)\, dt \\ &\geq 0, \end{aligned}$$

$$\begin{aligned} V(0) &= E\int_0^T ((G(B - G_x) - G_{xx} - (A - \Phi))u_x^2 + 2G_x u_x u_{xx} + G u_{xx}^2)(t, 0)\, dt \\ &\leq E\int_0^T (-N\lambda^3\mu^3\varphi^3(t, 0)\psi_x^3(t, 0)u_x^2(t, 0) - K\lambda\mu\varphi(t, 0)\psi_x(t, 0)u_{xx}^2(t, 0))\, dt \\ &\leq 0. \end{aligned}$$

Recall that $|\psi_x| > 0$ in $\bar{I} \setminus I_0$. It follows that

$$\begin{aligned} &E\int_{Q\setminus Q^{I_0}} (\lambda\mu^2\varphi u_{xx}^2 + \lambda^3\mu^4\varphi^3 u_x^2 + \lambda^5\mu^6\varphi^5 u^2)\, dx\, dt \\ &\quad \leq C_1(\psi)\Big(E\int_Q (\theta^2 f^2 + \lambda^3\mu^3\varphi^3\theta^2 g^2 + \lambda\mu\varphi\theta^2 g_x^2)\, dx\, dt \\ &\qquad + E\int_Q (\lambda\mu\varphi u_{xx}^2 + \lambda^3\mu^3\varphi^3 u_x^2 + \lambda^5\mu^5\varphi^5 u^2)\, dx\, dt\Big). \end{aligned}$$

From this, if we choose $\mu_0 = C_1(\psi) + 1$, then

$$\begin{aligned} &E\int_{Q\setminus Q^{I_0}} (\lambda\mu\varphi u_{xx}^2 + \lambda^3\mu^3\varphi^3 u_x^2 + \lambda^5\mu^5\varphi^5 u^2)\, dx\, dt \\ &\quad \leq C(\psi)\Big(E\int_Q (\theta^2 f^2 + \lambda^3\mu^3\varphi^3\theta^2 g^2 + \lambda\mu\varphi\theta^2 g_x^2)\, dx\, dt \\ &\qquad + E\int_{Q^{I_0}} (\lambda\mu\varphi u_{xx}^2 + \lambda^3\mu^3\varphi^3 u_x^2 + \lambda^5\mu^5\varphi^5 u^2)\, dx\, dt\Big). \end{aligned}$$

Thus,

$$\begin{aligned} &E\int_Q (\lambda\mu\varphi u_{xx}^2 + \lambda^3\mu^3\varphi^3 u_x^2 + \lambda^5\mu^5\varphi^5 u^2)\, dx\, dt \\ &\quad \leq C(\psi)\Big(E\int_Q (\theta^2 f^2 + \lambda^3\mu^3\varphi^3\theta^2 g^2 + \lambda\mu\varphi\theta^2 g_x^2)\, dx\, dt \\ &\qquad + E\int_{Q^{I_0}} (\lambda\mu\varphi u_{xx}^2 + \lambda^3\mu^3\varphi^3 u_x^2 + \lambda^5\mu^5\varphi^5 u^2)\, dx\, dt\Big) \end{aligned}$$

and consequently

$$E\int_Q(\lambda\varphi u_{xx}^2+\lambda^3\varphi^3u_x^2+\lambda^5\varphi^5u^2)\,dx\,dt$$
$$\leq C(\psi)\Big(E\int_Q(\theta^2f^2+\lambda^3\varphi^3\theta^2g^2+\lambda\varphi\theta^2g_x^2)\,dx\,dt$$
$$+E\int_{Q^{I_0}}(\lambda\varphi u_{xx}^2+\lambda^3\varphi^3u_x^2+\lambda^5\varphi^5u^2)\,dx\,dt\Big).$$

Replacing u by θy, we obtain (3.3).

Step 3. We shall eliminate the term $E\int_{Q^{I_0}}\lambda^3\theta^2\varphi^3y_x^2\,dx\,dt$ on the left-hand side of (3.3).

