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REMARKS ON CERTAIN METAPLECTIC GROUPS

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ABSTRACT. We study metaplectic coverings of the adelized group of a split connected reductive group G over a number field F. Assume its derived group G' is a simply connected simple Chevalley group. The purpose is to provide some naturally defined sections for the coverings with good properties which might be helpful when we carry some explicit calculations in the theory of automorphic forms on metaplectic groups. Specifically, we

1. construct metaplectic coverings of $G(\mathbb{A})$ from those of $G'(\mathbb{A})$;
2. for any non-archimedean place v, show the section for a covering of $G(F_v)$ constructed from a Steinberg section is an isomorphism, both algebraically and topologically in an open subgroup of $G(F_v)$;
3. define a global section which is a product of local sections on a maximal torus, a unipotent subgroup and a set of representatives for the Weyl group.

1. **Coverings of simple groups over a local fields.** Suppose F is a field of characteristic 0 and G is a connected, simply connected, simple Chevalley group. We also use G to denote the group of the rational points. Fix a maximal split torus H in G, together with the root system Σ relative to H and a set of simple roots Σ_0. Each root λ determines a homomorphism $n_\lambda\colon F \longrightarrow G$ whose image is a unipotent subgroup. Then G is generated by the collection of symbols $\{n_\lambda(x) : \lambda \in \Sigma, x \in F\}$ subject to the following conditions [7]:

(A) $n_\lambda(x)$ is additive in x.

(B) If λ and δ are roots and $\lambda + \delta \neq 0$, then

$$\bigl[n_\lambda(x), n_\delta(y)\bigr] = \prod n_{i\lambda+j\delta}(c_{ij}x^i y^j) \tag{1}$$

where the c_{ij} are certain integers. The product is over all positive integers i and j such that $i\lambda + j\delta$ is a root arranged in any fixed order.

(B′) Denote $w_\lambda(x) = n_\lambda(x)n_{-\lambda}(-x^{-1})n_\lambda(x)$ for $x \in F^\times$ then

$$w_\lambda(x)n_\lambda(y)w_\lambda(-x) = n_{-\lambda}(-x^{-2}y). \tag{2}$$

(C) Denote $h_\lambda(x) = w_\lambda(x)w_\lambda(-1)$ for $x \in F^\times$, then $h_\lambda(x)$ is multiplicative in x.

Suppose μ is an abelian group, we consider the following central extension

$$1 \longrightarrow \mu \longrightarrow \bar{G} \xrightarrow{\mathfrak{p}} G \longrightarrow 1.$$

By [7], $\bar{G}$ is generated by symbols $\{\bar{n}_\lambda(x) : \lambda \in \Sigma, x \in F\}$ subject to the conditions (A), (B) (or (B′) if the rank of G is equal to 1) and a set of relations

$$\prod_i \Bigl(\bar{h}_{\lambda_i}(x_i)\bar{h}_{\lambda_i}(y_i)\bar{h}_{\lambda_i}(x_i y_i)^{-1}\Bigr) = 1 \tag{3}$$

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where the product is over a finite set $\{\lambda_i\}$ of roots in some order and

$$\bar{w}_\lambda(x) = \bar{n}_\lambda(x)\bar{n}_{-\lambda}(-x^{-1})\bar{n}_\lambda(x), \qquad \bar{h}_\lambda(x) = \bar{w}_\lambda(x)\bar{w}_\lambda(-1).$$

We choose a section $\mathfrak{s}\colon G \longrightarrow \bar{G}$ as follows. Denote by N (resp. N') the subgroup of G (resp. $\bar{G}$) generated by $\{n_\lambda(x) : \lambda > 0, x \in F\}$ (resp. $\{\bar{n}_\lambda(x) : \lambda > 0, x \in F\}$). Since the restriction of the projection $\mathfrak{p}$ on N' is an isomorphism onto N, we define $\mathfrak{s}$ on N by

$$\mathfrak{s}|_N = (\mathfrak{p}|_{N'})^{-1}. \tag{4}$$

Fix an ordering in Σ_0. Then every element of H can be uniquely written as $\prod h_\lambda(x_\lambda)$ ($\lambda \in \Sigma_0$ and $x_\lambda \in F^\times$) in the fixed ordering. Define

$$\mathfrak{s}\Big(\prod h_\lambda(x_\lambda)\Big) = \prod \bar{h}_\lambda(x_\lambda). \tag{5}$$

