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# Boundedness of Calderón–Zygmund Operators on Non-homogeneous Metric Measure Spaces

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*Abstract.* Let $(\mathcal{X}, d, \mu)$ be a separable metric measure space satisfying the known upper doubling condition, the geometrical doubling condition, and the non-atomic condition that $\mu(\{x\}) = 0$ for all $x \in \mathcal{X}$. In this paper, we show that the boundedness of a Calderón–Zygmund operator $T$ on $L^2(\mu)$ is equivalent to that of $T$ on $L^p(\mu)$ for some $p \in (1, \infty)$, and that of $T$ from $L^1(\mu)$ to $L^{1,\infty}(\mu)$. As an application, we prove that if $T$ is a Calderón–Zygmund operator bounded on $L^2(\mu)$, then its maximal operator is bounded on $L^p(\mu)$ for all $p \in (1, \infty)$ and from the space of all complex-valued Borel measures on $\mathcal{X}$ to $L^{1,\infty}(\mu)$. All these results generalize the corresponding results of Nazarov et al. on metric spaces with measures satisfying the so-called polynomial growth condition.

## 1 Introduction

The classical theory of singular integrals of Calderón–Zygmund type started with the study of convolution operators on the Euclidean space associated with singular kernels and has been well developed into a large branch of analysis on metric spaces. One of the most interesting cases is the "space of homogeneous type" in the sense of Coifman and Weiss [3, 4]. Recall that a metric space $(\mathcal{X}, d)$ equipped with a nonnegative Borel measure $\mu$ is called a *space of homogeneous type* if $(\mathcal{X}, d, \mu)$ satisfies the following *measure doubling condition* that there exists a positive constant $C_\mu$ such that for any ball $B(x, r) \equiv \{y \in \mathcal{X} : d(x, y) < r\}$ with $x \in \mathcal{X}$ and $r \in (0, \infty)$,

$$(1.1) \qquad \mu(B(x, 2r)) \le C_\mu \mu(B(x, r)).$$

The measure doubling condition (1.1) was considered the cornerstone of any extension to abstract frameworks of the theory of singular integrals. However, recently, many results on the classical Calderón–Zygmund theory have still proved valid if the measure doubling condition is replaced by a less demanding condition; see, for example, [2, 12–14, 16–18] and the references therein.

In particular, let $\kappa \in (0, \infty)$, and let $\mathcal{X}$ be a separable metric space endowed with a metric $d$ and a nonnegative "$\kappa$ dimensional" Borel measure $\mu$ in the sense that there

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exists a positive constant $C_0$ such that for all $x \in \mathcal{X}$ and $r \in (0, \infty)$,

$$\mu(B(x, r)) \le C_0 r^{\kappa}. \tag{1.2}$$

Such a measure need not satisfy the doubling condition (1.1). In [13], Nazarov, Treil, and Volberg showed that if $T$ is a Calderón–Zygmund operator bounded on $L^2(\mu)$, then $T$ is bounded on $L^p(\mu)$ for all $p \in (1, \infty)$ and from $L^1(\mu)$ to $L^{1,\infty}(\mu)$, and the corresponding maximal operator $T^\sharp$ is also bounded on $L^p(\mu)$ for any $p \in (1, \infty)$ and from the *space* $\mathcal{M}(\mathcal{X})$ of all complex-valued Borel measures on $\mathcal{X}$ to $L^{1,\infty}(\mu)$. Moreover, Nazarov et al. [13] also proved that if $T$ is a Calderón–Zygmund operator bounded from $L^1(\mu)$ to $L^{1,\infty}(\mu)$, then $T$ is also bounded on $L^2(\mu)$.

Notice that measures satisfying the polynomial growth condition (1.2) are only different, not more general than measures satisfying (1.1). Thus, the Calderón–Zygmund theory with non-doubling measures is not in all respects a generalization of the corresponding theory of spaces of homogeneous type. In [9], Hytönen introduced a new class of metric measure spaces satisfying the so-called upper doubling condition and the geometrical doubling condition (see Definitions 1.1 and 1.3), and a notion of the space of regularized BMO. This new class of metric measure spaces is a simultaneous generalization of the spaces of homogeneous type and metric spaces with power bounded measures. Later, Hytönen and Martikainen [10] further established a version of the $T(b)$ theorem for Calderón–Zygmund operators in such spaces.

Let $(\mathcal{X}, d, \mu)$ be a separable metric space that satisfies the upper doubling condition, the geometrical doubling condition and the *non-atomic condition* that $\mu(\{x\}) = 0$ for all $x \in \mathcal{X}$. The goal of this paper is to generalize the corresponding results of Nazarov et al. in [13]. Precisely, in this paper we show that the boundedness of a Calderón–Zygmund operator $T$ on $L^2(\mu)$ is equivalent to that of $T$ on $L^p(\mu)$ for some $p \in (1, \infty)$, and that of $T$ from $L^1(\mu)$ to $L^{1,\infty}(\mu)$. As an application, we prove that if $T$ is a Calderón–Zygmund operator bounded on $L^2(\mu)$, then its maximal operator is bounded on $L^p(\mu)$ for all $p \in (1, \infty)$ and from the space of all complex-valued Borel measures on $\mathcal{X}$ to $L^{1,\infty}(\mu)$.

To state our main results, we first recall some necessary notions and notation. We begin with the definition of the upper doubling spaces in [9].

***Definition 1.1*** A metric measure space $(\mathcal{X}, d, \mu)$ is said to be *upper doubling* if $\mu$ is a Borel measure on $\mathcal{X}$ and there exists a *dominating function* $\lambda : \mathcal{X} \times (0, \infty) \to (0, \infty)$ and a positive constant $C_\lambda$ such that for each $x \in \mathcal{X}$, $r \to \lambda(x, r)$ is non-decreasing, and for all $x \in \mathcal{X}$ and $r \in (0, \infty)$,

$$\mu(B(x, r)) \le \lambda(x, r) \le C_\lambda \lambda(x, r/2). \tag{1.3}$$

***Remark 1.2*** (i) Obviously, a space of homogeneous type is a special case of upper doubling spaces, where one can take the dominating function $\lambda(x, r) \equiv \mu(B(x, r))$. On the other hand, a metric space $(\mathcal{X}, d, \mu)$ satisfying the polynomial growth condition (1.2) (in particular, $(\mathcal{X}, d, \mu) \equiv (\mathbb{R}^n, |\cdot|, \mu)$ with $\mu$ satisfying (1.2) for some $\kappa \in (0, n]$) is also an upper doubling measure space if we take $\lambda(x, r) \equiv C_0 r^{\kappa}$.

(ii) Let $(\mathcal{X}, d, \mu)$ be an upper doubling space and let $\lambda$ be a dominating function on $\mathcal{X} \times (0, \infty)$ as in Definition 1.1. It was shown in [11] that there exists another dominating function $\widetilde{\lambda}$ such that $\widetilde{\lambda} \le \lambda$, $C_{\widetilde{\lambda}} \le C_\lambda$, and, for all $x, y \in \mathcal{X}$ with $d(x, y) \le r$,

$$\widetilde{\lambda}(x, r) \le \widetilde{C}\widetilde{\lambda}(y, r). \tag{1.4}$$

Thus, in this paper, we always assume that $\lambda$ satisfies (1.4).

We now recall the notion of geometrically doubling spaces introduced in [9].

***Definition 1.3*** A metric space $(\mathcal{X}, d)$ is called *geometrically doubling* if there exists some $N_0 \in \mathbb{N} \equiv \{1, 2, \dots\}$ such that for any ball $B(x, r) \subseteq \mathcal{X}$, there exists a finite ball covering $\{B(x_i, r/2)\}_i$ of $B(x, r)$ such that the cardinality of this covering is at most $N_0$.

***Remark 1.4*** Let $(\mathcal{X}, d)$ be a metric space. In [9, Lemma 2.3], Hytönen showed that the following statements are equivalent:

(i) $(\mathcal{X}, d)$ is geometrically doubling.
(ii) For any $\epsilon \in (0, 1)$ and any ball $B(x, r) \subseteq \mathcal{X}$, there exists a finite ball covering $\{B(x_i, \epsilon r)\}_i$ of $B(x, r)$ such that the cardinality of this covering is at most $N_0\epsilon^{-n}$, where, and in what follows, $N_0$ is as in Definition 1.3 and $n \equiv \log_2 N_0$.
(iii) For every $\epsilon \in (0, 1)$, any ball $B(x, r) \subseteq \mathcal{X}$ can contain at most $N_0\epsilon^{-n}$ centers $\{x_i\}_i$ of disjoint balls with radius $\epsilon r$.
(iv) There exists $M \in \mathbb{N}$ such that any ball $B(x, r) \subseteq \mathcal{X}$ can contain at most $M$ centers $\{x_i\}_i$ of disjoint balls $\{B(x_i, r/4)\}_{i=1}^M$.

Now we recall the notions of standard kernels and corresponding Calderón–Zygmund operators in the current setting from [10]. Let $\mathscr{M}(\mathcal{X})$ be the *space of all complex-valued Borel measures on* $\mathcal{X}$. For a measure $\nu \in \mathscr{M}(\mathcal{X})$, we denote by $\|\nu\| \equiv \int_{\mathcal{X}} |d\nu(x)|$ the *total variation of* $\nu$ and by $\operatorname{supp} \nu$ the *smallest closed set* $F \subseteq \mathcal{X}$ *for which* $\nu$ *vanishes on* $\mathcal{X} \setminus F$ (such a smallest closed set always exists since $\mathcal{X}$ is separable; see [13, p. 466]). Also, for any function $f$, $\operatorname{supp} f$ means the *essential support of the function* $f$, namely, the smallest closed set $F \subseteq \mathcal{X}$ such that $f$ vanishes at $\mu$-almost every $x \in \mathcal{X} \setminus F$.

***Definition 1.5*** Let $\triangle \equiv \{(x, x) : x \in \mathcal{X}\}$. A *standard kernel* is a mapping $K\colon (\mathcal{X} \times \mathcal{X}) \setminus \triangle \to \mathbb{C}$ for which there exist positive constants $\tau \in (0, 1]$ and $C$ such that for all $x, y \in \mathcal{X}$ with $x \ne y$,

$$|K(x, y)| \le C\frac{1}{\lambda(x, d(x, y))}, \tag{1.5}$$

and that for all $x, \widetilde{x}, y \in \mathcal{X}$ with $d(x, y) \ge 2d(x, \widetilde{x})$,

$$|K(x, y) - K(\widetilde{x}, y)| + |K(y, x) - K(y, \widetilde{x})| \le C\frac{[d(x, \widetilde{x})]^\tau}{[d(x, y)]^\tau\lambda(x, d(x, y))}. \tag{1.6}$$

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A linear operator $T$ is called a *Calderón–Zygmund operator* with $K$ satisfying (1.5) and (1.6) if for all $f \in L_b^\infty(\mu)$, the *space of bounded functions with bounded support*, and $x \notin \operatorname{supp} f$,

$$Tf(x) \equiv \int_{\mathcal{X}} K(x, y) f(y) \, d\mu(y).$$

A new example of operators with kernel satisfying (1.5) and (1.6) is the so-called Bergman-type operator appearing in [19]; see also [10] for an explanation.

Assume that $T$ is a Calderón–Zygmund operator with $K$ satisfying (1.5) and (1.6). For any $\nu \in \mathscr{M}(\mathcal{X})$ with bounded support and $x \in \mathcal{X} \setminus \operatorname{supp} \nu$, define

$$T\nu(x) \equiv \int_{\mathcal{X}} K(x, y) \, d\nu(y).$$

Moreover, the maximal operator $T^\sharp$ associated with $T$ is defined as follows. For every $f \in L_b^\infty(\mu)$ and $\nu \in \mathscr{M}(\mathcal{X})$, we set, for all $x \in \mathcal{X}$,

$$T^\sharp f(x) \equiv \sup_{r>0} |T_r f(x)| \quad \text{and} \quad T^\sharp \nu(x) \equiv \sup_{r>0} |T_r \nu(x)|,$$

where for every $r > 0$,

$$T_r f(x) \equiv \int_{d(x, y)>r} K(x, y) f(y) \, d\mu(y) \quad \text{and} \quad T_r \nu(x) \equiv \int_{d(x, y)>r} K(x, y) \, d\nu(y).$$

The main result of this paper reads as follows.

***Theorem 1.6*** *Let $T$ be a Calderón–Zygmund operator with kernel $K$ satisfying* (1.5) *and* (1.6). *Then the following statements are equivalent:*

(i) *$T$ is bounded on $L^2(\mu)$; namely, there exists a positive constant $C$ such that for all $f \in L^2(\mu)$,*

$$\|Tf\|_{L^2(\mu)} \le C \|f\|_{L^2(\mu)}.$$

(ii) *$T$ is bounded on $L^p(\mu)$ for some $p \in (1, \infty)$; namely, there exists a positive constant $C(p)$, depending on $p$, such that for all $f \in L^p(\mu)$,*

$$\|Tf\|_{L^p(\mu)} \le C(p) \|f\|_{L^p(\mu)}.$$

(iii) *$T$ is bounded from $L^1(\mu)$ to $L^{1,\infty}(\mu)$; namely, there exists a positive constant $\widetilde{C}$ such that for all $f \in L^1(\mu)$,*

$$\|Tf\|_{L^{1,\infty}(\mu)} \le \widetilde{C} \|f\|_{L^1(\mu)}. \tag{1.7}$$

As an application of Theorem 1.6, we also obtain the following boundedness of the maximal operators associated with the Calderón–Zygmund operators.

***Corollary 1.7*** *Let $T$ be a Calderón–Zygmund operator with kernel $K$ satisfying* (1.5) *and* (1.6), *which is bounded on $L^2(\mu)$, and let $T^\sharp$ be the maximal operator associated with $T$. Then the following statements hold:*

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(i) *Let $p \in (1, \infty)$. There exists a positive constant $c$ such that for all $f \in L^p(\mu)$,*

$$\|T^\sharp f\|_{L^p(\mu)} \le c\|f\|_{L^p(\mu)}.$$

(ii) *There exists a positive constant $\widetilde{c}$ such that for all $\nu \in \mathscr{M}(\mathcal{X})$,*

$$\|T^\sharp \nu\|_{L^{1,\infty}(\mu)} \le \widetilde{c}\|\nu\|. \tag{1.8}$$

*Moreover, for all $f \in L^1(\mu)$,*

$$\|T^\sharp f\|_{L^{1,\infty}(\mu)} \le \widetilde{c}\|f\|_{L^1(\mu)}. \tag{1.9}$$

Together, Theorem 1.6 and Corollary 1.7 consist of a generalization of Nazarov–Treil–Volberg's [13, Theorems 1.1 and 10.1] from measures of type (1.2) to general upper doubling measures.