By the interpolation inequality, for any $\varepsilon>0$,

$$\int_{I_0}(\theta y)_x^2\,dx\leq\varepsilon\int_{I_0}(\theta y)_{xx}^2\,dx+\frac{C}{\varepsilon}\int_{I_0}(\theta y)^2\,dx,$$

where C depends only on I_0. Take ε to be $\varepsilon_2(\lambda/t(T-t))^{-2}$ in the above inequality, where ε_2 will be fixed later. Then

$$\int_{I_0}\theta^2y_x^2dx\leq\varepsilon_2\Big(\frac{\lambda}{t(T-t)}\Big)^{-2}\int_{I_0}(\theta y)_{xx}^2\,dx+\frac{C}{\varepsilon_2\big(\frac{\lambda}{t(T-t)}\big)^{-2}}\int_{I_0}(\theta y)^2\,dx$$
$$-\int_{I_0}\theta_x^2y^2\,dx-2\int_{I_0}\theta\theta_xyy_x\,dx,$$

from which

$$\int_{I_0}\theta^2y_x^2\,dx\leq\varepsilon\int_{I_0}\lambda^{-2}t^2(T-t)^2\theta^2y_{xx}^2\,dx+C\int_{I_0}\lambda^2t^{-2}(T-t)^{-2}\theta^2y^2\,dx.$$

Noting that there exist two positive constants C_1 and C_2 such that

$$\frac{C_1}{t(T-t)}\leq\varphi\leq\frac{C_2}{t(T-t)},$$

we find that

$$E\int_{Q^{I_0}}\lambda^3\theta^2\varphi^3y_x^2\,dx\,dt\leq\varepsilon E\int_{Q^{I_0}}\lambda\theta^2\varphi y_{xx}^2\,dx\,dt+CE\int_{Q^{I_0}}\lambda^5\theta^2\varphi^5y^2\,dx\,dt.\qquad(3.4)$$

Combining (3.4) and (3.3), we obtain (1.2).

This completes the proof of Theorem 1.1.

4. UCP for the linear stochastic KdV equation

In this section, we apply the Carleman estimate in Theorem 1.1 to obtain the UCP for the stochastic dispersion equation

$$\begin{cases}dy+(y_x+y_{xxx})\,dt=ay\,dt+by\,dw & \text{in } Q,\\ y(0,t)=0=y(1,t) & \text{in } (0,T),\\ y_x(1,t)=0 & \text{in } (0,T),\\ y(x,0)=y_0(x) & \text{in } I.\end{cases}\qquad(4.1)$$

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THEOREM 4.1. *Let* $a \in L^\infty_{\mathcal{F}}(0,T;W^{2,\infty}(I))$, $b \in L^\infty_{\mathcal{F}}(0,T;W^{3,\infty}(I))$ *P-a.s. If* y *is the solution of* (4.1) *and* $y \equiv 0$ *in* Q^{I_0}, *then* $y \equiv 0$ *in* Q *P-a.s.*

REMARK 4.2. The well posedness of (4.1) can be obtained by Theorem 2.2, the Banach fixed point theorem, energy estimates and the standard extension argument.

REMARK 4.3. The classical Holmgren uniqueness theorem does not work for stochastic partial differential equations.

PROOF. According to (1.2),

$$\begin{aligned} E\int_Q(\lambda\varphi\theta^2 y_{xx}^2 + \lambda^3\varphi^3\theta^2 y_x^2 + \lambda^5\varphi^5\theta^2 y^2)\,dx\,dt \\ \leq C\Big(E\int_Q(\theta^2(ay)^2 + \lambda^3\varphi^3\theta^2(by)^2 + \lambda\varphi\theta^2(by)_x^2)\,dx\,dt \\ + E\int_{Q^{I_0}}(\lambda\varphi\theta^2 y_{xx}^2 + \lambda^5\varphi^5\theta^2 y^2)\,dx\,dt\Big). \end{aligned}$$

If we take

$$\lambda \geq C(T, \|a\|_{L^\infty_{\mathcal{F}}(0,T;W^{2,\infty}(I))}, \|b\|_{L^\infty_{\mathcal{F}}(0,T;W^{3,\infty}(I))}),$$

where $C(T, \|a\|_{L^\infty_{\mathcal{F}}(0,T;W^{2,\infty}(I))}, \|b\|_{L^\infty_{\mathcal{F}}(0,T;W^{3,\infty}(I))})$ is large enough, then

$$E\int_Q(\lambda\varphi\theta^2 y_{xx}^2 + \lambda^3\varphi^3\theta^2 y_x^2 + \lambda^5\varphi^5\theta^2 y^2)\,dx\,dt \leq CE\int_{Q^{I_0}}(\lambda\varphi\theta^2 y_{xx}^2 + \lambda^5\varphi^5\theta^2 y^2)\,dx\,dt.$$

If $y \equiv 0$ in Q^{I_0} P-a.s., then

$$E\int_Q(\lambda\varphi\theta^2 y_{xx}^2 + \lambda^3\varphi^3\theta^2 y_x^2 + \lambda^5\varphi^5\theta^2 y^2)\,dx\,dt \leq 0.$$

Thus, $y \equiv 0$ in Q P-a.s. □

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