Denote by $\bar{H}$ the subgroup of $\bar{G}$ generated by $\{\bar{h}_\lambda(x) : \lambda \in \Sigma, x \in F^\times\}$. Denote by $\mathfrak{N}_{\bar{G}}(\bar{H})$ the the normalizer of $\bar{H}$ in $\bar{G}$. Then $\mathfrak{N}_{\bar{G}}(\bar{H})/\bar{H} \cong \mathfrak{N}_G(H)/H$ which we denote by W. Choose a system of representatives $\{w(\tau) : \tau \in W\}$ (resp. $\{\bar{w}(\tau) : \tau \in W\}$) in $\mathfrak{N}_G(H)/H$ (resp. $\mathfrak{N}_{\bar{G}}(\bar{H})$) for W as follows. If $\lambda \in \Sigma_0$, denote by τ_λ the simple reflection relative to λ. Any $\tau \in W$ can be written as a product $\prod_{i=1}^m \tau_{\lambda_i}$ where $\lambda_i \in \Sigma_0$ and m is the length $\ell(\tau)$ of τ. Define

$$w(\tau) = \prod_{i=1}^m w_{\lambda_i}(1) \quad \Big(\text{resp. } \bar{w}(\tau) = \prod_{i=1}^m \bar{w}_{\lambda_i}(1)\Big).$$

They are well-defined [3, p. 44]. Define

$$\mathfrak{s}\big(w(\tau)\big) = \bar{w}(\tau). \tag{6}$$

We then define the section on G according to the Bruhat decomposition

$$\mathfrak{s}(nhwn') = \mathfrak{s}(n)\mathfrak{s}(h)\mathfrak{s}(w)\mathfrak{s}(n') \quad \forall n, n' \in N, h \in H, w = w(\tau), \tau \in W. \tag{7}$$

Remark that $\mathfrak{s}$ does not depend on the particular Bruhat decomposition.

Denote by σ the 2-cocycle associated with the section $\mathfrak{s}$. For $\lambda \in \Sigma_0$, the subgroup G_λ generated by $\{n_\lambda(x), n_{-\lambda}(x) : x \in F\}$ is isomorphic to SL(2). The function

$$c_\lambda(x,y) = \sigma\big(h_\lambda(x), h_\lambda(y)\big) = \bar{h}_\lambda(x)\bar{h}_\lambda(y)\bar{h}_\lambda(xy)^{-1}$$

on $F^\times \times F^\times$ is a Steinberg cocycle [5, Section 8]. According to [5, p. 198], the value of σ on G_λ can be calculated explicitly. Every element in G_λ is uniquely of the form $g_1(u,t) = n_\lambda(u)h_\lambda(t)$, $u \in k$, $t \in k^\times$ or of the form $g_2(u,t,v) = n_\lambda(u)w_\lambda(t)n_\lambda(v)$, $u, v \in k$, $t \in k^\times$. We then have

$$\begin{aligned} \sigma\big(g_2(u,t,v), g_2(u',t',v')\big) &= \begin{cases} c_\lambda(-t,-t')^{-1} & \text{if } w = -(v+u') = 0 \\ c_\lambda(tw^{-1},w)^{-1}c_\lambda(tw^{-1},t') & \text{if } w \neq 0; \end{cases} \\ \sigma\big(g_2(u,t,v), g_1(u',t')\big) &= c_\lambda(t,t'^{-1}); \\ \sigma\big(g_1(u,t), g_2(u',t',v')\big) &= c_\lambda(t,t'); \\ \sigma\big(g_1(u,t), g_1(u',t')\big) &= c_\lambda(t,t'). \end{aligned} \tag{8}$$

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Recall that if furthermore λ is a long root, then c_λ uniquely determines the extension $\overline{G}$.

In the rest of section, we assume F is a p-adic field and the abelian group μ is given the discrete topology. The central extensions discussed in the sequel are topological. We first observe the following simple lemma.

LEMMA 1. *Suppose G is a locally compact totally disconnected group and $\overline{G}$ is a central extension of G by a discrete abelian group μ. Assume that for any open subgroup R, $[R, R]$ contains a neighborhood of 1. Then there is an open subgroup U of G together with a section $\mathfrak{s}_U : G \longrightarrow \overline{G}$, such that $\mathfrak{s}_U$ is an isomorphism on U, both algebraically and topologically. Furthermore, there is an open subgroup V of U such that $\mathfrak{s}_U|_V$ is uniquely determined by the above property.*

PROOF. The existence of such $\mathfrak{s}_U$ follows from [6, Theorem 2 e_4]. Choose an open subgroup $V \subset [U, U]$. The uniqueness of $\mathfrak{s}_U|_V$ follows from the fact that $\mathfrak{s}_U|_U$ is a homomorphism. ∎

Now we return to the simple group G over the p-adic field F. Denote by $\mathfrak{O}$ the ring of intergers in F and by ϖ a fixed generator of the prime ideal in $\mathfrak{O}$. Denote by U_i the subgroup of G generated by $\{n_\lambda(x) : x \in \varpi^i\mathfrak{O}, \lambda \in \Sigma\}$. Then U_i, $i > 0$ form a system of neighborhoods of 1 and the assumptions in Lemma 1 are satisfied. Denote $K = G(\mathfrak{O}) = U_0$.