This paper is organized as follows. Let $(\mathcal{X}, d, \mu)$ be a separable metric space satisfying Definitions 1.1 and 1.3, and the non-atomic condition. In Section 2, we make some preliminaries, including a Whitney-type Covering Lemma 2.2 and a Hörmander-type inequality, Lemma 2.4. In Section 3, we first establish a Cotlar type inequality and an endpoint estimate for $T$ in terms of the so-called *elementary measures*, which is an alternative to the Calderón–Zygmund decomposition introduced by Nazarov, Treil, and Volberg [13] in the case of $\mathcal{X} \equiv \mathbb{R}^n$ and the polynomial bound (1.2). As an application of these estimates and the non-atomic assumption, we further obtain (i) $\Rightarrow$ (ii), (i) $\Rightarrow$ (iii), and (ii) $\Rightarrow$ (iii) of Theorem 1.6. We remark that the non-atomic assumption is to guarantee that every $A \subseteq \mathcal{X}$ of positive $\mu$-measure can be further divided into two subsets, both of positive $\mu$-measure (see Definition 3.3 and Remark 3.4). Notice that the non-atomic condition is automatically true under the polynomial growth condition (1.2).

Section 4 is devoted to the proof of (iii) $\Rightarrow$ (i) of Theorem 1.6, while the proof of Corollary 1.7 is presented in Section 5. We point out that in [13], the size condition of a given Calderón–Zygmund kernel $K(x, y)$ is just related to the distance $d(x, y)$ of $x$ and $y$, which is a very important fact used in [13]. However, this may be false in our context, since $K(x, y)$ is controlled by $[\lambda(x, d(x, y))]^{-1}$ and $\lambda(x, d(x, y))$ depends not only on $d(x, y)$, but also on $x$. To overcome this difficulty, we first restrict $\mu$ to the closure of some ball, $\overline{B}(x_0, M)$ for some fixed $x_0 \in \mathcal{X}$ and large radius $M$, where, and in what follows, for an open ball $B$, $\overline{B}$ means the *closure of* $B$, and show that (iii) $\Rightarrow$ (i) of Theorem 1.6 holds for the restriction of $\mu$ with constant independent of $M$. Then by a limiting argument we obtain (iii) $\Rightarrow$ (i) of Theorem 1.6 for $\mu$. A similar method is used in the proof of Corollary 1.7 in Section 5. In Section 5, we also obtain an endpoint estimate for $T^\sharp$ via the elementary measures. Then, as in [13], using this and some tools of probability theory, we establish Corollary 1.7.

While this manuscript was in its final stages, we learned that (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii) of Theorem 1.6 and a variant of Lemma 3.1 in this paper were also independently obtained by Anh and Duong in [1] via a different approach modeled after the work of Tolsa [16] for measures of type (1.2) on $\mathbb{R}^n$. In fact, Anh and Duong in [1] first established a variant of the Calderón–Zygmund decomposition in this setting; then as an application, they further proved Theorem 1.6 and a variant of Lemma 3.1. Our approach, on the other hand, consists of extending the techniques of Nazarov, Treil, and Volberg [13].

Finally, we make some conventions on symbols. Throughout the paper, $C$, $\widetilde{C}$, $c$, and $\widetilde{c}$ stand for *positive constants* that are independent of the main parameters, but which may vary from line to line. Constants with subscripts, such as $C_1$ and $c_1$, do not change in different occurrences. Also, $C(\alpha, \beta, \dots)$ denotes a positive constant depending on $\alpha, \beta, \dots$. If $f \le Cg$, we then write $f \lesssim g$ or $g \gtrsim f$; and if $f \lesssim g \lesssim f$, we then write $f \sim g$. For any $q \in (1, \infty)$, let $q' \equiv q/(q-1)$ be the *conjugate index* of $q$. Sometimes, the *characteristic function* of a set $E$ in $\mathcal{X}$ is denoted by $\chi_E$ or $1_E$, depending on what seems convenient in a particular place. For $\rho \in (0, \infty)$ and $B \equiv B(x, r)$, the notation $\rho B \equiv B(x, \rho r)$ means the concentric dilation of $B$. For any $f \in L^1_{\rm loc}(\mu)$, *its average in a set* $E$ is denoted by

$$\langle f \rangle_E \equiv \frac{1}{\mu(E)} \int_E f(x)\, d\mu(x).$$

## 2 Preliminaries

In this section, we present some preliminary lemmas used in the rest of the paper. We begin with a covering lemma from [11] that is a simple corollary of [8, Theorem 1.2] and [9, Lemma 2.5].

***Lemma 2.1*** *Let $(\mathcal{X}, d)$ be a geometrically doubling metric space. Then every family $\mathcal{F}$ of balls of uniformly bounded diameter contains an at most countable disjointed subfamily $\mathcal{G}$ such that $\bigcup_{B\in\mathcal{F}} B \subseteq \bigcup_{B\in\mathcal{G}} 5B$.*

The following Whitney type covering lemma was included in [3, p. 70, Theorem (1.3)] (see also [4, p. 623, Theorem (3.2)] or [2]). We present the proof here for completeness.

***Lemma 2.2*** *Let $\Omega \subsetneq \mathcal{X}$ be a bounded open set. Then there exists a sequence $\{B_i\}_i$ of balls such that:*

*$(w)_i$ $\Omega = \bigcup_i B_i$ and $2B_i \subseteq \Omega$ for all $i$;*
*$(w)_{ii}$ there exists a positive constant $C$ such that for all $x \in \mathcal{X}$, $\sum_i \chi_{B_i}(x) \le C$;*
*$(w)_{iii}$ for all $i$, $(3B_i) \cap (\mathcal{X} \setminus \Omega) \ne \varnothing$.*

**Proof** For any $x \in \Omega$, let $\widehat{r}(x) \equiv \frac{1}{10} \operatorname{dist}(x, \mathcal{X}\setminus\Omega)$, where, and in what follows, for any $y$ and set $E$, $\operatorname{dist}(y, E) \equiv \inf_{z\in E} d(y, z)$. The function $\widehat{r}(x)$ is strictly positive, because $\Omega$ is open and the balls centered at $x$ form a basis of neighborhood of $x$. Then by Lemma 2.1, there exists a sequence $\{\widehat{B}_i\}_i \equiv \{B(x_i, \widehat{r}(x_i))\}_i$ of balls with $\{x_i\}_i \subseteq \Omega$ satisfying that $\{\widehat{B}_i\}_i$ are pairwise disjoint and $\{B_i\}_i \equiv \{5\widehat{B}_i\}_i$ forms a covering of $\Omega$. Moreover, for each $i$, set $r_i \equiv 5\widehat{r}(x_i)$. Then for any $i$ and $y \in 2B_i$, since $\mathcal{X}\setminus\Omega$ is closed, we have that

$$\operatorname{dist}(y, \mathcal{X}\setminus\Omega) \ge \operatorname{dist}(x_i, \mathcal{X}\setminus\Omega) - d(y, x_i) > \operatorname{dist}(x_i, \mathcal{X}\setminus\Omega) - 2r_i = 0.$$

This yields $y \in \Omega$ and hence $2B_i \subseteq \Omega$, which implies $(\mathrm{w})_{\mathrm{i}}$. On the other hand, since, by the definition of $r_i$, $3r_i = \frac{3}{2}\operatorname{dist}(x_i, \mathcal{X} \setminus \Omega)$, we then see that $(3B_i) \cap (\mathcal{X} \setminus \Omega) \neq \emptyset$, which implies $(\mathrm{w})_{\mathrm{iii}}$.

It remains to show $(\mathrm{w})_{\mathrm{ii}}$. To this end, we claim that for any $i$ and $x \in B_i \cap \Omega$,

$$\frac{1}{3}\operatorname{dist}(x, \mathcal{X} \setminus \Omega) < r_i < \operatorname{dist}(x, \mathcal{X} \setminus \Omega). \tag{2.1}$$

Indeed, by the fact that $\mathcal{X} \setminus \Omega$ is closed, we have

$$\operatorname{dist}(x_i, \mathcal{X} \setminus \Omega) \leq \operatorname{dist}(x, \mathcal{X} \setminus \Omega) + d(x, x_i),$$

which further implies that

$$\operatorname{dist}(x_i, \mathcal{X} \setminus \Omega) - r_i < \operatorname{dist}(x, \mathcal{X} \setminus \Omega). \tag{2.2}$$

Observe that by the definition of $r_i$, $\operatorname{dist}(x_i, \mathcal{X} \setminus \Omega) = 2r_i$. This together with (2.2) gives us that

$$r_i < \operatorname{dist}(x, \mathcal{X} \setminus \Omega). \tag{2.3}$$

On the other hand, by this, we also have

$$\operatorname{dist}(x, \mathcal{X} \setminus \Omega) \leq d(x, x_i) + \operatorname{dist}(x_i, \mathcal{X} \setminus \Omega) < 3r_i,$$

which combined with (2.3) implies (2.1), and hence the claim holds.

Now let $x \in \Omega$ and $B_i$ contain $x$. Then by (2.1), we see that

$$B_i \subseteq B\big(x, 2\operatorname{dist}(x, \mathcal{X} \setminus \Omega)\big).$$

On the other hand, observe that $\{\frac{1}{5}B_i\}_i = \{\widehat{B}_i\}_i$ are mutually disjoint. This, together with another application of (2.1), implies that $\{B(x_i, \frac{1}{15}\operatorname{dist}(x, \mathcal{X} \setminus \Omega))\}_i$ are also pairwise disjoint. From this and Remark 1.4(iii), we deduce that the cardinality of

$$\big\{B\big(x_i, \tfrac{1}{15}\operatorname{dist}(x, \mathcal{X} \setminus \Omega)\big)\big\}_i$$

contained in $B(x, 2\operatorname{dist}(x, \mathcal{X} \setminus \Omega))$ is at most $N_0 30^n$, and so is the cardinality of $\{B_i\}_i$ containing $x$. Thus, $(\mathrm{w})_{\mathrm{ii}}$ holds, which completes the proof of Lemma 2.2. ∎

Let $p \in (0, \infty)$, $f \in L^p_{\mathrm{loc}}(\mu)$ and $\nu \in \mathscr{M}(\mathcal{X})$. The *centered maximal functions* $\mathcal{M}_p f$ and $\mathcal{M}\nu$ are defined by setting, for all $x \in \mathcal{X}$,

$$\mathcal{M}_p f(x) \equiv \sup_{r>0}\left[\frac{1}{\mu(\overline{B}(x, 5r))}\int_{\overline{B}(x, r)} |f(y)|^p \, d\mu(y)\right]^{\frac{1}{p}}$$

and

$$\mathcal{M}\nu(x) \equiv \sup_{r>0} \frac{\nu(\overline{B}(x, r))}{\mu(\overline{B}(x, 5r))}.$$

If $p = 1$, we denote $\mathcal{M}_1$ simply by $\mathcal{M}$, which is called the *centered Hardy–Littlewood maximal operator*.

**Lemma 2.3** *The following statements hold.*

(i) *Let $p \in [1,\infty)$. Then $\mathcal{M}_p$ is bounded on $L^q(\mu)$ for all $q \in (p,\infty]$ and from $L^p(\mu)$ to $L^{p,\infty}(\mu)$.*

(ii) *Let $p \in (0,1)$. Then $\mathcal{M}_p$ is bounded on $L^{1,\infty}(\mu)$.*

(iii) *There exists a positive constant $C$ such that for all $\nu \in \mathscr{M}(\mathcal{X})$, $\mathcal{M}\nu \in L^{1,\infty}(\mu)$ and*

$$\|\mathcal{M}\nu\|_{L^{1,\infty}(\mu)} \le C\|\nu\|.$$

**Proof** The proof of (ii) mimics the proof of [13, Lemma 3.2], and the proof of (iii) is similar to that of boundedness of $\mathcal{M}$ from $L^1(\mu)$ to $L^{1,\infty}(\mu)$ in (i). Thus, it suffices to prove (i) by similarity. By [9, Lemma 2.5], any disjoint collection of open balls is at most countable. So is any disjoint collection of closed balls. Moreover, by an argument similar to that used in the proof of [9, Proposition 3.5], we see that $\mathcal{M}_p$ is bounded on $L^q(\mu)$ for all $q \in (p,\infty]$ and bounded from $L^p(\mu)$ to $L^{p,\infty}(\mu)$. This finishes the proof of Lemma 2.3. ∎

**Lemma 2.4** *Let $\eta \in \mathscr{M}(\mathcal{X})$ such that $\eta(\mathcal{X}) = 0$ and $\operatorname{supp}\eta \subseteq \overline{B}(x,\rho)$ for some $\rho \in (0,\infty)$ and $x \in \mathcal{X}$, and let $T$ be a Calderón–Zygmund operator with kernel $K$ satisfying* (1.5) *and* (1.6) *as in Definition* 1.5. *Then there exists a positive constant $C$, independent of $\eta$, $x$, and $\rho$, such that for all nonnegative Borel measures $\nu$ on $\mathcal{X}$,*

$$\int_{\mathcal{X}\setminus B(x,2\rho)} |T\eta(y)|\,d\nu(y) \le C\|\eta\|\mathcal{M}\nu(x). \tag{2.4}$$

*Moreover, for any $p \in [1,\infty)$ and $f \in L^p_{\mathrm{loc}}(\mu)$,*

$$\int_{\mathcal{X}\setminus B(x,2\rho)} |T\eta(y)||f(y)|\,d\mu(y) \le C\|\eta\|\mathcal{M}_p f(x) \tag{2.5}$$

*and*

$$\int_{\mathcal{X}\setminus B(x,2\rho)} |T\eta(y)|\,d\mu(y) \le C\|\eta\|, \tag{2.6}$$

*where $C$ is a positive constant, independent of $\eta$, $x$, $\rho$, and $f$.*

**Proof** By similarity, we only prove (2.4). By $\eta(\mathcal{X}) = 0$, $\operatorname{supp}\eta \subseteq \overline{B}(x,\rho)$, and (1.6), we have that for any $y \in \mathcal{X}\setminus B(x,2\rho)$,

$$\begin{aligned}|T\eta(y)| &= \left|\int_{\overline{B}(x,\rho)} K(y,\widetilde{x})\,d\eta(\widetilde{x})\right| = \left|\int_{\overline{B}(x,\rho)} [K(y,\widetilde{x}) - K(y,x)]\,d\eta(\widetilde{x})\right| \\ &\le \|\eta\| \sup_{\widetilde{x}\in\overline{B}(x,\rho)} |K(y,\widetilde{x}) - K(y,x)| \lesssim \|\eta\| \left[\frac{\rho}{d(x,y)}\right]^{\tau} \frac{1}{\lambda(x,d(x,y))}.\end{aligned}$$

Therefore, by (1.3), we have that

$$\begin{aligned}\int_{\mathcal{X}\setminus B(x,2\rho)} |T\eta(y)|\,d\nu(y) &\lesssim \|\eta\| \int_{\mathcal{X}\setminus B(x,2\rho)} \left[\frac{\rho}{d(x,y)}\right]^{\tau} \frac{1}{\lambda(x,d(x,y))}\,d\nu(y)\\ &\lesssim \|\eta\| \sum_{k=1}^{\infty} \int_{B(x,2^{k+1}\rho)\setminus B(x,2^k\rho)} \frac{1}{2^{k\tau}} \frac{1}{\lambda(x,2^k\rho)}\,d\nu(y)\\ &\lesssim \|\eta\| \sum_{k=1}^{\infty} \frac{1}{2^{k\tau}} \frac{\nu(B(x,2^{k+1}\rho))}{\mu(B(x,5\cdot 2^{k+1}\rho))}\\ &\lesssim \|\eta\| \sum_{k=1}^{\infty} \frac{1}{2^{k\tau}} \mathcal{M}\nu(x) \lesssim \|\eta\| \mathcal{M}\nu(x),\end{aligned}$$

which completes the proof of Lemma 2.4. ∎

## 3 Proof of Theorem 1.6, Part I

This section is devoted to the proof of the implications (i) $\Rightarrow$ (ii), (i) $\Rightarrow$ (iii), and (ii) $\Rightarrow$ (iii) of Theorem 1.6. To this end, we first establish an endpoint estimate for $T$ via the so-called elementary measures which are finite linear combinations of unit point masses with positive coefficients. We begin with the following Cotlar type inequality inspired by [13].

