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About A. N. Kolmogorov's work on the entropy of dynamical systems

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In the fall of 1957 A. N. Kolmogorov started lecturing on the theory of dynamical systems and supervised a seminar on the same theme at the Mechanical–Mathematical Department of Moscow State University. He began his lectures with the theory of systems with a pure point spectrum, which he approached from a probabilistic point of view. This approach, undoubtedly, has many advantages. In the seminar we studied Ito's theory of multiple stochastic integrals and, under the supervision of A. N. Kolmogorov, I. V. Girsanov constructed an example of a Gaussian dynamical system with a simple continuous spectrum. At one of the meetings of the seminar, still before the advent of entropy, Kolmogorov suggested a proof of an assertion, which today would read: the unitary operator induced by a K-automorphism has a countable Lebesgue spectrum. At this point Kolmogorov was studying Shannon's theory of information and the concept of the capacity of functional spaces. Judging by his well known article, we can say that the first of these, and all that is related to it, played a big role in the development of information theory in our country. The investigation of capacity is connected with Kolmogorov's work on Hilbert's 13th problem and was summarized in his well-known survey co-authored by V. M. Tihomirov.

The concept of entropy appeared unexpectedly. At a lecture, which incidentally I missed due to illness, he explained how to prove the fact that two Bernoulli generators must have the same entropy. At one point his proof used the fact that the entropy of a product of Bernoulli partitions is equal to the sum of the entropy of the factors.

In the winter of 1958, Kolmogorov spent half a year in France, leaving us the text of his article published later in Doklady Akademia Nauk. Besides the general properties of the entropy of partitions, which somehow or other had appeared earlier, the concept of K-systems and the definition of entropy of K-systems appeared in this article. Today it seems to me that the reason for such an isolation of K-systems can be explained by the fact that at that time there was a clear perception of the existence of some sort of boundary between dynamical systems arising in probability theory from those arising from differential equations, these lacked only a metric invariant which would be able to differentiate between them. The advent of entropy gave us such an invariant.

On the other hand, I was taken by the idea that all finite partitions give rise t stationary processes. As a probabilist, I was trying to understand which processe could arise from differential dynamical systems. Thinking about these questions le to the definition of entropy for an arbitrary automorphism of the torus.

At that time our close contacts with V. A. Rokhlin began. He was then living i Kolomna, near Moscow. Rokhlin read Kolmogorov's article and invited me t discuss it with him. I told him about my definition of entropy, and he offered t compute the entropy of a group automorphism of the two-dimensional torus – which at that time I had not yet heard.

I cannot say precisely when the theorem about the entropy of generating partitior appeared. The proof followed the same path as the proof Kolmogorov had give in his lectures. In the place where Komogorov wrote an equality because of Bernoull ness, one can always write an inequality. Now this seems a triviality, but at th time it took intense determination. Frequent conversations with M. C. Pinsker we also very helpful.

After the proof of the theorem about generators, the path to computing the entrop of automorphisms of the two-dimensional torus was clear. Since it was thought th only systems arising in probability theory had positive entropy, I was trying to prov that the system was of zero entropy. When I visited Kolmogorov in Komarovk ('dacha') I showed him my pictures, and he immediately said that the entropy mu be positive. After that I quickly arrived at the final answer.

We used the ideas from Kolmogorov's work many more times. In the last sectio there is an example of a flow with positive, but finite entropy, built in purel probabilistic terms. Later on, when geodesic flows on surfaces of negative curvatu were studied, it quickly became obvious that the flows, and Kolmogorov's exampl had similar characteristics.

In the history of mathematics it is difficult to find an example of an article whic not only decides a classical problem, but at the same time opens a totally new are of research.

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