LEMMA 2. *There is an open subgroup U of G such that $\mathfrak{s}$ is an isomorphism on U, both algebraically and topologically. Furthermore, $\mathfrak{s}(U)$ is normal in $\overline{K}$.*

PROOF. Choose a $U = U_k$ and $\mathfrak{s}_U$ as in Lemma 1. Define a section $\mathfrak{s}'$ as follows. First, $\mathfrak{s}' = \mathfrak{s}$ on N and H. Second, fix a $\tau \in W$ and hence $w = w(\tau)$. Denote by N_τ the subgroup of N generated by $\{n_\lambda(x) : \lambda > 0, \tau(\lambda) < 0, x \in F\}$. Since $U \cap NHw(\tau)N \neq \emptyset$ [8, p. 127, Theorem 23], define for $u = n_\tau hw(\tau)n \in U$, $n_\tau \in N_\tau$, $h \in H$, $n \in N$,

$$\mathfrak{s}'(u, w) = \mathfrak{s}'(h)^{-1}\mathfrak{s}'(n_\tau)^{-1}\mathfrak{s}_U(u)\mathfrak{s}'(n)^{-1}. \tag{9}$$

Observe that $\mathfrak{p}\big(\mathfrak{s}'(u, w)\big) = w$ and $\mathfrak{s}'(u, w)$ is continuous in u for $u \in U \cap NHw(\tau)N$ [5, p. 200] and hence must be constant. Define $\mathfrak{s}'(w) = \mathfrak{s}'(u, w)$, $u \in U$. By the Bruhat decomposition, we can define the section $\mathfrak{s}'$ on G by

$$\mathfrak{s}'\big(n_\tau hw(\tau)n\big) = \mathfrak{s}(n_\tau)\mathfrak{s}(h)\mathfrak{s}'\big(w(\tau)\big)\mathfrak{s}(n), \quad \forall \tau \in W, n_\tau \in N_\tau, h \in H, n \in N.$$

It is easy to see that $\mathfrak{s}' = \mathfrak{s}_U$ on U, so $\mathfrak{s}'$ satisfies the first property in this lemma.

We show $\mathfrak{s}' = \mathfrak{s}$. It is enough to show $\mathfrak{s}'\big(w(\tau)\big) = \mathfrak{s}\big(w(\tau)\big)$, for any $\tau \in W$. If τ is a reflection relative to a simple root λ, a straightforward calculation by (8) shows that $\mathfrak{s}$ is an isomorphism in a neighborhood hence must equal $\mathfrak{s}'$ there by the uniqueness in Lemma 1. It then follows $\mathfrak{s}'\big(w_\lambda(1)\big) = \mathfrak{s}\big(w_\lambda(1)\big)$. Now we show $\mathfrak{s}'\big(w(\tau)\big) = \mathfrak{s}\big(w(\tau)\big)$ by induction on $\ell(\tau)$. Suppose this is true for $\ell(\tau) = k - 1$. If $w(\tau) = h^{-1}n_1un_2$, for $h \in H$, $n_1, n_2 \in N$, $u \in U$, then $\mathfrak{s}\big(w(\tau)\big) = \mathfrak{s}(h)^{-1}\overline{n}_1\mathfrak{s}_U(u)\overline{n}_2$, $\overline{n}_1, \overline{n}_2 \in N'$. We consider $\tau \cdot \tau_{\lambda_k}$ and

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assume $\ell(\tau\tau_{\lambda_k}) = k$. Suppose n is the order of the group of all roots of unity in F. Choose an $x \in -F^{\times n}$ close to 0. Observe that for any $h \in H$, $\lambda \in \Sigma$, $\overline{h}_\lambda(-x)\mathfrak{s}(h) = \mathfrak{s}\big(h_\lambda(-x)h\big)$. By applying the relations in [3, Lemme 5.2.] we get

$$\begin{aligned}\mathfrak{s}\big(w(\tau)\big)\overline{w}_{\lambda_k}(1) &= \mathfrak{s}\big(w(\tau)\big)\overline{h}_{\lambda_k}(-x^{-1})^{-1}\overline{n}_{\lambda_k}(-x^{-1})\overline{n}_{-\lambda_k}(x)\overline{n}_{\lambda_k}(-x^{-1})\\ &= \overline{h}_{\tau(\lambda_k)}(-x^{-1})^{-1}\overline{n}_3\mathfrak{s}\big(w(\tau)\big)\overline{n}_{-\lambda_k}(x)\overline{n}_{\lambda_k}(-x^{-1})\\ &= \overline{h}_{\tau(\lambda_k)}(-x^{-1})^{-1}\overline{n}_3\big(\mathfrak{s}(h)^{-1}\overline{n}_1\mathfrak{s}_U(u)\overline{n}_2\big)\overline{n}_{-\lambda_k}(x)\overline{n}_{\lambda_k}(-x^{-1})\\ &= \mathfrak{s}\big(hh_{\tau(\lambda_k)}(-x^{-1})\big)^{-1}\overline{n}_4\mathfrak{s}_U(u)\overline{n}_{-\lambda_k}(x)\overline{n}_5.\end{aligned}$$