***Lemma 3.1*** *Let $T$ be a Calderón–Zygmund operator with kernel $K$ satisfying* (1.5) *and* (1.6)*, which is bounded on $L^2(\mu)$. Then there exist positive constants $C$ and $c$ such that for any $f \in L^\infty_b(\mu)$ and $x \in \operatorname{supp}\mu$,*

$$T^\sharp(f)(x) \le C\mathcal{M}(Tf)(x) + c\mathcal{M}_2(f)(x). \tag{3.1}$$

**Proof** Let $x \in \operatorname{supp}\mu$, $r \in (0,\infty)$, and $r_j \equiv 5^j r$ and $\mu_j \equiv \mu(\overline{B}(x,r_j))$ for $j \in \mathbb{Z}_+ \equiv \mathbb{N}\cup\{0\}$. We claim that there exists some $j \in \mathbb{N}$ such that $\mu_{j+1} \le 4C_\lambda^6 \mu_{j-1}$, where $C_\lambda$ is as in (1.3). Otherwise, by (1.3), we would have that for every $j \in \mathbb{N}$,

$$\mu_0 < (4C_\lambda^6)^{-j}\mu_{2j} = (4C_\lambda^6)^{-j}\mu\big(\overline{B}(x,r_{2j})\big) \lesssim (4C_\lambda^6)^{-j}\lambda(x,5^{2j}r) \lesssim 5^{-j}\lambda(x,r).$$

Letting $j \to 0$, we have $\mu(\overline{B}(x,r)) = 0$, which contradicts the fact that $\mu(\overline{B}(x,r)) > 0$ for each $r > 0$ and each $x \in \operatorname{supp}\mu$. Thus, the claim holds.

Let $k \in \mathbb{N}$ be the smallest integer such that $\mu_{k+1} \le 4C_\lambda^6\mu_{k-1}$ and $R \equiv r_{k-1} \equiv 5^{k-1}r$. Then we see that

$$\mu\big(\overline{B}(x,25R)\big) \lesssim \mu\big(\overline{B}(x,R)\big). \tag{3.2}$$

Observe that for all $j \in \{1,\dots,k\}$, we have that $\mu_{j+1} \le (2C_\lambda^3)^{j+2-k}\mu_k$ and

$$\lambda(x,r_k) \le (C_\lambda^3)^{\max\{0,k-j-1\}}\lambda(x,r_{j+1}).$$

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Let $f \in L_b^\infty(\mu)$. From this, (1.5), (1.3), and the Hölder inequality, we deduce that

$$
\begin{aligned}
(3.3)\qquad |T_r f(x) - T_{5R} f(x)| &\le \int_{\overline{B}(x,\,5R)\setminus\overline{B}(x,\,r)} |K(x,y)||f(y)|\,d\mu(y)\\
&= \sum_{j=1}^{k} \int_{\overline{B}(x,\,r_j)\setminus\overline{B}(x,\,r_{j-1})} |K(x,y)||f(y)|\,d\mu(y)\\
&\lesssim \sum_{j=1}^{k} \frac{\mu(\overline{B}(x,r_{j+1}))}{\lambda(x,r_{j+1})} \mathcal{M}(f)(x)\\
&\lesssim \sum_{j=1}^{k} 2^{j-k} \mathcal{M}(f)(x) \lesssim \mathcal{M}(f)(x).
\end{aligned}
$$

Let

$$V_R(x) \equiv \frac{1}{\mu(\overline{B}(x,R))} \int_{\overline{B}(x,R)} Tf(y)\,d\mu(y).$$

Then we have

$$(3.4)\qquad |V_R(x)| \lesssim \mathcal{M}(Tf)(x).$$

On the other hand, observe that

$$
\begin{aligned}
T_{5R} f(x) &= \int_{\mathcal{X}\setminus\overline{B}(x,\,5R)} K(x,y) f(y)\,d\mu(y) = \int_{\mathcal{X}} K(x,y)\chi_{\mathcal{X}\setminus\overline{B}(x,\,5R)}(y) f(y)\,d\mu(y)\\
&= T\big(f\chi_{\mathcal{X}\setminus\overline{B}(x,\,5R)}\big)(x) = \Big\langle \delta_x, T\big(f\chi_{\mathcal{X}\setminus\overline{B}(x,\,5R)}\big)\Big\rangle\\
&= \big\langle T^*\delta_x, f\chi_{\mathcal{X}\setminus\overline{B}(x,\,5R)}\big\rangle = \int_{\mathcal{X}\setminus\overline{B}(x,\,5R)} T^*\delta_x(y) f(y)\,d\mu(y),
\end{aligned}
$$

where, and in what follows, $\delta_x$ denotes the *Dirac measure at* $x$, and for a linear operator $T$, $T^*$ means the *adjoint operator of* $T$. By writing

$$
\begin{aligned}
V_R(x) &= \frac{1}{\mu(\overline{B}(x,R))} \int_{\mathcal{X}} \chi_{\overline{B}(x,\,R)}(y) T(f)(y)\,d\mu(y)\\
&= \frac{1}{\mu(\overline{B}(x,R))} \int_{\mathcal{X}} \chi_{\overline{B}(x,\,R)}(y) T\big(f\chi_{\overline{B}(x,\,5R)}\big)(y)\,d\mu(y)\\
&\quad + \int_{\mathcal{X}} T^*\Big(\frac{\chi_{\overline{B}(x,\,R)}}{\mu(\overline{B}(x,R))}\Big)(y) f(y)\chi_{\mathcal{X}\setminus\overline{B}(x,\,5R)}(y)\,d\mu(y),
\end{aligned}
$$

we obtain that

$$
\begin{aligned}
(3.5)\qquad |T_{5R} f(x) - V_R(x)| &\le \left| \int_{\mathcal{X}\setminus\overline{B}(x,\,5R)} T^*\Big(\delta_x - \frac{\chi_{\overline{B}(x,R)}}{\mu(\overline{B}(x,R))}\,d\mu\Big)(y) f(y)\,d\mu(y) \right|\\
&\quad + \left| \frac{1}{\mu(\overline{B}(x,R))} \int_{\mathcal{X}} \big[Tf\chi_{\overline{B}(x,\,5R)}(y)\big]\, \chi_{\overline{B}(x,\,R)}(y)\,d\mu(y) \right|\\
&\equiv \mathrm{L}_1 + \mathrm{L}_2.
\end{aligned}
$$

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By (2.5), we have $\mathrm{L}_1 \lesssim \mathcal{M}(f)(x)$. From the Hölder inequality, the boundedness of $T$ on $L^2(\mu)$, and (3.2), we further deduce that

$$\begin{aligned}\mathrm{L}_2 &\leq \left[\mu\left(\overline{B}(x,R)\right)\right]^{-\frac{1}{2}} \left[\int_{\mathcal{X}} \left|T\left(f\chi_{\overline{B}(x,\,5R)}\right)(y)\right|^2 d\mu(y)\right]^{\frac{1}{2}} \\ &\lesssim \left[\mu\left(\overline{B}(x,R)\right)\right]^{-\frac{1}{2}} \left[\int_{\overline{B}(x,\,5R)} |f(y)|^2\, d\mu(y)\right]^{\frac{1}{2}} \lesssim \mathcal{M}_2(f)(x).\end{aligned}$$

Then combining the estimates for $\mathrm{L}_1$ and $\mathrm{L}_2$ and using (3.5), (3.4), and (3.3), we have that for any $r \in (0,\infty)$,

$$\begin{aligned}|T_r f(x)| &\leq |T_r f(x) - T_{5R} f(x)| + |T_{5R} f(x) - V_R(x)| + |V_R(x)| \\ &\lesssim \mathcal{M}_2(f)(x) + \mathcal{M}(Tf)(x).\end{aligned}$$

Taking the supremum over $r \in (0,\infty)$, we obtain (3.1), and hence complete the proof of Lemma 3.1. ■

***Remark 3.2*** We point out that if we replace the boundedness of $T$ on $L^2(\mu)$ in Lemma 3.1 by the boundedness of $T$ on $L^q(\mu)$ for some $q \in (1,\infty)$, then (3.1) still holds with $\mathcal{M}_2$ replaced by $\mathcal{M}_q$.

To prove Theorem 1.6, we still need to recall the notion of non-atomic spaces; see, for example, [6].

***Definition 3.3*** A subset $A$ of a measure space $(\mathcal{X},\mu)$ is called an *atom* if $\mu(A) > 0$ and each $B \subseteq A$ has measure either equal to zero or equal to $\mu(A)$. A measure space $(\mathcal{X},\mu)$ is called *non-atomic* if it contains no atoms.

***Remark 3.4*** We know from Definition 3.3 that $\mathcal{X}$ is non-atomic if and only if for any $A \subseteq \mathcal{X}$ with $\mu(A) > 0$, there exists a proper subset $B \subsetneq A$ with $\mu(B) > 0$ and $\mu(A \setminus B) > 0$. By this, it is straightforward that if $\mu(\{x\}) = 0$ for any $x \in \mathcal{X}$, then $(\mathcal{X},\mu)$ is a non-atomic space. Moreover, it is known that if $(\mathcal{X},\mu)$ is a non-atomic measure space, then for any sets $A_0 \subseteq A_1 \subseteq \mathcal{X}$ such that $0 < \mu(A_1) < \infty$ and $\mu(A_0) \leq t \leq \mu(A_1)$ for some $t \in (0,\infty)$, there exists a set $E$ such that $A_0 \subseteq E \subseteq A_1$ and $\mu(E) = t$; see, for example, [6, p. 65].

We say that $\nu$ is an *elementary measure* if it is of the form

$$\nu \equiv \sum_{i=1}^{N} \alpha_i \delta_{x_i},$$

where $N \in \mathbb{N}$, $\delta_{x_i}$ is the Dirac measure at some $x_i \in \mathcal{X}$ and $\alpha_i > 0$ for $i \in \{1,\ldots,N\}$. To prove Theorem 1.6, we first establish an endpoint estimate for $T$ on these elementary measures. This generalizes [13, Theorem 5.1], where it was proven for polynomially bounded measures as in (1.2) on $\mathbb{R}^n$.

***Theorem 3.5*** *Let $T$ be a Calderón–Zygmund operator with kernel $K$ satisfying* (1.5) *and* (1.6)*, which is bounded on $L^2(\mu)$. Then there exist positive constants $C_1$ and $C_2$ such that for all elementary measures $\nu$,*

$$\|T\nu\|_{L^{1,\infty}(\mu)} \le \left[C_1 + C_2\|T\|_{L^2(\mu)\to L^2(\mu)}\right]\|\nu\|. \tag{3.6}$$

**Proof** Without loss of generality, we may normalize $\nu$ such that $\|\nu\| = \sum_{i=1}^N \alpha_i = 1$, and hence we only need prove

$$\|T\nu\|_{L^{1,\infty}(\mu)} \le C_1 + C_2\|T\|_{L^2(\mu)\to L^2(\mu)}. \tag{3.7}$$

Since for $t \in (0, 1/\mu(\mathcal{X})]$, we have

$$t\mu\big(\{x \in \mathcal{X} : |T\nu(x)| > t\}\big) \le t\mu(\mathcal{X}) \le 1.$$

Therefore it remains to consider the case $t \in (1/\mu(\mathcal{X}), \infty)$. Let $\overline{B}(x_1, \rho_1)$ be the *smallest closed ball* such that $\mu(\overline{B}(x_1, \rho_1)) \ge \alpha_1/t$. Indeed, since the function $\rho \to \mu(\overline{B}(x, \rho))$ is increasing and continuous from the right, and greater than $1/t \ge \alpha_1/t$ for sufficiently large $\rho > 0$, such $\rho_1$ exists and is strictly positive. Then

$$\mu(B(x_1, \rho_1)) = \lim_{\rho\to\rho_1-0} \mu(\overline{B}(x_1, \rho)) \le \frac{\alpha_1}{t}.$$

Since $(\mathcal{X}, \mu)$ is non-atomic, by Remark 3.4, we can find a Borel set $E_1$ such that

$$B(x_1, \rho_1) \subseteq E_1 \subseteq \overline{B}(x_1, \rho_1)$$

and $\mu(E_1) = \frac{\alpha_1}{t}$.

Let $\overline{B}(x_2, \rho_2)$ be the *smallest closed ball* such that $\mu(\overline{B}(x_2, \rho_2) \setminus E_1) \ge \alpha_2/t$. Similarly, for the corresponding open ball $B(x_2, \rho_2)$, we have $\mu(B(x_2, \rho_2) \setminus E_1) \le \alpha_2/t$ and henceforth find a Borel set $E_2$ with the property:

$$\big(B(x_2, \rho_2) \setminus E_1\big) \subseteq E_2 \subseteq \big(\overline{B}(x_2, \rho_2) \setminus E_1\big)$$

and $\mu(E_2) = \frac{\alpha_2}{t}$.

Repeating the process, for $i \in \{3, \cdots, N\}$, we have $\overline{B}(x_i, \rho_i)$ and $E_i$ such that $\overline{B}(x_i, \rho_i)$ is the *smallest closed ball* satisfying that $\mu(\overline{B}(x_i, \rho_i) \setminus \bigcup_{l=1}^{i-1} E_l) \ge \alpha_i/t$,

$$\Big(B(x_i, \rho_i) \setminus \bigcup_{l=1}^{i-1} E_l\Big) \subseteq E_i \subseteq \Big(\overline{B}(x_i, \rho_i) \setminus \bigcup_{l=1}^{i-1} E_l\Big)$$

and $\mu(E_i) = \frac{\alpha_i}{t}$. Let $E \equiv \bigcup_{i=1}^N E_i$. Then by the fact that $\sum_{i=1}^N \alpha_i = 1$, together with the choices of $\{B(x_i, \rho_i)\}_{i=1}^N$ and $\{E_i\}_{i=1}^N$, we see that

$$\bigcup_{i=1}^N B(x_i, \rho_i) \subseteq E \subseteq \bigcup_{i=1}^N \overline{B}(x_i, \rho_i)$$

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and $\mu(E) = \frac{1}{t}$.

