J. Appl. Prob. **44,** 1056–1067 (2007)
Printed in England

DISTRIBUTIONS OF THE LONGEST EXCURSIONS IN A TIED DOWN SIMPLE RANDOM WALK AND IN A BROWNIAN BRIDGE

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Abstract

Expressions for the joint distribution of the longest and second longest excursions as well as the marginal distributions of the three longest excursions in the Brownian bridge are obtained. The method, which primarily makes use of the weak convergence of the random walk to the Brownian motion, principally gives the possibility to obtain any desired joint or marginal distribution. Numerical illustrations of the results are also given.

Keywords: Tied down random walk; Brownian bridge; ranked excursion length; weak convergence; generating function; Kummer function

2000 Mathematics Subject Classification: Primary 60J65
Secondary 60C05

1. Introduction

The aim of this paper is to calculate distributions of the longest excursions, i.e. ranked excursion lengths, in tied down random walks and especially in the Brownian motion tied down at time $t = 1$. The results for the Brownian bridge are given as asymptotic results when passing to the limit in a random walk. Several previous studies deal with properties of excursions in general. Our special interest here lies in results for distributions of ranked excursion lengths in tied down random walks and the Brownian bridge. Analytical expressions for the longest excursion, V_1, are obtained in, for example, [3], [6], and [7]. It is interesting to note that explicit formulae for $\mathrm{P}(V_1 \leq x)$ are only available for the interval $\frac{1}{4} \leq x \leq 1$. However, a semi-analytical result due to Rosén was found in [7] that covers the full interval $0 \leq x \leq 1$. We extend this result to, in principle, any desired order of excursion as well as to joint distributions. The results are expressed in terms of infinite sums, where the Kummer function and its zeros play a central role. We show that these formulae, in spite of their complexity, are readily manageable in standard mathematical software. Furthermore, it is often sufficient for accurate results to include only the first term in the sums, and thereby only the first zero in the Kummer function.

In Section 2 we start with some results regarding exact formulae for the distribution of the longest excursion in a tied down simple random walk, and then recapitulate formulae for the distribution of the longest excursion in a Brownian bridge. Our main results, semi-analytical formulae for the joint distribution of the longest and second longest excursions as well as the marginal distributions for the three longest excursions, are given in Section 3. To conclude, we illustrate the results numerically and graphically in Section 4.

Received 22 June 2006; revision received 29 September 2007.
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2. Longest excursion in a tied down random walk and a Brownian bridge

Let $\{S_i\}_{i=0}^{2n}$ be a simple random walk starting at 0. Define an excursion as a zero-free interval that starts and ends at 0. Let L_{2n} be the longest excursion length until time $2n$. Introduce the notation

$$\left[\sum_{k=0}^{\infty} a_k z^k\right]_m = \sum_{k=0}^{m} a_k z^k \quad \text{and} \quad [z^n]\sum_{k=0}^{\infty} a_k z^k = a_n.$$

The following result is given in [7, p. 22].

Proposition 2.1.

$$\mathrm{P}(L_{2n} \le 2m \mid S_{2n} = 0) = [z^n]\frac{1}{[\sqrt{1-z}]_m} \Big/ u_{2n}, \tag{2.1}$$

where

$$u_{2n} = \mathrm{P}(S_{2n} = 0) = \binom{2n}{n}\left(\frac{1}{2}\right)^{2n}.$$

Proof. Let

$$v_{2n,m} = \mathrm{P}(L_{2n} \le 2m,\, S_{2n} = 0),$$
$$f_{2k} = \mathrm{P}(S_1 \ne 0, \dots, S_{2k-1} \ne 0,\ S_{2k} = 0).$$

Introduce the generating functions

$$F_m(z) = \sum_{k=1}^{m} f_{2k} z^k = [1 - \sqrt{1-z}]_m \quad \text{and} \quad V_m(z) = \sum_{k=0}^{\infty} v_{2k,m} z^k.$$

The equality in the first formula can be found in, for example, [4, p. 273]. We have

$$v_{0,m} = 1, \qquad v_{2n,m} = \sum_{k=1}^{m} f_{2k} v_{2n-2k,m}, \quad n \ge 1.$$

Thus, the convolution property for generating functions gives

$$V_m(z) = \frac{1}{1 - F_m(z)} = \frac{1}{[\sqrt{1-z}]_m}.$$

We obtain the assertion by extracting the appropriate term in $V_m(z)$ and conditioning on $S_{2n} = 0$.