In the above expressions, $\overline{n}_3$, $\overline{n}_4$ and $\overline{n}_5$ are certain elements in N'. Since $\mathfrak{s}_U(u)\overline{n}_{-\lambda_k}(x) \in \mathfrak{s}_U(U)$, we conclude $\mathfrak{s}\big(w(\tau)w_{\lambda_k}(1)\big) = \mathfrak{s}'\big(w(\tau)w_{\lambda_k}(1)\big)$. This complete the proof of the first assertion.

For the second assertion, observe that for any $k \in K$, the section $\mathfrak{s}(k)\mathfrak{s}(k^{-1}uk)\mathfrak{s}(k)^{-1}$ is also a homomorphism on U, hence must equal $\mathfrak{s}$ by the uniqueness. ∎

2**. Coverings of certain reductive groups over local field.** Suppose F is a field of characteristic 0 and μ is an abelian group. Suppose G is a connected reductive group split over F whose derived group $G' = [G, G]$ is a simply connected simple Chevalley group. Here we denote $[g, h] = ghg^{-1}h^{-1}$, $g, h \in G$ and denote by $[G, G]$ the subgroup of G generated by $\{[g, h] : g, h \in G\}$. We also use G to denote the group of rational points of G. Fix maximal split tori H and S of G' and G respectively such that $S = H \times T$ for a subtorus T of S. It is not hard to show that

LEMMA 3. *The group G is isomorphic to the semi-direct product $G' \rtimes T$ as algebraic groups, with G' normal in G.* ∎

To discuss coverings of G, we first make the following definition. We call the data $(\overline{G'}, \overline{T}, \alpha)$ an admissible couple of extensions if

1. $\overline{G'}$ is a central extension of G' by μ with the projection $\mathfrak{p}_0\colon \overline{G'} \longrightarrow G'$;
2. $\overline{T}$ is a central extension of T by μ;
3. α is an action of T on $\overline{G'}$ such that

(10) $\quad \alpha(t)(\xi) = \xi, \quad \forall t \in T, \xi \in \mu; \quad \mathfrak{p}_0\big(\alpha(t)(g)\big) = t\mathfrak{p}_0(g)t^{-1}, \quad \forall t \in T, g \in \overline{G'}.$

The above action α gives rise to an action of $\overline{T}$ on $\overline{G'}$ by the requirement that μ acts on $\overline{G'}$ trivially.

Given an admissible couple of extensions $(\overline{G'}, \overline{T}, \alpha)$, we can construct a central extension $\overline{G}$ of G by μ as follows. Define the semi-direct product $\overline{G'} \rtimes \overline{T}$ relative to the action α. The multiplication is given by $(g_1, t_1)(g_2, t_2) = \big(g_1\alpha(t_1)(g_2), t_1t_2\big)$, for $(g_i, t_i) \in \overline{G'} \rtimes \overline{T}$, $i = 1, 2$. The group μ is a central subgroup of $\overline{G'} \rtimes \overline{T}$ via the map $\xi \longmapsto (\xi, \xi^{-1})$, $\forall \xi \in \mu$. It is then immediate that the quotient group $(\overline{G'} \rtimes \overline{T})/\mu$ is a central extension of $T \rtimes G' = G$.

Conversely, suppose $\overline{G}$ is a central extension of G by μ. Restrictions give central extensions $\overline{G'}$ and $\overline{T}$ of G' and T by μ respectively. Define an action α of T on $\overline{G'}$ by $\alpha(t)(g) = \mathfrak{s}_T(t)g\mathfrak{s}_T(t)^{-1}$, where $t \in T$, $g \in G'$ and $\mathfrak{s}_T$ is a section $T \longrightarrow \overline{T}$. Then $(\overline{T}, \overline{G'}, \alpha)$ is an admissible couple of extensions.

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Two admissible couples of extensions $(\overline{G'}, \overline{T}, \alpha)$ and $(\tilde{G}', \tilde{T}, \beta)$ are said to be equivalent if there are equivalences of group extensions $\phi_0: \overline{G'} \longrightarrow \tilde{G}'$ and $\phi_T: \overline{T} \longrightarrow \tilde{T}$ such that $\phi_0\big(\alpha(t)(g)\big) = \beta(t)\big(\phi_0(g)\big)$. It is routine to show the following lemma.