Outside $E$, let us compare $T\nu$ to $t\sigma$, where

$$\sigma \equiv \sum_{i=1}^{N} \chi_{\mathcal{X}\setminus\overline{B}(x_i,\,2\rho_i)} T(\chi_{E_i}\,d\mu).$$

We have

$$\begin{aligned} \text{(3.8)} \qquad T\nu - t\sigma &= T\Big(\sum_{i=1}^{N} \alpha_i \delta_{x_i}\Big) - t\sum_{i=1}^{N} \chi_{\mathcal{X}\setminus\overline{B}(x_i,\,2\rho_i)} T(\chi_{E_i}\,d\mu) \\ &= \sum_{i=1}^{N} \big[\alpha_i\, T\delta_{x_i} - t\chi_{\mathcal{X}\setminus\overline{B}(x_i,\,2\rho_i)} T(\chi_{E_i}\,d\mu)\big] \equiv \sum_{i=1}^{N} \varphi_i. \end{aligned}$$

Notice that for any $i$,

$$\begin{aligned} \text{(3.9)} \qquad &\int_{\mathcal{X}\setminus E} |\varphi_i(x)|\,d\mu(x) \\ &= \int_{\mathcal{X}\setminus \bigcup_{i=1}^{N} E_i} \big|\alpha_i\, T\delta_{x_i}(x) - t\chi_{\mathcal{X}\setminus\overline{B}(x_i,\,2\rho_i)}(x) T(\chi_{E_i}\,d\mu)(x)\big|\,d\mu(x) \\ &\le \int_{\mathcal{X}\setminus\overline{B}(x_i,\,2\rho_i)} \big|\alpha_i\, T\delta_{x_i}(x) - t\chi_{\mathcal{X}\setminus\overline{B}(x_i,\,2\rho_i)}(x) T(\chi_{E_i}\,d\mu)(x)\big|\,d\mu(x) \\ &\quad + \int_{\overline{B}(x_i,\,2\rho_i)\setminus B(x_i,\,\rho_i)} \cdots \\ &= \int_{\mathcal{X}\setminus\overline{B}(x_i,\,2\rho_i)} \big|T(\alpha_i\delta_{x_i} - t\chi_{E_i}\,d\mu)(x)\big|\,d\mu(x) \\ &\quad + \int_{\overline{B}(x_i,\,2\rho_i)\setminus B(x_i,\,\rho_i)} \alpha_i |T\delta_{x_i}(x)|\,d\mu(x) \equiv \mathrm{J}_1 + \mathrm{J}_2. \end{aligned}$$

For each $i$, using (2.6) and $\mu(E_i) = \frac{\alpha_i}{t}$, we see that

$$\mathrm{J}_1 \lesssim \|\alpha_i\delta_{x_i} - t\chi_{E_i}\,d\mu\| \lesssim \alpha_i.$$

Moreover, from (1.5), (1.4) and (1.3), we deduce that

$$\begin{aligned} \mathrm{J}_2 &\lesssim \int_{\overline{B}(x_i,\,2\rho_i)\setminus B(x_i,\,\rho_i)} \frac{\alpha_i}{\lambda(x, d(x, x_i))}\,d\mu(x) \\ &\lesssim \int_{\overline{B}(x_i,\,2\rho_i)\setminus B(x_i,\,\rho_i)} \frac{\alpha_i}{\lambda(x_i, d(x, x_i))}\,d\mu(x) \lesssim \alpha_i \frac{\mu(\overline{B}(x_i,\,2\rho_i))}{\lambda(x_i, \rho_i)} \lesssim \alpha_i. \end{aligned}$$

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By the estimates of $\mathrm{J}_1$ together with $\mathrm{J}_2$ and (3.9), we obtain that $\int_{\mathcal{X}\setminus E}|\varphi_i|\,d\mu \lesssim \alpha_i$, which, together with (3.8) and the fact that $\sum_{i=1}^N \alpha_i = 1$, further implies that there exists a positive constant $C_3$ such that

$$(3.10)\qquad \int_{\mathcal{X}\setminus E} |T\nu(x) - t\sigma(x)|\,d\mu(x) \le \sum_{i=1}^{N} \int_{\mathcal{X}\setminus E} |\varphi_i(x)|\,d\mu(x) \le C_3.$$

Via (3.10), to accomplish the proof of Theorem 3.5, it suffices to show that there exist positive constants $C_4$ and $C_5$ such that $C_6 \equiv C_4 + C_5\|T\|_{L^2(\mu)\to L^2(\mu)}$ satisfying

$$(3.11)\qquad \mu(\{x \in \mathcal{X} : |\sigma(x)| > C_6\}) \le \frac{2}{t}.$$

Indeed, assume that (3.11) holds for the moment. Then from $\mu(E) = \frac{1}{t}$, (3.10) and (3.11), we deduce that

$$\begin{aligned}
&\mu\Big(\big\{x \in \mathcal{X} : |T\nu(x)| > (C_3 + C_6)t\big\}\Big)\\
&\quad\le \mu\Big(\{x \in \mathcal{X}\setminus E : |T\nu(x)| > (C_3 + C_6)t\}\Big) + \mu(E)\\
&\quad\le \mu\Big(\{x \in \mathcal{X}\setminus E : |T\nu(x) - t\sigma(x)| > C_3 t\}\Big)\\
&\qquad + \mu\Big(\{x \in \mathcal{X} : |\sigma(x)| > C_6\}\Big) + \mu(E) \le \frac{4}{t}.
\end{aligned}$$

This implies (3.7), and hence finishes the proof of Theorem 3.5, up to the verification of (3.11), which we do in the following lemma. ∎

***Lemma 3.6*** *The estimate* (3.11) *holds.*

**Proof** We first claim that there exist $C_4$ and $C_5$ such that for any set $F$ with $\mu(F) = \frac{1}{t}$,

$$(3.12)\qquad \left|\int_{\mathcal{X}} \sigma(x)\chi_F(x)\,d\mu(x)\right| \le \frac{1}{t}\left[C_4 + C_5\|T\|_{L^2(\mu)\to L^2(\mu)}\right].$$

Indeed, let $F$ be such a set. Then the definition of $\sigma$ gives us that

$$\begin{aligned}
(3.13)\qquad \int_{\mathcal{X}} \sigma(x)\chi_F(x)\,d\mu(x) &= \sum_{i=1}^{N} \int_{\mathcal{X}} T\chi_{E_i}(x)\chi_{F\setminus \overline{B}(x_i,\,2\rho_i)}(x)\,d\mu(x)\\
&= \sum_{i=1}^{N} \int_{\mathcal{X}} \chi_{E_i}(x)T^*\chi_{F\setminus \overline{B}(x_i,\,2\rho_i)}(x)\,d\mu(x).
\end{aligned}$$

From (1.4) and (1.3), it follows that for all $x \in E_i \subseteq \overline{B}(x_i, \rho_i)$ and $y \in \overline{B}(x_i,\, 2\rho_i) \setminus \overline{B}(x,\, \rho_i)$, $\lambda(x_i, \rho_i) \lesssim \lambda(y, d(x, y))$, which, together with (1.5) and (1.4), further im-

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plies that for all $x \in E_i \subseteq \overline{B}(x_i, \rho_i)$,

$$\begin{aligned}\left| T^*\chi_{F\setminus\overline{B}(x_i,\,2\rho_i)}(x) - T^*\chi_{F\setminus\overline{B}(x,\,\rho_i)}(x)\right| &\le \int_{\overline{B}(x_i,\,2\rho_i)\setminus\overline{B}(x,\,\rho_i)} |K(y,x)|\,d\mu(y)\\ &\lesssim \int_{\overline{B}(x_i,\,2\rho_i)\setminus\overline{B}(x,\,\rho_i)} \frac{1}{\lambda(y,d(x,y))}\,d\mu(y)\\ &\lesssim \frac{\mu(\overline{B}(x_i,2\rho_i))}{\lambda(x_i,\rho_i)} \lesssim 1.\end{aligned}$$

This, combined with the fact that $T^*\chi_{F\setminus\overline{B}(x,\,\rho_i)}(x) \le (T^*)^\sharp\chi_F(x)$ and Lemma 3.1, yields that for all $x \in E_i \subseteq \overline{B}(x_i, \rho_i)$,

$$\begin{aligned}\left|T^*\chi_{F\setminus\overline{B}(x_i,\,2\rho_i)}(x)\right| &\le \left|T^*\chi_{F\setminus\overline{B}(x_i,\,2\rho_i)}(x) - T^*\chi_{F\setminus\overline{B}(x,\,\rho_i)}(x)\right| + \left|T^*\chi_{F\setminus\overline{B}(x,\,\rho_i)}(x)\right|\\ &\lesssim 1 + (T^*)^\sharp\chi_F(x) \lesssim 1 + \mathcal{M}(T^*\chi_F)(x).\end{aligned}$$

Furthermore, by this, (3.13), $E = \bigcup_{i=1}^N E_i$ (disjoint union), and $\mu(E) = \frac{1}{t}$, we have that

$$\begin{aligned}\left|\int_{\mathcal{X}} \sigma(x)\chi_F(x)\,d\mu(x)\right| &\le \sum_{i=1}^N \left|\int_{\mathcal{X}} \chi_{E_i}(x)\left[T^*\chi_{F\setminus\overline{B}(x_i,\,2\rho_i)}\right](x)\,d\mu(x)\right|\\ &\lesssim \sum_{i=1}^N \int_{\mathcal{X}} \chi_{E_i}(x)[1+\mathcal{M}(T^*\chi_F)(x)]\,d\mu(x)\\ &\sim \frac{1}{t} + \int_{\mathcal{X}} \chi_E(x)\mathcal{M}(T^*\chi_F)(x)\,d\mu(x).\end{aligned}\tag{3.14}$$

Since $T$ is bounded on $L^2(\mu)$, by duality, we see that $T^*$ is also bounded on $L^2(\mu)$ and

$$\|T^*\|_{L^2(\mu)\to L^2(\mu)} = \|T\|_{L^2(\mu)\to L^2(\mu)}.$$

From this fact, Lemma 2.3(i), $\mu(F) = \frac{1}{t} = \mu(E)$, and the Hölder inequality, we further deduce that

$$\begin{aligned}\int_{\mathcal{X}} \chi_E(x)\mathcal{M}(T^*\chi_F)(x)\,d\mu(x) &\le \|\chi_E\|_{L^2(\mu)}\|\mathcal{M}(T^*\chi_F)\|_{L^2(\mu)}\\ &\le \|\chi_E\|_{L^2(\mu)}\|\mathcal{M}\|_{L^2(\mu)\to L^2(\mu)}\|T^*\|_{L^2(\mu)\to L^2(\mu)}\|\chi_F\|_{L^2(\mu)}\\ &= \frac{1}{t}\|\mathcal{M}\|_{L^2(\mu)\to L^2(\mu)}\|T\|_{L^2(\mu)\to L^2(\mu)},\end{aligned}$$

which together with (3.14) gives that there exist $C_4$ and $C_5$ satisfying (3.12). Therefore, claim (3.12) holds.

Suppose that $\mu\big(\{x \in \mathcal{X} : |\sigma(x)| > C_6\}\big) > 2/t$. Then either

$$\mu\big(\{x \in \mathcal{X} : \sigma(x) > C_6\}\big) > \frac{1}{t} \tag{3.15}$$

or

$$\mu\big(\{x \in \mathcal{X} : \sigma(x) < -C_6\}\big) > \frac{1}{t}.$$

Without loss of generality, we may only consider (3.15) by similarity. Pick some set $F \subseteq \mathcal{X}$ with $\mu(F) = 1/t$ such that $\sigma(x) > C_6$ everywhere on $F$ (such $F$ exists because of Remark 3.4). Then apparently,

$$\int_{\mathcal{X}} \sigma(x)\chi_F(x)\, d\mu(x) > \frac{C_6}{t}. \tag{3.16}$$

Thus, we get a contradiction by combining (3.12) with (3.16), which implies (3.11), and hence completes the proof of Lemma 3.6. ∎

***Remark 3.7*** (i) Theorem 3.5 also holds with finite linear combinations of Dirac measures with arbitrary real coefficients. Indeed, every such measure $\nu$ can be represented as $\nu = \nu_+ - \nu_-$, where $\nu_+$ and $\nu_-$ are finite linear combinations of Dirac measures with positive coefficients and $\|\nu\| = \|\nu_+\| + \|\nu_-\|$. Therefore, $\|T\nu\|_{L^{1,\infty}(\mu)} \le 2(C_1 + C_2\|T\|_{L^2(\mu)\to L^2(\mu)})\|\nu\|$.

(ii) If we replace the assumption of Theorem 3.5 that $T$ is bounded on $L^2(\mu)$ by the assumption that $T$ is bounded on $L^q(\mu)$ for some $q \in (1,\infty)$, then via a slight modification of the proof Theorem 3.5, we have (3.6) with $\|T\|_{L^2(\mu)\to L^2(\mu)}$ replaced by $\|T\|_{L^q(\mu)\to L^q(\mu)}$.

**Proof of Theorem 1.6, Part I** In this part, we show that Theorem 1.6(i) implies Theorem 1.6(ii) and (iii) and that Theorem 1.6(ii) implies Theorem 1.6(iii).

We first assume that (i) holds and show that (ii) and (iii) hold. By the Marcinkiewicz interpolation theorem and a duality argument, we obtain (ii) via (iii). Therefore, we only need to prove (iii). To this end, observe that for any $f \in L^1(\mu)$, $f = f^+ - f^-$, where $f^+ \equiv \max\{f, 0\} \ge 0$ and $f^- \equiv \max\{-f, 0\} \ge 0$. Moreover, if we let $C_b(\mathcal{X})$ be the *space of all continuous functions with bounded support*, by [9, Proposition 3.4] and its proof, we see that for any $f \in L^1(\mu)$ and $f \ge 0$, there exist $\{f_j\}_{j\in\mathbb{N}} \subseteq C_b(\mathcal{X})$ and $f_j \ge 0$ for all $j \in \mathbb{N}$ such that $\|f_j - f\|_{L^1(\mu)} \to 0$ as $j \to \infty$. By these observations, combined with the linear property of $T$, we see that to show (iii), it suffices to prove that (1.7) holds for all $f \in C_b(\mathcal{X})$ and $f \ge 0$.