Using (2.1), it is possible to obtain the distribution for the longest excursion in a tied down simple random walk. It is easy to plot these distributions with standard mathematical software; see Figure 1 in Section 4.

By passing to the limit in a tied down simple random walk we obtain a Brownian bridge; see [2] and the references therein. For the longest excursion length, V_1, in a Brownian bridge, it was proved in [7, p. 23] that

$$\mathrm{P}(V_1 \le x) = 2 - \frac{1}{\sqrt{x}}, \qquad \frac{1}{2} \le x \le 1.$$

Pitman and Yor [6, p. 868] obtained the following explicit formula for the density of V_1, covering the case in which $\frac{1}{4} \le x \le 1$:

$$g(x) = q_1(x) - q_2(x) + q_3(x),$$

where

$$q_1(x) = \frac{1}{2x^{3/2}},$$
$$q_2(x) = \mathbf{1}_{\{x\le 1/2\}}\,\frac{1}{\pi x^{3/2}}\biggl(-\pi + 2\sqrt{\frac{1-2x}{x}} + 2\arcsin\sqrt{\frac{x}{1-x}}\biggr),$$
$$q_3(x) = \mathbf{1}_{\{x\le 1/3\}}\,\frac{3}{4\pi x^{3/2}}\biggl(2 + 2\pi + \frac{2}{x} - 8\sqrt{\frac{1-2x}{x}} - 8\arcsin\sqrt{\frac{x}{1-x}}\biggr).$$

3. Joint and marginal distributions for the longest excursions

We will derive the joint and marginal distributions for the longest excursions in the Brownian bridge using the generating function of the number of excursions of certain lengths. The generating function is obtained by convergence of the corresponding generating functions of tied down random walks.

3.1. Generating function of the number of excursions of certain lengths

Let $2Y_1, 2Y_2, \ldots$ denote the lengths of the excursions in a symmetric simple random walk. Here, the Ys are independent and identically distributed with

$$\mathrm{P}(Y = k) = \frac{1}{2k-1}\binom{2k}{k}\frac{1}{2^{2k}}, \qquad k = 1, 2, \ldots;$$

see, e.g. [4, p. 78].

For fixed $0 < x_r \le x_{r-1} \le \cdots \le x_1 < x_0 = \infty$, let $N_j^{(n)}$ denote the number of excursions with lengths in $(2nx_j, 2nx_{j-1}]$ in a random walk tied down at time $2n$. We have the weak convergence

$$(N_1^{(n)}, N_2^{(n)}, \ldots, N_r^{(n)}) \to (N_1, N_2, \ldots, N_r), \qquad n \to \infty,$$

where N_j denote the number of excursions in a Brownian bridge with lengths in $(x_j, x_{j-1}]$; cf. Csáki and Hu [2] and the references therein.

Proposition 3.1. *We have*

$$\mathrm{E}\bigl(z_1^{N_1^{(n)}} \cdots z_r^{N_r^{(n)}}\bigr) = \frac{1}{2\mathrm{i}\pi\,\mathrm{P}(S_{2n} = 0)} \int_{-\mathrm{i}\pi n}^{\mathrm{i}\pi n} \mathrm{e}^{s}\,\frac{E_n(s) - (E_n(s))^{n+1}}{1 - E_n(s)}\,\mathrm{d}s,$$

where

$$E_n(s) = \mathrm{E}\biggl(\mathrm{e}^{-sY/n} \prod_{j=1}^{r} z_j^{\mathbf{1}_{\{nx_{j-1}\ge Y > nx_j\}}}\biggr).$$