LEMMA 4. *There is a one to one correspondence between the set of equivalence classes of central extensions $\overline{G}$ of G by μ and the set of equivalence classes of admissible couples $(\overline{G'}, \overline{T}, \alpha)$ given by the following two inverse maps:*

$$\Phi: \overline{G} \longmapsto (\overline{G'}, \overline{T}, \text{ conjugation in } \overline{G});$$
$$\Phi^{-1}: (\overline{G'}, \overline{T}, \alpha) \longmapsto (\overline{G'} \rtimes \overline{T})/\mu. \qquad \blacksquare$$

Now fix a central extension $\overline{G}$ of G by μ. Suppose $(\overline{G'}, \overline{T}, \alpha)$ is the associated admissible couple of extensions as in the above lemma. Define the section $\mathfrak{s}_0$ for the extension $\overline{G'}$ by (4), (5), (6), (7). Fix a section $\mathfrak{s}_T$ for the extension $\overline{T}$. Define a section $\mathfrak{s}$ on G by

$$\mathfrak{s}(gt) = \mathfrak{s}_0(g)\mathfrak{s}_T(t), \quad g \in G', t \in T. \tag{11}$$

Denote by σ the two cocycle associated with $\mathfrak{s}$. We have for any $g \in G'$, $t \in T$

$$\alpha(t)(g) = \mathfrak{s}(tgt^{-1})\sigma(t, g). \tag{12}$$

Then α being a group action implies that for any $g, g_1, g_2 \in G'$, $t, t_1, t_2 \in T$,

$$\sigma(t_1t_2, g) = \sigma(t_1, t_2gt_2^{-1})\sigma(t_2, g); \tag{13}$$
$$\sigma(t, g_1g_2)\sigma(g_1, g_2) = \sigma(t, g_1)\sigma(t, g_2)\sigma(tg_1t^{-1}, tg_2t^{-1}); \tag{14}$$
$$\sigma(g_1t_1, g_2t_2) = \sigma(g_1, t_1g_2t_1^{-1})\sigma(t_1, t_2)\sigma(t_1, g_2). \tag{15}$$

Denote by ϵ_λ the character on T such that $\alpha(t)\big(n_\lambda(x)\big) = n_\lambda\big(\epsilon_\lambda(t)x\big)$.

LEMMA 5. *Suppose μ is finite. Fix a positive root λ. For any $t \in T$, $x \in F^\times$,*

$$\sigma\big(t, n_\lambda(x)\big) = 1; \tag{16}$$
$$\sigma\big(t, w_\lambda(x)\big) = 1; \tag{17}$$
$$\sigma\big(t, h_\lambda(x)\big) = c_\lambda\big(x, \epsilon_\lambda(t)\big)^{-1}. \tag{18}$$

PROOF. By our construction of $\mathfrak{s}_0$, $\sigma(g, n) = \sigma(n, g) = 1$ for any $n \in N$ and $g \in G'$. A straightforward calculation by (8) shows that

$$\sigma\big(tw_\lambda(x)t^{-1}, tw_\lambda(-1)t^{-1}\big)\sigma\big(w_\lambda(x), w_\lambda(-1)\big)^{-1} = c_\lambda\big(x, \epsilon_\lambda(t)\big)^{-1}.$$

Applying (14), we get for any simple root λ, any $x, y \in F^\times$,

$$\sigma\big(t, n_\lambda(x+y)\big) = \sigma\big(t, n_\lambda(x)\big)\sigma\big(t, n_\lambda(y)\big) \tag{19}$$
$$\sigma\big(t, n_{-\lambda}(x+y)\big) = \sigma\big(t, n_{-\lambda}(x)\big)\sigma\big(t, n_{-\lambda}(y)\big) \tag{20}$$
$$\sigma\big(t, w_\lambda(x)\big) = \sigma\big(t, n_\lambda(2x)\big)\sigma\big(t, n_{-\lambda}(-x^{-1})\big) \tag{21}$$
$$\sigma\big(t, h_\lambda(x)\big) = \sigma\big(t, n_\lambda(2x-2)\big)\sigma\big(t, n_{-\lambda}(-x^{-1}+1)\big)c_\lambda\big(x, \epsilon_\lambda(t)\big)^{-1}. \tag{22}$$

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The identity (19) implies that $\sigma\big(t, n_\lambda(x)\big) = \sigma\big(t, n_\lambda(x/n)\big)^n = 1$ which is (16). Similarly, (20) implies $\sigma\big(t, n_{-\lambda}(x)\big) = 1$. Identities (21) and (22) give (17) and (18) respectively. ∎

It is immediate by the above lemma that

$$\sigma(g, n) = \sigma(n, g) = 1, \quad \forall n \in N, g \in G.$$

PROPOSITION 1. *Denote by $L(G', \mu)$ the subgroup of $H^2(G', \mu)$ consisting of elements corresponding to bilinear Steinberg cocycles. There is an injective homomorphism*

$$\Phi\colon H^2(G, \mu) \hookrightarrow H^2(G', \mu) \times H^2(T, \mu) \tag{23}$$

whose image contains $L(G', \mu) \times H^2(T, \mu)$.