Let $t > 0$, $G \equiv \{x \in \mathcal{X} : f(x) > t\}$, $f^t \equiv f\chi_G$, and $f_t \equiv f\chi_{\mathcal{X}\setminus G}$. Then $Tf = Tf^t + Tf_t$. Notice that

$$\int_{\mathcal{X}} [f_t(x)]^2\, d\mu(x) \le t\int_{\mathcal{X}} f_t(x)\, d\mu(x) \le t\|f\|_{L^1(\mu)}.$$

This and the boundedness of $T$ on $L^2(\mu)$ yield that

$$\int_{\mathcal{X}} |Tf_t(x)|^2\, d\mu(x) \le \|T\|^2_{L^2(\mu)\to L^2(\mu)} t\|f\|_{L^1(\mu)},$$

which implies that

$$\mu\Big(\big\{x\in\mathcal{X}:|Tf_t(x)|>t\|T\|_{L^2(\mu)\to L^2(\mu)}\big\}\Big)\le\frac{\|f\|_{L^1(\mu)}}{t}. \tag{3.17}$$

We now estimate $Tf^t$. Since, by $f\in C_b(\mathcal{X})$, $G$ is a bounded open set, by Lemma 2.2, there exists a sequence $\{B_i\}_i$ of balls with finite overlap such that $G=\bigcup_i B_i$ and $2B_i\subseteq G$ for all $i$. Without loss of generality, we may assume that the cardinality of $\{B_i\}_i$ is just $\mathbb{N}$. Then the fact that $\{B_i\}_{i\in\mathbb{N}}$ has the finite overlap implies that

$$f^t=\sum_{i\in\mathbb{N}}f\frac{\chi_{B_i}}{\sum_{j\in\mathbb{N}}\chi_{B_j}}\equiv\sum_{i\in\mathbb{N}}f_i.$$

Then it is easy to see that $f_i\ge 0$ for all $i\in\mathbb{N}$. For any $N\in\mathbb{N}$ and $i\in\{1,2,\dots,N\}$, define $f^{(N)}\equiv\sum_{i=1}^N f_i$ and

$$\alpha_i\equiv\int_{\mathcal{X}}f_i(y)\,d\mu(y)=\int_{B_i}f(y)\,d\mu(y).$$

Then $\alpha_i\ge 0$ for all $i\in\mathbb{N}$. By $G=\cup_{i\in\mathbb{N}}B_i$ and the finite overlap property of $\{B_i\}_{i\in\mathbb{N}}$, we have

$$\sum_{i=1}^\infty\alpha_i\le\sum_{i=1}^\infty\int_{B_i}f(y)\,d\mu(y)\lesssim\int_G f(y)\,d\mu(y)\lesssim\|f\|_{L^1(\mu)}. \tag{3.18}$$

Pick $x_i\in B_i$ and define $\nu^{(N)}\equiv\sum_{i=1}^N\alpha_i\delta_{x_i}$. We obtain that $\|\nu^{(N)}\|=\sum_{i=1}^N\alpha_i$. By (3.18), the fact that $2B_i\subseteq G$ for all $i\in\mathbb{N}$, and (2.6), there exists a positive constant $C_7$ such that

$$\begin{aligned}\int_{\mathcal{X}\setminus G}&\big|Tf^{(N)}(x)-T\nu^{(N)}(x)\big|\,d\mu(x)\\&=\int_{\mathcal{X}\setminus G}\Big|T\Big(\sum_{i=1}^N[f_i\,d\mu-\alpha_i\delta_{x_i}]\Big)(x)\Big|\,d\mu(x)\\&\le\sum_{i=1}^N\int_{\mathcal{X}\setminus 2B_i}\big|T(f_i\,d\mu-\alpha_i\delta_{x_i})(x)\big|\,d\mu(x)\lesssim\sum_{i=1}^N\alpha_i\le C_7\|f\|_{L^1(\mu)}.\end{aligned} \tag{3.19}$$

On the other hand, by Theorem 3.5, we see that

$$\mu\Big(\big\{x\in\mathcal{X}:|T\nu^{(N)}(x)|>(C_1+C_2\|T\|_{L^2(\mu)\to L^2(\mu)})t\big\}\Big)\le\frac1t\|\nu^{(N)}\|\le\frac1t\|f\|_{L^1(\mu)},$$

from which, together with (3.19), we deduce that

$$\begin{aligned}\mu\Big(\big\{x\in\mathcal{X}\setminus G:&|Tf^{(N)}(x)|>(C_7+C_1+C_2\|T\|_{L^2(\mu)\to L^2(\mu)})t\big\}\Big)\\&\le\mu\Big(\big\{x\in\mathcal{X}\setminus G:|Tf^{(N)}(x)-T\nu^{(N)}(x)|>C_7t\big\}\Big)\\&\quad+\mu\Big(\big\{x\in\mathcal{X}\setminus G:|T\nu^{(N)}(x)|>(C_1+C_2\|T\|_{L^2(\mu)\to L^2(\mu)})t\big\}\Big)\le\frac2t\|f\|_{L^1(\mu)}.\end{aligned}$$

This, combined with the fact that $\mu(G) \le \|f\|_{L^1(\mu)}/t$, implies that

$$(3.20)\quad \mu\Big(\big\{x \in \mathcal{X} : |Tf^{(N)}(x)| > (C_7 + C_1 + C_2\|T\|_{L^2(\mu)\to L^2(\mu)})t\big\}\Big) \le \frac{3}{t}\|f\|_{L^1(\mu)}.$$

Observe that $f^{(N)} \to f^t$ in $L^2(\mu)$ as $N \to \infty$. From the $L^2(\mu)$-boundedness of $T$, we then deduce that $Tf^{(N)} \to Tf^t$ also in $L^2(\mu)$ as $N \to \infty$. By this fact and (3.20), we have

$$\mu\Big(\big\{x \in \mathcal{X} : |Tf^t(x)| > (C_7 + C_1 + C_2\|T\|_{L^2(\mu)\to L^2(\mu)})t\big\}\Big) \le \frac{3}{t}\|f\|_{L^1(\mu)},$$

from which, together with (3.17), it follows that there exist positive constants $C_8$ and $C_9$ such that

$$\sup_{t>0} t\,\mu(\{x \in \mathcal{X} : |Tf(x)| > t\}) \le \big(C_8 + C_9\|T\|_{L^2(\mu)\to L^2(\mu)}\big)\,\|f\|_{L^1(\mu)}.$$

This implies (1.7), and hence finishes the proof of the implication (i) ⇒ (iii).

Now assume that (ii) holds. Then by Remark 3.7(ii) and a similar proof of (i) ⇒ (iii), we see that (iii) holds. We omit the details, which completes Part I of the proof of Theorem 1.6. ■

## 4 Proof of Theorem 1.6, Part II

This section is devoted to proving (iii) ⇒ (i) of Theorem 1.6. To do so, we first establish the boundedness of $T^\sharp$ from $L^1(\mu)$ to $L^{1,\infty}(\mu)$, which implies that $\{T_r\}_{r\in(0,\infty)}$ is uniformly bounded from $L^1(\mu)$ to $L^{1,\infty}(\mu)$. By restricting $\mu$ to $\mu_M$, where $\mu_M$ is the restriction of $\mu$ to a given ball $\overline{B}(x_0, M)$ for some $x_0 \in \mathcal{X}$ and $M \in (0,\infty)$, we will prove that for any $r \in (0,\infty)$ and $p \in (1,\infty)$, $T_r$ is bounded on $L^p(\mu_M)$. Then, using a smooth truncation argument, we will further show that $\{T_r\}_{r\in(0,\infty)}$ is uniformly bounded from $L^2(\mu)$ to $L^2(\mu_M)$ with the constant independent of $M$. By letting $M \to \infty$, $\{T_r\}_{r\in(0,\infty)}$ is uniformly bounded on $L^2(\mu)$. An argument involving the random dyadic cubes from [10] will yield the desired conclusion.

***Theorem 4.1*** *Let $T$ be a Calderón–Zygmund operator with kernel $K$ satisfying* (1.5) *and* (1.6)*, which is bounded from $L^1(\mu)$ to $L^{1,\infty}(\mu)$. Then there exists a positive constant $C$ such that for any $f \in L^1(\mu)$,*

$$\|T^\sharp f\|_{L^{1,\infty}(\mu)} \le C\|f\|_{L^1(\mu)}.$$

**Proof** Let $p \in (0,1)$. By Lemma 2.3(i) and (ii), we see that $\mathcal{M}$ is bounded from $L^1(\mu)$ to $L^{1,\infty}(\mu)$, and $\mathcal{M}_p$ is bounded on $L^{1,\infty}(\mu)$. Then by the boundedness of $T$ from $L^1(\mu)$ to $L^{1,\infty}(\mu)$, to show Theorem 4.1, we only need to prove that for any $f \in L^\infty_b(\mu)$ and $x \in \mathcal{X}$,

$$[T^\sharp f(x)]^p \lesssim [\mathcal{M}_p Tf(x)]^p + [\mathcal{M}f(x)]^p.$$

Moreover, it suffices to prove that for any $r > 0$, $f \in L_b^\infty(\mu)$ and $x \in \mathcal{X}$,

$$|T_r f(x)|^p \lesssim [\mathcal{M}_p Tf(x)]^p + [\mathcal{M} f(x)]^p. \tag{4.1}$$

To this end, for any $j \in \mathbb{N}$, let $r_j \equiv 5^j r$ and $\mu_j \equiv \mu(\overline{B}(x, r_j))$ be as in the proof of Lemma 3.1. Again let $k$ be the *smallest positive integer* such that $\mu_{k+1} \le 4C_\lambda^6 \mu_{k-1}$ and $R \equiv r_{k-1} = 5^{k-1} r$. Similarly to the proof of (3.3), we see that

$$|T_r f(x) - T_{5R} f(x)| \lesssim \mathcal{M} f(x). \tag{4.2}$$

Let $f_1 \equiv f\chi_{\overline{B}(x,5R)}$ and $f_2 \equiv f - f_1$. For any $u \in \overline{B}(x, R)$, if $K$ is the kernel associated with $T$, then by (1.6) and (1.3), we see that

$$\begin{aligned}\left|Tf_2(x) - Tf_2(u)\right| &\le \int_{d(x,y)>5R} \left|K(x, y) - K(u, y)\right| |f(y)| d\mu(y)\\ &\lesssim \sum_{k=1}^\infty \Big[\frac{d(x,u)}{5^k R}\Big]^\tau \int_{\overline{B}(x,5^{k+1}R)} \frac{|f(y)|}{\lambda(x, 5^k R)}\, d\mu(y) \lesssim \mathcal{M} f(x).\end{aligned}$$

This, combined with (4.2) and the fact that

$$Tf_2(x) = \int_{\mathcal{X}} K(x, y) f_2(y)\, d\mu(y) = T_{5R} f(x),$$

implies that

$$\begin{aligned}|T_r f(x)| &\le |T_r f(x) - T_{5R} f(x)| + |T_{5R} f(x) - Tf_2(u)| + |Tf_2(u)|\\ &\lesssim \mathcal{M} f(x) + |Tf(u)| + |Tf_1(u)|,\end{aligned}$$

from which, together with $p \in (0, 1)$, it further follows that for all $u \in \overline{B}(x, R)$,

$$|T_r f(x)|^p \lesssim \big[\mathcal{M} f(x)\big]^p + |Tf(u)|^p + |Tf_1(u)|^p. \tag{4.3}$$

Since $T$ is bounded from $L^1(\mu)$ to $L^{1,\infty}(\mu)$, by the Kolmogorov inequality (see, for example, [5, p. 102]), we obtain that

$$\frac{1}{\mu(\overline{B}(x,R))} \int_{\overline{B}(x,R)} |Tf_1(u)|^p\, d\mu(u) \lesssim \frac{1}{[\mu(\overline{B}(x,R))]^p} \Big[\int_{\overline{B}(x,R)} |f_1(u)|\, d\mu(u)\Big]^p. \tag{4.4}$$

Taking the average of the variable $u$ over $\overline{B}(x, R)$ on both sides of (4.3), and using (4.4), the Hölder inequality, and (3.2), we see that

$$\begin{aligned}|T_r f(x)|^p &\lesssim \big[\mathcal{M} f(x)\big]^p + \big[\mathcal{M}_p(Tf)(x)\big]^p + \frac{1}{\mu(\overline{B}(x,R))} \int_{\overline{B}(x,R)} |Tf_1(u)|^p\, d\mu(u)\\ &\lesssim \big[\mathcal{M} f(x)\big]^p + \big[\mathcal{M}_p(Tf)(x)\big]^p\\ &\quad + \frac{1}{[\mu(\overline{B}(x,25R))]^p} \Big[\int_{\overline{B}(x,5R)} |f(u)|\, d\mu(u)\Big]^p\\ &\lesssim \big[\mathcal{M} f(x)\big]^p + \big[\mathcal{M}_p(Tf)(x)\big]^p,\end{aligned}$$

which implies (4.1), and hence completes the proof Theorem 4.1. ■

Let $x_0 \in \mathcal{X}$ and $M \in (0, \infty)$. We now obtain the boundedness of the truncated operators $\{T_r\}_{r\in(0,\infty)}$ on $L^p(\mu_M)$ for all $p \in (1, \infty)$. Notice that the set $\mathcal{X} \setminus \overline{B}(x_0, M)$ has $\mu_M$-measure zero by definition, and hence we may agree that any $f \in L^p(\mu_M)$ satisfies $f|_{\mathcal{X}\setminus\overline{B}(x_0,M)} \equiv 0$. With this agreement, observe that

$$T_r f(x) = \int_{d(x,y)>r} K(x, y)f(y)\, d\mu(y) = \int_{d(x,y)>r} K(x, y)f(y)\, d\mu_M(y)$$

for $f \in L^p(\mu_M)$, so we may also replace $\mu$ by $\mu_M$ in the formula of $T_r f$ when considering functions $f \in L^p(\mu_M)$. Finally, observe that $\mu_M$ also satisfies the upper doubling condition with the same dominating function $\lambda$, so that all results shown for $\mu$ apply equally well to $\mu_M$, with constants uniform with respect to $M$.

***Lemma 4.2*** *Let $p \in (1, \infty)$ and $r \in (0, \infty)$. Let $M \in (0, \infty)$ and $\mu_M$ be as above. Then there exists a positive constant $\widetilde{C}$, depending on $M$ and $r$, such that for all $f \in L^p(\mu_M)$,*

$$\|T_r f\|_{L^p(\mu_M)} \le \widetilde{C}\|f\|_{L^p(\mu_M)}.$$

**Proof** We first claim that there exists a positive constant $C$ such that for all $x \in \overline{B}(x_0, M)$,

$$|T_r f(x)| \le C[\lambda(x, r)]^{-1/p}\|f\|_{L^p(\mu_M)}. \tag{4.5}$$

To this end, let $B_0 \equiv B(x, r)$. Then (1.5) together with the Hölder inequality gives that

$$|T_r f(x)| \lesssim \left[\int_{\mathcal{X}\setminus B_0} \frac{d\mu(y)}{[\lambda(x, d(x, y))]^{p'}}\right]^{\frac{1}{p'}} \|f\|_{L^p(\mu_M)}. \tag{4.6}$$

We prove the claim by inductively constructing an auxiliary sequence of radii, $\{r_0, r_1, r_2, \dots\}$, such that $r_0 = r$ and $r_{i+1}$ is the smallest $2^k r_i$ with $k \in \mathbb{N}$ satisfying

$$\lambda(x, 2^k r_i) > 2\lambda(x, r_i), \tag{4.7}$$

whenever such a $k$ exists. We consider the following two cases.