Proof. We have

$$\begin{aligned}
&\mathrm{E}\bigl(z_1^{N_1^{(n)}} \cdots z_r^{N_r^{(n)}}\bigr)\,\mathrm{P}(S_{2n} = 0)\\
&\quad= \sum_{k=1}^{n} \mathrm{E}\biggl(\mathbf{1}_{\{\sum_{l=1}^{k} Y_l = n\}} \prod_{j=1}^{r} z_j^{\sum_{l=1}^{k}\mathbf{1}_{\{nx_{j-1}\ge Y_l > nx_j\}}}\biggr)\\
&\quad= \sum_{k=1}^{n} \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}tn}\,\mathrm{E}\biggl(\mathrm{e}^{\mathrm{i}t\sum_{l=1}^{k} Y_l} \prod_{j=1}^{r} z_j^{\sum_{l=1}^{k}\mathbf{1}_{\{nx_{j-1}\ge Y_l > nx_j\}}}\biggr)\,\mathrm{d}t.
\end{aligned}$$

As the Ys are independent and identically distributed, the last expression is equal to

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}tn}\sum_{k=1}^{n}\bigg(\mathrm{E}\bigg(\mathrm{e}^{\mathrm{i}tY}\prod_{j=1}^{r} z_j^{\mathbf{1}_{\{nx_{j-1}\geq Y>nx_j\}}}\bigg)\bigg)^k\,\mathrm{d}t.$$

Changing the variable $-\mathrm{i}tn$ to s and identifying the geometric sum, the assertion follows.

Lemma 3.1. *We have*

$$\sqrt{\pi n}(1-E_n(s)) \to \int_0^\infty \frac{1-\mathrm{e}^{-sy}\prod_{j=1}^r z_j^{\mathbf{1}_{\{x_{j-1}\geq y>x_j\}}}}{2y^{3/2}}\,\mathrm{d}y, \qquad n\to\infty.$$

Proof. By Stirling's formula we obtain

$$\sqrt{\pi k}\binom{2k}{k}4^{-k}\to 1, \qquad k\to\infty.$$

Furthermore, by convergence of Riemann sums we have

$$\begin{aligned}
&\sqrt{\pi n}(1-E_n(s))\\
&\quad=\sqrt{\pi n}\sum_{k=1}^{\infty}\bigg(1-\mathrm{e}^{-sk/n}\prod_{j=1}^{r} z_j^{\mathbf{1}_{\{x_{j-1}\geq k/n>x_j\}}}\bigg)\frac{1}{2k-1}\binom{2k}{k}4^{-k}\\
&\quad=\sum_{k=1}^{\infty}\bigg(1-\mathrm{e}^{-sk/n}\prod_{j=1}^{r} z_j^{\mathbf{1}_{\{x_{j-1}\geq k/n>x_j\}}}\bigg)\bigg(\frac{n}{k}\bigg)^{3/2}\frac{k}{2k-1}\sqrt{\pi k}\binom{2k}{k}4^{-k}\frac{1}{n}\\
&\quad\to\int_0^\infty \frac{1-\mathrm{e}^{-sy}\prod_{j=1}^r z_j^{\mathbf{1}_{\{x_{j-1}\geq y>x_j\}}}}{2y^{3/2}}\,\mathrm{d}y, \qquad n\to\infty.
\end{aligned}$$

We introduce the function $K(s)=M(-\frac{1}{2},\frac{1}{2},-s)$, where $M(a,b,s)$ is the Kummer function; see Appendix A.

Lemma 3.2. *For* $\mathrm{Re}(s)\geq 0$ *and* $x>0$, *we obtain*

$$\int_0^\infty \frac{1-\mathrm{e}^{-sy}}{2y^{3/2}}\,\mathrm{d}y=\sqrt{\pi s}, \tag{3.1}$$

$$\int_0^x \frac{1-\mathrm{e}^{-sy}}{2y^{3/2}}\,\mathrm{d}y=\frac{-1+K(sx)}{\sqrt{x}}, \tag{3.2}$$

$$\int_x^\infty \frac{1-z\mathrm{e}^{-sy}}{2y^{3/2}}\,\mathrm{d}y=z\sqrt{\pi s}+\frac{1-zK(sx)}{\sqrt{x}}. \tag{3.3}$$