PROOF. Lemma 4 gives the map Φ in the proposition. Once we fix central extensions $\overline{T}$ and $\overline{G'}$, the action of T on G' is uniquely determined by (12), (14) and Lemma 5. So Φ is injective.

We now show the image of Φ contains $L(G, \mu) \times H^2(T, \mu)$. This is equivalent to constructing a group action α of T on $\overline{G'}$ satisfying (10) if we are given a covering $\overline{G'}$ whose corresponding Steinberg cocycle is bilinear.

Define the action α of T on $N'_\lambda = \{\overline{n}_\lambda(x) : \lambda \in \Sigma, x \in F\}$ in the obvious way: $\alpha(t)\big(\overline{n}_\lambda(x)\big) = \overline{n}_\lambda\big(\epsilon_\lambda(t)x\big)$. Since N_λ, $\lambda \in \Sigma$, generate G', we can extend α to $T \times G'$ if the relations (A), (B), (B′), (C) and (3) still hold after we replace each $\overline{n}_\lambda(x)$, by $\overline{n}_\lambda\big(\epsilon_\lambda(t)x\big)$, $\lambda \in \Sigma$, $x \in F$. We then need to check (10). All the above can be seen by straightforward calculations. ∎

We remark that Lemma 5 and hence Proposition 1 are still true even if μ is infinite. However, we do not need it in the following discussion.

In the rest of this section we suppose F is a completion of a number field containing the group μ_r of all n-th roots of unity. Then G becomes a topological group. It is not hard to see that in Lemma 4, Φ and Φ^{-1} send topological extensions to topological extensions, so Proposition 1 is true for topological extensions. By [5, Theorem 3.1], a topological Steinberg cocycle on F valued in a group of roots of unity in F is always bilinear. So we have

COROLLARY 1. *Suppose F is a p-adic field containing the group μ_n of n-th roots of unity. There is one and only one central extension $\overline{G}$ of G by μ_n such that its restrictions to G' and T are the given central extensions $\overline{G'}$ and $\overline{T}$ respectively, both by μ_n.*

Denote by K the subgroup of G generated by $G'(\mathfrak{O})$ and $T(\mathfrak{O})$. Then K is a maximal compact subgroup of G. Suppose $\overline{G}$ is a topological covering of G.

COROLLARY 2. *There is a section $\mathfrak{s}_T\colon T \longrightarrow \overline{T}$ such that the section $\mathfrak{s}$ defined by (11) is an isomorphism on an open subgroup U, both algebraically and topologically. Furthermore, $\mathfrak{s}(U)$ is normal in $\overline{K}$.*

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PROOF. By Lemma 1, there is an open subgroup U of $\overline{G}$ and a section $\mathfrak{s}_U\colon U \longrightarrow \overline{U}$ which is also a homomorphism. Extend $\mathfrak{s}_U|_{T\cap U}$ to a section $\mathfrak{s}_T$ on T. It is then not hard to see that $\mathfrak{s}_T$ satisfies all the properties in the corollary. ∎

It follows from the above corollary that for any open normal subgroup V of K contained in U, $\mathfrak{s}(V)$ is normal in $\overline{K}$.

3. **Global metaplectic groups.** Suppose μ is a finite abelian group and F is a number field. Denote by $\mathbb{A}$ the ring of adeles over F. Suppose G is a connected reductive group split over F whose derived group G' is a connected, simply connected simple Chevalley group. As in Section 2, denote by $S = H \times T$ the maximal split torus in G and write $G = G' \rtimes T$. We consider metaplectic coverings $\overline{G}(\mathbb{A})$ of $G(\mathbb{A})$ by μ. By definition, they are central extensions of $G(\mathbb{A})$ by μ which split over $G(F)$.

If L is a subgroup of $G(\mathbb{A})$ we denote by $\overline{L}$ the inverse image under the natural projection $\overline{G}(\mathbb{A}) \longrightarrow G(\mathbb{A})$. By [4, I.1.2. Remark], the following properties on the covering $\overline{G(\mathbb{A})}$ are satisfied:

1. For any non-archimedean place v, there is an open subgroup U_v where the covering $\overline{G(F_v)}$ splits.
2. For almost all places v, the above U_v can be chosen to be K_v.