*Case* (i) For each $i \in \mathbb{Z}_+$, there exists $k \in \mathbb{N}$ such that (4.7) holds. In this case, $r_{i+1}$ will be the smallest $2^k r_i$ satisfying (4.7) for all $k \in \mathbb{N}$, and $\{B_i\}_{i\in\mathbb{N}} \equiv \{B(x, r_i)\}_{i\in\mathbb{N}}$. Now by (1.3) and the fact that $2^i\lambda(x, r) \le \lambda(x, r_i)$ for all $i \in \mathbb{Z}_+$, we have that

$$\begin{aligned}\int_{\mathcal{X}\setminus B_0} \frac{d\mu(y)}{[\lambda(x, d(x, y))]^{p'}} &\lesssim \sum_{i=0}^{\infty} \frac{\mu(B_{i+1})}{[\lambda(x, r_{i+1})]^{p'}} \lesssim \sum_{i=0}^{\infty} \frac{1}{[\lambda(x, r_{i+1})]^{p'-1}}\\ &\lesssim \sum_{i=0}^{\infty} \frac{1}{[2^i\lambda(x, r)]^{p'-1}} \sim \frac{1}{[\lambda(x, r)]^{p'-1}},\end{aligned} \tag{4.8}$$

and hence

$$\left[\int_{\mathcal{X}\setminus B_0} \frac{d\mu(y)}{[\lambda(x, d(x, y))]^{p'}}\right]^{\frac{1}{p'}} \lesssim [\lambda(x, r)]^{-\frac{1}{p}},$$

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which, combined with (4.6), implies (4.5), and the claim holds in this case.

*Case* (ii) For some $i_0 \in \mathbb{Z}_+$, (4.7) holds for all $i < i_0$ but does not hold for $i_0$. In this case, if $i_0 \in \mathbb{N}$, we let $\{B_i\}_{i=1}^{i_0}$ be as in Case (i), $r_{i_0+1} \equiv \infty$ and $B_{i_0+1} \equiv \mathcal{X}$; otherwise, if $i_0 = 0$, we then let $r_1 \equiv \infty$ and $B_1 \equiv \mathcal{X}$. Then we see that $\lambda(x, 2^k r_{i_0}) \le 2\lambda(x, r_{i_0})$ for all $k \in \mathbb{N}$ and

$$\mu(\mathcal{X}) \equiv \lim_{t\to\infty} \mu(B(x,t)) \le \lim_{t\to\infty} \lambda(x,t) \equiv \lambda(x,\infty) \le 2\lambda(x, r_{i_0}),$$

which, together with (1.3) and the fact that $2^i\lambda(x,r) \le \lambda(x,r_i)$ for all $i \le i_0$, gives (4.8) in this case, and the claim holds.

If $x \in \operatorname{supp} \mu_M = \overline{B}(x_0, M)$, then $\operatorname{supp} \mu_M \subseteq B(x, 3M)$. By this and the definition of $\operatorname{supp} \mu_M$, we get that

$$\mu_M(\mathcal{X}) = \mu_M(B(x, 3M)) \le \lambda(x, 3M) \le C_\lambda^{1+\log_2(3M/r)}\lambda(x,r),$$

thus

$$\frac{1}{\lambda(x,r)} \le \frac{C_\lambda^{3+\log_2(M/r)}}{\mu_M(\mathcal{X})}.$$

By this fact, we obtain that

$$\int_{\mathcal{X}} \frac{d\mu_M(\mathcal{X})}{\lambda(x,r)} \le \frac{C_\lambda^{3+\log_2(M/r)}}{\mu_M(\mathcal{X})} \int_{\mathcal{X}} d\mu_M(x) \le C_\lambda^{3+\log_2(M/r)}.$$

From this and (4.5), it follows that

$$\begin{aligned}\|T_r f\|_{L^p(\mu_M)} &\lesssim \|f\|_{L^p(\mu_M)} \Big[\int_{\mathcal{X}} \frac{d\mu_M(x)}{\lambda(x,r)}\Big]^{\frac{1}{p}} \\ &\lesssim \|f\|_{L^p(\mu_M)} \big[C_\lambda^{3+\log_2(M/r)}\big]^{\frac{1}{p}} = \widetilde{C}(M,r)\|f\|_{L^p(\mu_M)}.\end{aligned}$$

This finishes the proof of Lemma 4.2. ■

We will need the following result, which shows that two bounded Calderón–Zygmund operators having the same kernel can at most differ by a multiplication operator.

***Proposition 4.3*** *Let $T$ and $\widetilde{T}$ be Calderón-Zygmund operators that have the same kernel $K$ satisfying* (1.5) *and* (1.6) *and that are both bounded from $L^p(\mu)$ to $L^{p,\infty}(\mu)$ for some $p \in [1,\infty)$. Then there exists $b \in L^\infty(\mu)$ such that for all $f \in L^p(\mu)$,*

$$Tf - \widetilde{T}f = bf \quad \text{and} \quad \|b\|_{L^\infty(\mu)} \le \|T - \widetilde{T}\|_{L^p(\mu)\to L^{p,\infty}(\mu)}.$$

The proof will rely on the following lemma.

***Lemma 4.4*** *For a suitable $\delta \in (0,1)$, there exists a sequence of countable Borel partitions, $\{Q_\alpha^k\}_{\alpha\in\mathscr{A}_k}$, $k \in \mathbb{Z}$, of $\mathcal{X}$ with the following properties:*

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(i) *For some* $x_\alpha^k \in \mathcal{X}$ *and constants* $0 < c_1 < c_2 < \infty$, $B(x_\alpha^k, c_1\delta^k) \subseteq Q_\alpha^k \subseteq B(x_\alpha^k, c_2\delta^k)$;

(ii) $\{Q_\alpha^{k+1}\}_{\alpha\in\mathscr{A}_{k+1}}$ *is a refinement of* $\{Q_\alpha^k\}_{\alpha\in\mathscr{A}_k}$.

*Moreover, it may be arranged that*

$$\mu\Big(\bigcup_{\alpha, k} \partial Q_\alpha^k\Big) = 0, \tag{4.9}$$

*where for a set* $Q$, $\partial Q \equiv \{x \in \mathcal{X} : d(x, Q) = d(x, \mathcal{X} \setminus Q) = 0\}$ *is the boundary.*

**Proof** Let $\{Q_\alpha^k\}_{\alpha, k\in\mathbb{Z}}$ be the *random dyadic cubes* constructed in [10], so in fact $Q_\alpha^k = Q_\alpha^k(\omega)$, where $\omega$ is a point of an underlying *probability space* $\Omega$. We use $\mathbb{P}$ to denote a *probability measure* on $\Omega$ (as constructed in [10]), so that $\mathbb{P}(A)$ is the *probability of the event* $A \subset \Omega$. By the construction given in [10], these sets automatically satisfy the other claims for all $\omega \in \Omega$, and it remains to show that we can choose $\omega \in \Omega$ so as to also satisfy (4.9).

The "side-length" of $Q_\alpha^k$ is defined $\ell(Q_\alpha^k) \equiv \delta^k$, where $\delta \in (0, 1)$ is a fixed parameter entering the construction. For $\varepsilon \in (0, \infty)$, let

$$\delta_\varepsilon Q \equiv \{x : d(x, Q) \le \varepsilon\ell(Q)\} \bigcap \{x : d(x, \mathcal{X} \setminus Q) \le \varepsilon\ell(Q)\}.$$

It was shown in [10, Lemma 10.1] that there exists an $\eta > 0$ such that for any fixed $x \in \mathcal{X}$ and $k \in \mathbb{Z}$,

$$\mathbb{P}\Big(x \in \bigcup_\alpha \delta_\varepsilon Q_\alpha^k\Big) \lesssim \varepsilon^\eta.$$

In particular, by taking the limit as $\varepsilon \to 0$, we obtain that

$$\mathbb{P}\Big(x \in \bigcup_\alpha \partial Q_\alpha^k\Big) = 0.$$

Then it is possible to sum the zero probabilities over $k \in \mathbb{Z}$ to deduce

$$\mathbb{P}\Big(x \in \bigcup_{k, \alpha} \partial Q_\alpha^k\Big) = 0.$$

Now we can compute (the integration variable of the $d\mathbb{P}$-integrals is $\omega \in \Omega$, the random variable implicit in the random dyadic cubes $Q_\alpha^k = Q_\alpha^k(\omega)$):

$$\begin{aligned}\int_\Omega \mu\Big(\bigcup_{k, \alpha} \partial Q_\alpha^k\Big)\, d\mathbb{P} &= \int_\Omega \int_{\mathcal{X}} 1_{\bigcup_{k,\alpha} \partial Q_\alpha^k}(x)\, d\mu(x)\, d\mathbb{P} = \int_{\mathcal{X}} \int_\Omega 1_{\bigcup_{k,\alpha} \partial Q_\alpha^k}(x)\, d\mathbb{P}\, d\mu(x)\\ &= \int_{\mathcal{X}} \mathbb{P}\Big(x \in \bigcup_{k,\alpha} \partial Q_\alpha^k\Big)\, d\mu(x) = 0.\end{aligned}$$

So the integral of $\mu(\cup_{k,\alpha}\partial Q_\alpha^k(\omega)) \ge 0$ is zero. This means that $\mu(\bigcup_{k,\alpha} \partial Q_\alpha^k(\omega)) = 0$ for $\mathbb{P}$-almost every $\omega \in \Omega$. Now we just fix one such $\omega$, and for this choice, the boundaries of the corresponding dyadic cubes $Q_\alpha^k = Q_\alpha^k(\omega)$ have $\mu$-measure zero. This implies (4.9) and hence finishes the proof of Lemma 4.4. ∎

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**Proof of Proposition 4.3** Let $S \equiv T - \widetilde{T}$. Then $S$ is bounded from $L^p(\mu)$ to $L^{p,\infty}(\mu)$ for some $p \in [1, \infty)$ as in the proposition, and it has kernel 0. We will prove that for all $M \in \mathbb{N}$ and all $f \in L^p(\mu)$ with $\operatorname{supp} f \subseteq B_M \equiv \overline{B}(x_0, M)$, and $\mu$-almost every $x \in \mathcal{X}$,

$$Sf(x) = f(x)S(1_{B_M})(x) \equiv f(x)b_M(x) \tag{4.10}$$

and

$$\|b_M\|_{L^\infty(\mu_M)} \le \|S\|_{L^p(\mu)\to L^{p,\infty}(\mu)}, \tag{4.11}$$

where $\mu_M \equiv \mu|_{B_M}$.

Suppose for the moment that (4.10) and (4.11) are already verified. If $M < M'$, then for all $f \in L^p(\mu)$ with $\operatorname{supp} f \subseteq B_M \subseteq B_{M'}$, we have $fb_M = Sf = fb_{M'}$ almost everywhere on $B_M$. Since this is true for all such $f$, we must have $b_{M'} = b_M$ on $B_M$, and hence we can unambiguously define $b(x)$ for all $x \in \mathcal{X}$ by setting $b(x) \equiv b_M(x)$ for $x \in B_M$. The uniform bound (4.11) implies that $\|b\|_{L^\infty(\mu)} \le \|S\|_{L^p(\mu)\to L^{p,\infty}(\mu)}$, and we have $Sf = bf$ for all $f \in L^p(\mu)$ with bounded support. Finally, by density this holds for all $f \in L^p(\mu)$. Thus, proving (4.10) and (4.11) will prove the proposition, and we turn to this task.

Now we prove (4.10). Let us consider functions of the form

$$\sum_\alpha x_\alpha^k 1_{Q_\alpha^k \cap B_M}, \tag{4.12}$$

where $\{Q_\alpha^k\}_{\alpha,k}$ are the dyadic cubes with zero-measure boundaries as provided by Lemma 4.4. Since $(\mathcal{X}, d)$ is geometrically doubling and $B_M$ is bounded, we see that only finitely many $Q_\alpha^k$ intersect $B_M$, and hence the sum in (4.12) may taken to be finite.

We claim that for $\mu$-almost every $x \in \mathcal{X}$,

$$S\big(1_{Q_\alpha^k \cap B_M}\big)(x) = 1_{Q_\alpha^k \cap B_M}(x) \cdot S(1_{B_M})(x). \tag{4.13}$$

Indeed, observe first that for $\mu$-almost every $x \in \mathcal{X}$,

$$S(1_{B_M})(x) = S\Big(\sum_\beta 1_{Q_\beta^k \cap B_M}\Big)(x) = \sum_\beta S\big(1_{Q_\beta^k \cap B_M}\big)(x). \tag{4.14}$$

On the other hand, the assumption that $S$ has kernel 0 means that for any $f \in L_b^\infty(\mu)$ and $\mu$-almost every $x \notin \operatorname{supp} f$,

$$Sf(x) = \int_{\mathcal{X}} 0 f(y)\, d\mu(y) = 0.$$

This gives that

$$\begin{aligned}\operatorname{supp}\,(S(1_{Q_\beta^k \cap B_M})) &\subseteq \operatorname{supp} 1_{Q_\beta^k \cap B_M} = \overline{Q_\beta^k \cap B_M}\\ &\subseteq \overline{Q_\beta^k} \cup \overline{B}_M = (Q_\beta^k \cap \overline{B}_M) \cup (\partial Q_\beta^k \cap \overline{B}_M).\end{aligned}$$

Recall that $Q^k_\alpha$ and $Q^k_\beta$ are disjoint if $\alpha \neq \beta$, which together with (4.9) implies that almost every $x \in Q^k_\alpha \cap B_M$ is outside $\operatorname{supp}(S(1_{Q^k_\beta} \cap B_M))$. Hence $S(1_{Q^k_\beta \cap B_M})(x) = 0$ for $\mu$-almost every $x \in Q^k_\alpha \cap B_M$, and thus, for $\mu$-almost every $x \in \mathcal{X}$,

$$1_{Q^k_\alpha \cap B_M}(x) S\big(1_{Q^k_\beta \cap B_M}\big)(x) = \delta_{\alpha\beta} 1_{Q^k_\alpha \cap B_M}(x) S\big(1_{Q^k_\alpha \cap B_M}\big)(x) = \delta_{\alpha\beta} S\big(1_{Q^k_\alpha \cap B_M}\big)(x),$$

where $\delta_{\alpha\beta} \equiv 1$ if $\alpha = \beta$ and $\delta_{\alpha\beta} \equiv 0$ otherwise, and the last equality follows from the fact that $1_{Q^k_\alpha \cap B_M}(x) = 1$ for $\mu$-almost every $x \in \operatorname{supp}(S(1_{Q^k_\alpha \cap B_M}))$. Multiplying (4.14) by $1_{Q^k_\alpha \cap B_M}$ gives

$$1_{Q^k_\alpha \cap B_M}(x) S(1_{B_M})(x) = \sum_\beta 1_{Q^k_\alpha \cap B_M}(x) S(1_{Q^k_\beta \cap B_M})(x) = S(1_{Q^k_\alpha \cap B_M})(x),$$

which is precisely (4.13).

Now it is easy to complete the proof of (4.10). For any $f$ of the form (4.12), it follows from (4.13) that

$$Sf = \sum_\alpha x^k_\alpha S(1_{Q^k_\alpha \cap B_M}) = \sum_\alpha x^k_\alpha 1_{Q^k_\alpha \cap B_M} S(1_{B_M}) = f S(1_{B_M}). \tag{4.15}$$

On the other hand, recall that martingale convergence implies that for any $f \in L^1(\mu)$,

$$\mathbb{E}_k f \equiv \sum_\alpha \langle f \rangle_{Q^k_\alpha} 1_{Q^k_\alpha} \to f$$

for $\mu$-almost every $x \in \mathcal{X}$ and in $L^p(\mu)$ as $k \to \infty$. If $f \in L^p(\mu)$ is general, apply (4.15) to $\mathbb{E}_k f \cdot 1_{B_M}$. Then as $k \to \infty$, we have $\mathbb{E}_k f \cdot 1_{B_M} \to f \cdot 1_{B_M}$ in $L^p(\mu)$, hence $S(\mathbb{E}_k f \cdot 1_{B_M}) \to S(f \cdot 1_{B_M})$ in $L^{p,\infty}(\mu)$, and thus almost everywhere for a subsequence. Also, by (4.15), we obtain that

$$S(\mathbb{E}_k f \cdot 1_{B_M}) = \mathbb{E}_k f \cdot 1_{B_M} \cdot S(1_{B_M}) \to f \cdot 1_{B_M} \cdot S(1_{B_M})$$

for $\mu$-almost every $x \in \mathcal{X}$. As a result, for all $f \in L^p(\mu)$,

$$S(f \cdot 1_{B_M}) = f \cdot 1_{B_M} \cdot S(1_{B_M}) \equiv f \cdot 1_{B_M} \cdot b_M,$$

where $b_M \equiv S(1_{B_M}) \in L^{p,\infty}(\mu)$, since $1_{B_M} \in L^p(\mu)$. Thus, (4.10) holds for all $f \in L^p(\mu)$ with $\operatorname{supp} f \subseteq B_M$.

It remains to prove (4.11). Let $\lambda \in (0, \infty)$, $f \equiv 1_{\{|b_M| > \lambda\} \cap B_M}$ and

$$B \equiv \|S\|_{L^p(\mu) \to L^{p,\infty}(\mu)}.$$

Then $\|f\|_{L^p(\mu)} = [\mu(\{x \in \mathcal{X} : |b_M(x)| > \lambda\} \cap B_M)]^{1/p}$. By this, (4.10) and the

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boundedness of $S$ from $L^p(\mu)$ to $L^{p,\infty}(\mu)$, we see that

$$\begin{aligned}\lambda\Big[\mu\big(\{x\in\mathcal{X}:|b_M(x)|>\lambda\}\cap B_M\big)\Big]^{1/p}\\ &=\lambda\Big[\mu\big(\{x\in\mathcal{X}:|b_M(x)f(x)|>\lambda\}\big)\Big]^{1/p}\\ &=\lambda\Big[\mu\big(\{x\in\mathcal{X}:|Sf(x)|>\lambda\}\big)\Big]^{1/p}\\ &\le\|Sf\|_{L^{p,\infty}(\mu)}\le B\|f\|_{L^p(\mu)}\\ &=B\Big[\mu\big(\{x\in\mathcal{X}:|b_M(x)|>\lambda\}\cap B_M\big)\Big]^{1/p}.\end{aligned}$$

This means that either $\mu(\{x\in\mathcal{X}:|b_M(x)|>\lambda\}\cap B_M)=0$ or $\lambda\le B$, which is the same as $\|b_M\|_{L^\infty(\mu_M)}\le B$. This implies (4.11), and hence finishes the proof of Proposition 4.3. ■

From Proposition 4.3, we easily deduce the following consequence.

***Lemma 4.5*** *Let $T$ and $\widetilde{T}$ be Calderón–Zygmund operators having the same kernel $K$ satisfying* (1.5) *and* (1.6) *and which are both bounded from $L^1(\mu)$ to $L^{1,\infty}(\mu)$. Assume that $\widetilde{T}$ is bounded on $L^2(\mu)$. Then $T$ is also bounded on $L^2(\mu)$.*

**Proof** By Proposition 4.3, we have $Tf=\widetilde{T}f+bf$, where $b\in L^\infty(\mu)$. Hence

$$\|Tf\|_{L^2(\mu)}\le\|\widetilde{T}f\|_{L^2(\mu)}+\|bf\|_{L^2(\mu)}\le\big(\|\widetilde{T}\|_{L^2(\mu)\to L^2(\mu)}+\|b\|_{L^\infty(\mu)}\big)\|f\|_{L^2(\mu)},$$

which completes the proof of Lemma 4.5. ■

**Proof of Theorem 1.6, Part II** In this part, we show that Theorem 1.6(iii) implies Theorem 1.6(i). Let $\mu_M\equiv\mu|_{\overline{B}(x_0,M)}$ be as before. The assumption clearly implies that $T$ is bounded from $L^1(\mu_M)$ to $L^{1,\infty}(\mu_M)$, with a norm bound independent of $M$. We will then prove that $T$ is bounded on $L^2(\mu_M)$, still with a bound independent of $M$. By the density of boundedly supported $L^2_{\rm loc}(\mu)$-functions in $L^2(\mu)$ and the monotone convergence, this suffices to conclude the proof of (iii) $\Rightarrow$ (i) of Theorem 1.6. Thus, from now on we work with the measure $\mu_M$, recalling that it satisfies, uniformly in $M$, the same assumptions as $\mu$, so that everything shown for $\mu$ above applies equally well to $\mu_M$.

By Theorem 4.1, we see that $T^\sharp$ is bounded from $L^1(\mu_M)$ to $L^{1,\infty}(\mu_M)$, which implies that $\{T_r\}_{r\in(0,\infty)}$ is uniformly bounded from $L^1(\mu_M)$ to $L^{1,\infty}(\mu_M)$, and the bound (denoted by $N_1$) depends only on the norm of $T$ as the operator from $L^1(\mu)$ to $L^{1,\infty}(\mu)$.

Let $p\in(1,\infty)$. It follows from Lemma 4.2 that for any $r\in(0,\infty)$, $T_r$ is bounded on $L^p(\mu_M)$ with $p\in(1,\infty)$, but with the norm a priori depending on $M$ and $r$. We claim, however, that $\{T_r\}_{r\in(0,\infty)}$ is uniformly bounded on $L^2(\mu_M)$. That is, if we denote the corresponding norm by $N_p(r,M)$, then we have that there exists a positive constant $C$ depending on $N_1$, but not on $r$ or $M$, such that

$$N_2(r,M)\le C. \tag{4.16}$$

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To this end, we define for any $r \in (0,\infty)$ and $x \in \mathcal{X}$,

$$T_r^\psi f(x) \equiv \int_{\mathcal{X}} K(x,y)\psi\Big(\frac{d(x,y)}{r}\Big) f(y)\,d\mu(y),$$

where $\psi$ is a smooth function on $(0,\infty)$ such that $\operatorname{supp}\psi \subseteq [1/2,\infty)$, $\psi(t) \in [0,1]$ for all $t \in (0,\infty)$, and $\psi(t) \equiv 1$ when $t \in [1,\infty)$, and $K$ is the kernel of $T$. It follows, from the definition of $T_r^\psi$, (1.5) and (1.3), that for any $x \in \mathcal{X}$,

$$\begin{aligned}\big|\,T_r f(x) - T_r^\psi f(x)\big| &\le \int_{\overline{B}(x,r)\setminus B(x,r/2)} |K(x,y)||f(y)|\,d\mu(y)\\ &\lesssim \int_{\overline{B}(x,r)} \frac{|f(y)|}{\lambda(x,r/2)}\,d\mu(y) \lesssim \mathcal{M}f(x).\end{aligned}$$

This fact, together with Lemma 2.3(i), implies that the boundedness of $T_r$ on $L^p(\mu_M)$ for $p \in (1,\infty)$ or from $L^1(\mu_M)$ to $L^{1,\infty}(\mu_M)$ is equivalent to that of $T_r^\psi$. Moreover, if $\{T_r\}_{r\in(0,\infty)}$ is uniformly bounded on $L^p(\mu_M)$ or from $L^1(\mu_M)$ to $L^{1,\infty}(\mu_M)$, then so is $\{T_r^\psi\}_{r\in(0,\infty)}$; and vice verse.

Now we denote by $\widetilde{N}_p(r,M)$ the norm of $T_r^\psi$ on $L^p(\mu_M)$ and by $\widetilde{N}_1$ the (finite) supremum over $r$ and $M$ of the norms of $T_r^\psi$ from $L^1(\mu_M)$ to $L^{1,\infty}(\mu_M)$. Then to show (4.16), we only need to prove that

$$\widetilde{N}_2(r,M) \le \widetilde{C} \tag{4.17}$$

for some positive constant $\widetilde{C}$ independent of $r$ and $M$.

We now prove (4.17). Observe that for each $r$, $T_r^\psi$ is bounded on $L^2(\mu_M)$ and from $L^1(\mu_M)$ to $L^{1,\infty}(\mu_M)$. Then from the Marcinkiewicz interpolation theorem, we deduce that $T_r^\psi$ is bounded on $L^{\frac43}(\mu_M)$ and $\widetilde{N}_{\frac43}(r,M) \lesssim \widetilde{N}_1^{\frac12}[\widetilde{N}_2(r,M)]^{\frac12}$. By duality, the right-hand side also gives the bound for the norm of $(T_r^\psi)^*$ on $L^4(\mu_M)$. Observe that

$$(T_r^\psi)^*(g)(x) = \int_{\mathcal{X}} \overline{K(y,x)\psi\Big(\frac{d(x,y)}{r}\Big)}\, g(y)\,d\mu_M(y).$$

Then $(T_r^\psi)^*$ is also a Calderón–Zygmund operator. Thus $(T_r^\psi)^*$ is bounded from $L^1(\mu_M)$ to $L^{1,\infty}(\mu_M)$, and the norm is bounded by $c\widetilde{N}_1^{\frac12}[\widetilde{N}_2(r,M)]^{\frac12} + \widetilde{c}$ for some positive constants $c$ and $\widetilde{c}$. Another application of the Marcinkiewicz interpolation theorem yields that the norm of $(T_r^\psi)^*$ on $L^{\frac43}(\mu_M)$ is also bounded by $c\widetilde{N}_1^{\frac12}[\widetilde{N}_2(r,M)]^{\frac12} + \widetilde{c}$. By duality, we further see that $\widetilde{N}_4(r,M) \le c\widetilde{N}_1^{\frac12}[\widetilde{N}_2(r,M)]^{\frac12} + \widetilde{c}$. Using interpolation again, we have that $\widetilde{N}_2(r,M) \le c\widetilde{N}_1^{\frac12}[\widetilde{N}_2(r,M)]^{\frac12} + \widetilde{c}$, from which (4.17) follows. Thus, (4.16) holds and the claim is true.

As a result of (4.16), we see that $\{T_r\}_{r\in(0,\infty)}$ is uniformly bounded on $L^2(\mu_M)$, with bounds also uniform in $M$. By letting $M \to \infty$, we have that $\{T_r\}_{r\in(0,\infty)}$ is uniformly bounded on $L^2(\mu)$. Then there exists a weak limit $\tilde{T}$ bounded on $L^2(\mu)$ and some sequence $r_i \to 0$ as $i \to \infty$. That is, for all $f \in L^2(\mu)$ and $g \in L^2(\mu)$,

$$\langle g, \tilde{T}f\rangle = \lim_{r_i\to 0}\langle g, T_{r_i}f\rangle.$$

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By a standard argument (see, for example, [7, Proposition 8.1.11]), it is easy to check that $\tilde{T}$ is a Calderón–Zygmund operator with the same kernel $K$ as $T$. It follows, from (i) $\Rightarrow$ (iii) of Theorem 1.6 for the operator $\tilde{T}$, that $\tilde{T}$ is also bounded from $L^1(\mu)$ to $L^{1,\infty}(\mu)$. Applying Lemma 4.5, we have that $T$ is also bounded on $L^2(\mu)$. This finishes the proof of (iii) $\Rightarrow$ (i) of Theorem 1.6 and hence the proof of Theorem 1.6. ■

## 5 Proof of Corollary 1.7

As an application of Theorem 1.6, we prove Corollary 1.7 in this section. We begin with an inequality for $T^\sharp$ on the elementary measures.