We then can choose a section $\mathfrak{s}_T\colon T(\mathbb{A}) \longrightarrow \overline{T(\mathbb{A})}$ with the following property. Denote $\mathfrak{s}_{T,v} = \mathfrak{s}_T|_{T(F_v)}$. The following two conditions must be satisfied:

1. For any non-archimedean place v, there is an open subgroup $U_{T,v}$ of $T(F_v)$ on which $\mathfrak{s}_{T,v}$ is an isomorphism both algebraically and topologically;
2. For almost all places v, the above $U_{T,v}$ is $T(\mathfrak{O}_v)$;
3. The map $\mathfrak{s}_T|_{T(F)}$ is a homomorphism.

The existence of such $\mathfrak{s}$ is obvious.

Denote by V a finite set of places containing all archimedean ones such that for any $v \notin V$, the residue class field of F_v has at least four elements and v is relatively prime to the order of the group of all roots of unity in F. By [5, Lemma 11.3], for any $v \notin V$, $\overline{G}$ splits over $G'(\mathfrak{O}_v)$. Since $[G'(\mathfrak{O}_v), G'(\mathfrak{O}_v)] = G'(\mathfrak{O}_v)$, there is a unique homomorphism $\kappa_v\colon G'(\mathfrak{O}_v) \longrightarrow \overline{G'(\mathfrak{O}_v)}$. Denote $\mathfrak{W} = \{w(\tau) : \tau \in W\}$ which is a set of representatives of the Weyl group W.

LEMMA 6. *For $v \notin V$, the homomorphism κ_v defined above equals $\mathfrak{s}_v$ on $H(\mathfrak{O}_v)$, $N(F_v)$ and $\mathfrak{W}$.*

PROOF. We need to check $\kappa_v = \mathfrak{s}_v$ on $H_\lambda(\mathfrak{O}_v)$, $N_\lambda(F_v)$ and $w_\lambda(1)$ for any simple root λ. Then everything boils down to the case of SL(2). The calculations are straightforward by the Kubota's construction of κ_v (refer to [2, Theorem 2]). ∎

We extend the κ_v to a section, which we still denote by κ_v, on $G(F_v)$ by the conditions $\kappa_v(gt) = \kappa_v(g)\mathfrak{s}_v(t)$, for any $g \in G'(\mathfrak{O}_v)$ and $t \in T(\mathfrak{O}_v)$; $\kappa_v = \mathfrak{s}_v$ on $G(F_v) - G(\mathfrak{O}_v)$.

COROLLARY 3. *For $v \notin V$, the section κ_v defined above is a Borel section and equals $\mathfrak{s}_v$ on $S(\mathfrak{O}_v)$, $N(F_v)$, $\mathfrak{W}$ and $G(F_v) - G(\mathfrak{O}_v)$.* ∎

Denote $K_v^* = \kappa_v(K_v)$. Denote by $\prod_v \overline{G(F_v)}[K_v^*]$ the restrictive direct product of $\overline{G(F_v)}$ relative to K_v^* ($v \notin V$). It is easy to see the following two groups are isomorphic:

$$\overline{G(\mathbb{A})} \cong \prod_v \overline{G(F_v)}[K_v^*]/M$$

where M is the subgroup generated by the elements of the form $(\ldots, \xi, \ldots, \xi^{-1}, \ldots)$, $\xi \in \mu$, with ξ and ξ^{-1} at the v-th place and the w-th place respectively. For simplicity, we identify the two groups.

We define the global section $\mathfrak{s}$ for the covering by $\mathfrak{s}(g) = \prod_{v \in V} \mathfrak{s}_v(g_v) \cdot \prod_{v \notin V} \kappa_v(g_v)$ for $g = \prod_v g_v \in G(\mathbb{A})$. Remark that if F is a totally imaginary then $\mathfrak{s}$ is an isomorphism on an open subgroup of $G(\mathbb{A})$.

Denote by σ the two cocycle associated with the section $\mathfrak{s}$. By Corollary 3, we get

COROLLARY 4. *The global cocycle σ equals $\prod_v \sigma_v$ on $S(\mathbb{A}) \times S(\mathbb{A})$ where each σ_v is the local two cocycle associated with the section $\mathfrak{s}_v$ for the local covering $\overline{G(F_v)}$.* ∎

Fix a long root λ and call the function $\sigma\big(h_\lambda(\cdot), h_\lambda(\cdot)\big)$ on $\mathbb{A}^\times \times \mathbb{A}^\times$ the Steinberg cocycle for $\overline{G(\mathbb{A})}$. For each natural number n such that F contains the group μ_n of all n-th roots of unity, we denote by $\overline{{}^nG'(\mathbb{A})}$ the covering of $G'(\mathbb{A})$ given by the Steinberg cocycle $(\cdot,\cdot) = \prod_v (\cdot,\cdot)_v$, where $(\cdot,\cdot)_v$ is the n-th Hilbert symbol. The following lemma is a variation of [3, Théorème 11.3].