***Lemma 5.1*** *Let $p \in (0,1)$ and let $T$ be a Calderón–Zygmund operator with kernel $K$ satisfying* (1.5) *and* (1.6), *which is bounded on $L^2(\mu)$. Then there exist positive constants $C$ and $C(p)$ such that for all elementary measures $\nu = \sum_i \alpha_i \delta_{x_i}$ and $x \in$ supp $\mu$,*

$$\left[T^\sharp \nu(x)\right]^p \le C\left[\mathcal{M}_p T\nu(x)\right]^p + C(p)[\mathcal{M}\nu(x)]^p. \tag{5.1}$$

**Proof** As in Lemma 3.1, let $r \in (0,\infty)$, $r_j \equiv 5^j r$, $\mu_j \equiv \mu(\overline{B}(x, r_j))$ for $j \in \mathbb{Z}_+$, let $k$ be the *smallest positive integer* such that $\mu_{k+1} \le 4C_\lambda^6 \mu_{k-1}$, and $R \equiv r_{k-1} = 5^{k-1}r$. Similarly to the proof of (3.3), we have

$$|T_r\nu(x) - T_{5R}\nu(x)| \lesssim \mathcal{M}\nu(x). \tag{5.2}$$

Now decompose the measure $\nu$ as $\nu = \nu_1 + \nu_2$, where

$$\nu_1 \equiv \sum_{i: x_i \in \overline{B}(x,\,5R)} \alpha_i \delta_{x_i} \quad \text{and} \quad \nu_2 \equiv \sum_{i: x_i \notin \overline{B}(x,\,5R)} \alpha_i \delta_{x_i}.$$

Applying (2.4) to $T^*$, we have that for any $\widetilde{x} \in \overline{B}(x, R)$,

$$\begin{aligned}
|T_{5R}\nu(x) - T\nu_2(\widetilde{x})| &= \left| \int_{\mathcal{X}} K(x,y)\chi_{\mathcal{X}\setminus\overline{B}(x,\,5R)}(y)\,d\nu(y) - T\nu_2(\widetilde{x}) \right| \\
&= \left| \int_{\mathcal{X}} K(x,y)\,d\nu_2(y) - T\nu_2(\widetilde{x}) \right| \\
&= |T\nu_2(x) - T\nu_2(\widetilde{x})| = |\langle \delta_x,\, T\nu_2\rangle - \langle \delta_{\widetilde{x}},\, T\nu_2\rangle| \\
&\le \int_{\mathcal{X}} |T^*(\delta_x - \delta_{\widetilde{x}})(y)|\,d\nu_2(y) \\
&\le \int_{\mathcal{X}\setminus\overline{B}(x,\,5R)} |T^*(\delta_x - \delta_{\widetilde{x}})(y)|\,d\nu(y) \lesssim \mathcal{M}\nu(x).
\end{aligned}$$

This implies that

$$\mathrm{H}_1 \equiv \frac{1}{\mu(\overline{B}(x,R))} \int_{\overline{B}(x,\,R)} |T_{5R}\nu(x) - T\nu_2(\widetilde{x})|^p\,d\mu(\widetilde{x}) \lesssim [\mathcal{M}\nu(x)]^p. \tag{5.3}$$

On the other hand, write

$$
\begin{aligned}
\mathrm{H}_2 &\equiv \frac{1}{\mu(\overline{B}(x,R))}\int_{\overline{B}(x,R)} |T\nu_2(\widetilde{x}) - T\nu(\widetilde{x})|^p \, d\mu(\widetilde{x}) \\
&= \frac{1}{\mu(\overline{B}(x,R))}\int_{\overline{B}(x,R)} |T\nu_1(\widetilde{x})|^p \, d\mu(\widetilde{x}) \\
&= \frac{1}{\mu(\overline{B}(x,R))}\int_0^\infty ps^{p-1}\mu\Big(\big\{\widetilde{x}\in\overline{B}(x,R) : |T\nu_1(\widetilde{x})| > s\big\}\Big)\, ds.
\end{aligned}
$$

Since $T$ is bounded on $L^2(\mu)$, by Theorem 3.5, we have that for every $s \in (0,\infty)$,

$$
\mu\Big(\big\{\widetilde{x}\in\overline{B}(x,R) : |T\nu_1(\widetilde{x})| > s\big\}\Big) \lesssim \min\Big(\mu\big(\overline{B}(x,R)\big),\, \frac{\|\nu_1\|}{s}\Big). \tag{5.4}
$$

Observe that $\|\nu_1\| = \nu(\overline{B}(x,5R))$. This, together with (5.4), the definition of $\mathcal{M}\nu$, and (3.2), gives that

$$
\begin{aligned}
\mu\Big(\big\{\widetilde{x}\in\overline{B}(x,R) : |T\nu_1(\widetilde{x})| > s\big\}\Big) &\lesssim \mu\big(\overline{B}(x,R)\big)\,\min\Big(1,\ \frac{1}{s}\frac{\nu(\overline{B}(x,5R))}{\mu(\overline{B}(x,R))}\Big) \\
&\lesssim \mu\big(\overline{B}(x,R)\big)\,\min\Big(1,\ \frac{1}{s}\mathcal{M}\nu(x)\Big),
\end{aligned}
$$

which further implies that

$$
\begin{aligned}
\mathrm{H}_2 &\lesssim \int_0^\infty ps^{p-1}\min\Big(1, \frac{1}{s}\mathcal{M}\nu(x)\Big)\, ds \\
&\sim \int_0^{\mathcal{M}\nu(x)} ps^{p-1}\, ds + \int_{\mathcal{M}\nu(x)}^\infty ps^{p-2}\mathcal{M}\nu(x)\, ds \lesssim \big[\mathcal{M}\nu(x)\big]^p.
\end{aligned}
$$

From this, combined with (5.3), we deduce that

$$
\frac{1}{\mu(\overline{B}(x,R))}\int_{\overline{B}(x,R)} \big|T_{5R}\nu(x) - T\nu(\widetilde{x})\big|^p \, d\mu(\widetilde{x}) \lesssim \mathrm{H}_1 + \mathrm{H}_2 \lesssim [\mathcal{M}\nu(x)]^p.
$$

Using this and (5.2), we see that

$$
\begin{aligned}
|T_r\nu(x)|^p &= \frac{1}{\mu(\overline{B}(x,R))}\int_{\overline{B}(x,R)} |T_r\nu(x)|^p \, d\mu(\widetilde{x}) \\
&\le \frac{1}{\mu(\overline{B}(x,R))}\int_{\overline{B}(x,R)} \big[|T_r\nu(x) - T_{5R}\nu(x)|^p \\
&\qquad + |T_{5R}\nu(x) - T\nu(\widetilde{x})|^p + |T\nu(\widetilde{x})|^p\big]\, d\mu(\widetilde{x}) \\
&\lesssim [\mathcal{M}\nu(x)]^p + \frac{1}{\mu(\overline{B}(x,R))}\int_{\overline{B}(x,R)} |T\nu(\widetilde{x})|^p \, d\mu(\widetilde{x}) \\
&\lesssim [\mathcal{M}\nu(x)]^p + \big[\mathcal{M}_p T\nu(x)\big]^p.
\end{aligned}
$$

Taking the supremum over $r > 0$, we see that (5.1) holds, which completes the proof of Lemma 5.1. ∎

As a result of Lemma 5.1, by Theorem 3.5 and Lemma 2.3(i) and (ii), we have the following corollary.

***Proposition 5.2*** *Let $T$ be a Calderón–Zygmund operator with kernel $K$ satisfying* (1.5) *and* (1.6)*, which is bounded on $L^2(\mu)$. Then there exists a positive constant $C$ such that for all elementary measures $\nu \in \mathscr{M}(\mathfrak{X})$,*

$$\|T^\sharp \nu\|_{L^{1,\infty}(\mu)} \le C\|\nu\|.$$

**Proof of Corollary 1.7** By Theorem 1.6, Remark 3.2, Lemma 2.3(i), and a density argument, we have (i). To prove (ii), it suffices to prove (1.8), since for any $f \in L^1(\mu)$, if we define $d\nu \equiv f d\mu$, then we see that $\nu \in \mathscr{M}(\mathfrak{X})$ and (1.9) follows from (1.8). Moreover, recall that for any complex measure $\nu \in \mathscr{M}(\mathfrak{X})$, $|\nu|(\mathfrak{X}) < \infty$; see, for example, [15, Theorem 6.4]. Then, by considering the Jordan decompositions of real and imaginary parts of $\nu$, we only need to prove (1.8) for any finite nonnegative measure.

To this end, assume that $\nu$ is a finite nonnegative measure and fix $t > 0$. We show that

$$\mu\big(\{x \in \mathfrak{X} : |T^\sharp \nu(x)| > t\}\big) \lesssim \frac{\|\nu\|}{t}.$$

Let $R > 0$ and consider the truncated maximal operator $T^\sharp_R \nu \equiv \sup_{r>R} |T_r \nu|$. Since $T^\sharp_R \nu(x)$ increases to $T^\sharp \nu(x)$ pointwise on $\mathfrak{X}$ as $R \to 0$, it suffices to show that there exists a positive constant $C$ such that for every $R > 0$,

$$\mu\Big(\Big\{x \in \mathfrak{X} : |T^\sharp_R \nu(x)| > t\Big\}\Big) \le \frac{C\|\nu\|}{t}. \tag{5.5}$$

In what follows, we use $\mathbb{P}$ to denote a *probability measure* on a *probability space* $\Omega$, $\mathbb{P}(A)$ the *probability of the event* $A \subset \Omega$, $\mathbb{E}(\xi)$ the *mathematical expectation of a random variable* $\xi \in L^1(\mathbb{P})$, and $\mathbb{V}(\xi) \equiv \mathbb{E}[(\xi - \mathbb{E}\xi)^2] = \mathbb{E}\xi^2 - (\mathbb{E}\xi)^2$ the *variance of* $\xi \in L^2(\mathbb{P})$.

For each $N \in \mathbb{N}$, consider the *random elementary measure* $\nu_N \equiv \frac{\|\nu\|}{N}\sum_{i=1}^N \delta_{x_i}$, where the random points $\{x_i\}_{i=1}^N \subseteq \mathfrak{X}$ are independent and $\mathbb{P}(\{x_i \in E\}) = \nu(E)/\|\nu\|$ for every Borel set $E \subseteq \mathfrak{X}$. This immediately implies that

$$\mathbb{E} f(x_i) = \frac{1}{\|\nu\|}\int_X f(z)\,d\nu(z)$$

for $f = 1_E$ by definition, for simple functions $f$ by linearity, and finally for all $f \in L^1(\nu)$ by approximation. From this, we deduce that for every $x \in \mathfrak{X}$ and $r > R$,

$$\mathbb{E}[(T_r\delta_{x_i})(x)] = \frac{1}{\|\nu\|} T_r\nu(x). \tag{5.6}$$

Indeed,

$$\begin{aligned}\|\nu\| \cdot \mathbb{E}[(T_r\delta_{x_i})(x)] &= \int_X (T_r\delta_z)(x)\,d\nu(z) = \int_X \int_{d(y,z)>r} K(x,y)\,d\delta_z(y)\,d\nu(z)\\ &= \int_X 1_{d(x,z)>r} K(x,z)\,d\nu(z) = T_r\nu(x).\end{aligned}$$

Thus, (5.6) holds.

Fix some $x_0 \in \mathcal{X}$ and $M \in (R, \infty)$. On the other hand, from (1.4) and (1.3), we deduce that for any $x \in \overline{B}(x_0, M)$,

$$\lambda(x_0, M) \lesssim \lambda(x, M) \lesssim C_\lambda^{1+\log_2(M/R)} \lambda(x, R),$$

where $C_\lambda$ is as in (1.3). By this, the fact that $r > R$, (5.6), and (1.5), we have that for any $x \in \overline{B}(x_0, M)$,

$$\begin{aligned} \text{(5.7)} \quad \mathbb{V}[T_r\delta_{x_i}(x)] \le \mathbb{E}\big[|T_r\delta_{x_i}(x)|^2\big] &= \int_\Omega \Big[\int_{\mathcal{X}} K(x,y)\, d\delta_{x_i}(y)\Big]^2 d\mathbb{P} \\ &= \int_\Omega [K(x,x_i)]^2 \chi_{\mathcal{X}\setminus \overline{B}(x,r)}(x_i)\, d\mathbb{P} \lesssim \frac{1}{[\lambda(x,r)]^2} \lesssim \frac{C_\lambda^{2[1+\log_2(M/R)]}}{[\lambda(x_0,M)]^2}. \end{aligned}$$

Moreover, by (5.6), we see that

$$\text{(5.8)} \qquad \mathbb{E}[(T_r\nu_N)(x)] = \sum_{i=1}^N \frac{\|\nu\|}{N} \mathbb{E}[(T_r\delta_{x_i})(x)] = T_r\nu(x).$$

This, together with the Cauchy inequality and (5.7), implies that there exists a positive constant $c$, independent of $x_0$, $M$, $r$, $R$, and $N$, such that

$$\begin{aligned} \mathbb{V}[T_r\nu_N(x)] = \frac{\|\nu\|^2}{N^2} \mathbb{V}\Big[\sum_{i=1}^N T_r\delta_{x_i}(x)\Big] &\le \frac{\|\nu\|^2}{N} \sum_{i=1}^N \mathbb{V}\big[T_r\delta_{x_i}(x)\big] \\ &\le c\frac{\|\nu\|^2}{N} \frac{C_\lambda^{2[1+\log_2(M/R)]}}{[\lambda(x_0,M)]^2}. \end{aligned}$$

Fix a number $\gamma \in (0, \infty)$ small enough. From the fact above, the Chebyshev inequality, and (5.8), we deduce that for every point $x \in \overline{B}(x_0, M)$ such that $|T_r\nu(x)| > t$,

$$\begin{aligned} \mathbb{P}(\{|T_r\nu_N(x)| \le (1-\gamma)t\}) &\le \mathbb{P}(\{|T_r\nu_N(x) - T_r\nu(x)| > \gamma t\}) \\ &\le \frac{\mathbb{V}(T_r\nu_N)(x)}{\gamma^2 t^2} \le c\frac{1}{\gamma^2 t^2} \frac{\|\nu\|^2}{N} \frac{C_\lambda^{2[1+\log_2(M/R)]}}{[\lambda(x_0,M)]^2} \le \gamma, \end{aligned}$$

provided $N \ge c\frac{\|\nu\|^2}{\gamma^3 t^2} \frac{C_\lambda^{2[1+\log_2(M/R)]}}{[\lambda(x_0,M)]^2}$. Since $r > R$ is arbitrary, we infer that for each $x \in \mathcal{X}$ satisfying $T_R^\sharp \nu(x) > t$,

$$\mathbb{P}\Big(\big\{ T_R^\sharp \nu_N(x) \le (1-\gamma)t \big\}\Big) \le \gamma.$$

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Let $E$ be any given Borel set with $\mu(E) < \infty$ such that $T_R^\sharp \nu(x) > t$ for every $x \in E$. Then

$$\mathbb{E}\Big(\mu\big(\{x \in E : T_R^\sharp \nu_N(x) \le (1-\gamma)t\}\big)\Big) = \int_E \mathbb{P}\Big(\big\{T_R^\sharp \nu_N(x) \le (1-\gamma)t\big\}\Big)\,d\mu(x)$$
$$\le \gamma\mu(E).$$

Thus, there exists at least one choice of points $\{x_i\}_{i=1}^N$ such that

$$\mu(\{x \in E : T_R^\sharp \nu_N(x) \le (1-\gamma)t\}) \le \gamma\mu(E),$$

and therefore, $\mu(\{x \in E : T_R^\sharp \nu_N(x) > (1-\gamma)t\}) \ge (1-\gamma)\mu(E)$. From this, together with Proposition 5.2, it follows that

$$\mu(E) \le \frac{1}{1-\gamma}\mu\Big(\big\{x \in E : T_R^\sharp \nu_N(x) > (1-\gamma)t\big\}\Big)$$
$$\le \frac{1}{(1-\gamma)^2 t}\|T_R^\sharp \nu_N\|_{L^{1,\infty}(\mu)} \lesssim \frac{1}{(1-\gamma)^2 t}\|\nu_N\| \lesssim \frac{1}{(1-\gamma)^2 t}\|\nu\|.$$

Since $\gamma > 0$ is arbitrary, we obtain that $\mu(E) \lesssim \frac{\|\nu\|}{t}$. As $E$ is an arbitrary subset of finite measure of the set of the points $x \in \mathcal{X}$ for which $T_R^\sharp \nu(x) > t$, we obtain (5.5), which completes the proof of Corollary 1.7. ∎

***Remark 5.3*** If we replace the assumption of Corollary 1.7 that $T$ is bounded on $L^2(\mu)$ by the assumption that $T$ is bounded on $L^q(\mu)$ for some $q \in (1,\infty)$, then Corollary 1.7 still holds.

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