LEMMA 7. *For any metaplectic covering of $\overline{G'(\mathbb{A})}$ by a finite abelian group μ, there are a natural number n and injective homomorphisms $\phi_\mu \colon \mu_n \longrightarrow \mu$ and $\phi \colon \overline{{}^nG'(\mathbb{A})} \longrightarrow \overline{G'(\mathbb{A})}$ such that the following diagram commutes:*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mu_n & \longrightarrow & \overline{{}^nG'(\mathbb{A})} & \longrightarrow & G'(\mathbb{A}) & \longrightarrow & 1 \\
 & & \Big\downarrow{\scriptstyle \phi_\mu} & & \Big\downarrow{\scriptstyle \phi} & & \Big\| & & \\
1 & \longrightarrow & \mu & \longrightarrow & \overline{G'(\mathbb{A})} & \longrightarrow & G'(\mathbb{A}) & \longrightarrow & 1.
\end{array}$$

The number n is uniquely determined. ∎

Applying the above lemma to our reductive group G, we get

COROLLARY 5. *For any metaplectic covering $\overline{G(\mathbb{A})}$ of $G(\mathbb{A})$ by a finite abelian group μ, there is a uniquely determined positive integer n such that the restriction of $\overline{G(\mathbb{A})}$ to $G'(\mathbb{A})$ is a trivial covering of $\overline{{}^nG'(\mathbb{A})}$.* ∎

We summarize the results in this note in the following theorem.

THEOREM 1. *Suppose μ is a finite abelian group and G is a connected reductive group split over a number field F. Assume its derived group G' is simply connected simple Chevalley group. Then any metaplectic covering $\overline{G(\mathbb{A})}$ of $G(\mathbb{A})$ by μ is given by (up to isomorphisms) a positive integer n and a metaplectic covering $\overline{T(\mathbb{A})}$ of $T(\mathbb{A})$ by μ such that the Steinberg symbol for $\overline{G(\mathbb{A})}$ is given by the product $\prod_v (\cdot,\cdot)_v$ of local n-th Hilbert symbols.*

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For each place v, define a section $\mathfrak{s}_{0,v}\colon G'(F_v) \longrightarrow \overline{G'(F_v)}$ by (4), (5), (6), (7). We can choose a section $\mathfrak{s}_{T,v}\colon T(F_v) \longrightarrow \overline{T(F_v)}$ such that if $\mathfrak{s}_v\colon G(F_v) \longrightarrow \overline{G(F_v)}$ defined as (11), then for any non-archimedean place v, $\mathfrak{s}_v$ is an isomorphism on an open subgroup U_v of $G(F_v)$ and its image is normal in $\overline{K_v}$.

There is a global section $\mathfrak{s}\colon G(\mathbb{A}) \longrightarrow \overline{G(\mathbb{A})}$ such that

(1) for any non-archimedean v, $\mathfrak{s}|_{U_v} = \mathfrak{s}_v|_{U_v}$;

(2) $\mathfrak{s}|_{N(\mathbb{A})} = \prod_v \mathfrak{s}_v|_{N(F_v)}$ is a homomorphism;

(3) $\mathfrak{s}|_{S(\mathbb{A})} = \prod_v \mathfrak{s}_v|_{S(F_v)}$;

(4) $\mathfrak{s}|_{\mathfrak{W}} = \prod_v \mathfrak{s}_v|_{\mathfrak{W}}$. ∎

We can also give the explicit construction of an isomorphism from $G(F)$ into $\overline{G(\mathbb{A})}$ as follows. Define a map ψ on $G(F)$ by

$$\psi(g) = \mathfrak{s}(n)\mathfrak{s}(h)\mathfrak{s}(w)\mathfrak{s}(n'), \quad g = nhwn' \in G(F), \quad n, n' \in N(F), h \in H(F), w \in \mathfrak{W}.$$

By the above theorem, $\mathfrak{s} = \prod_v \mathfrak{s}_v$ on $H(\mathbb{A})$. By the reciprocity law for the Hilbert symbol, $\mathfrak{s}$ is a homomorphism on $H(F)$ and hence on $G'(F)$. It then follows from our construction of $\mathfrak{s}_T$ that we have the following corollary.

COROLLARY 6. *The map ψ is a homomorphism from $G(F)$ into $\overline{G(\mathbb{A})}$.* ∎

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