Proof. Integrating by parts and using the gamma function, (3.1) follows. By series expansion we have

$$\begin{aligned}
\int_0^x \frac{1-\mathrm{e}^{-sy}}{2y^{3/2}}\,\mathrm{d}y &= -\sum_{k=1}^{\infty}\frac{(-s)^k}{k!}\int_0^x \frac{y^{k-3/2}}{2}\,\mathrm{d}y\\
&= -\sum_{k=1}^{\infty}\frac{(-s)^k}{k!}\frac{x^{k-1/2}}{2k-1}\\
&= \frac{-1+K(sx)}{\sqrt{x}}.
\end{aligned}$$

Finally,

$$\begin{aligned}\int_x^\infty \frac{1-z\mathrm{e}^{-sy}}{2y^{3/2}}\,\mathrm{d}y &= (1-z)\int_x^\infty \frac{1}{2y^{3/2}}\,\mathrm{d}y + z\int_x^\infty \frac{1-\mathrm{e}^{-sy}}{2y^{3/2}}\,\mathrm{d}y \\ &= \frac{(1-z)}{\sqrt{x}} + z\biggl(\sqrt{\pi s} - \frac{-1+K(sx)}{\sqrt{x}}\biggr) \\ &= z\sqrt{\pi s} + \frac{1-zK(sx)}{\sqrt{x}}.\end{aligned}$$

The following theorem for the Brownian bridge gives the generating function for the number of excursions $\{N_j\}_{j=1}^r$ with lengths in the respective intervals $(x_j, x_{j-1}]$.

Theorem 3.1. *We have*

$$\mathrm{E}(z_1^{N_1}\cdots z_r^{N_r}) = \frac{1}{2\mathrm{i}\sqrt{x_r}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{\mathrm{e}^{s/x_r}}{K(s) - z_1 A(s) - \sum_{j=2}^r z_j B_j(s)}\,\mathrm{d}s,$$

where

$$\begin{aligned}A(s) &= K\biggl(\frac{sx_1}{x_r}\biggr)\sqrt{\frac{x_r}{x_1}} - \sqrt{\pi s}, \\ B_j(s) &= K\biggl(\frac{sx_j}{x_r}\biggr)\sqrt{\frac{x_r}{x_j}} - K\biggl(\frac{sx_{j-1}}{x_r}\biggr)\sqrt{\frac{x_r}{x_{j-1}}}.\end{aligned}$$

Proof. By Stirling's formula we obtain $\mathrm{P}(S_{2n}=0) \sim 1/\sqrt{\pi n}$. It follows, from Proposition 3.1 and Lemma 3.1, that

$$\mathrm{E}(z_1^{N_1}\cdots z_r^{N_r}) = \frac{1}{2\mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{\mathrm{e}^{s}}{\int_0^\infty((1-\mathrm{e}^{-sy}\prod_{j=1}^r z_j^{\mathbf{1}_{\{x_{j-1}\geq y>x_j\}}})/2y^{3/2})\,\mathrm{d}y}\,\mathrm{d}s.$$

By (3.1)–(3.3), we obtain

$$\begin{aligned}&\int_0^\infty \frac{1-\mathrm{e}^{-sy}\prod_{j=1}^r z_j^{\mathbf{1}_{\{x_{j-1}\geq y>x_j\}}}}{2y^{3/2}}\,\mathrm{d}y \\ &\quad= \int_0^{x_r}\frac{1-\mathrm{e}^{-sy}}{2y^{3/2}}\,\mathrm{d}y + \sum_{j=1}^r\int_{x_j}^{x_{j-1}}\frac{1-\mathrm{e}^{-sy}z_j}{2y^{3/2}}\,\mathrm{d}y \\ &\quad= \frac{-1+K(sx_r)}{\sqrt{x_r}} + z_1\sqrt{\pi s} + \frac{1-z_1K(sx_1)}{\sqrt{x_1}} \\ &\qquad + \sum_{j=2}^r\biggl[\frac{-1+z_jK(sx_{j-1})}{\sqrt{x_{j-1}}} - \frac{-1+z_jK(sx_j)}{\sqrt{x_j}}\biggr] \\ &\quad= \frac{K(sx_r)}{\sqrt{x_r}} - z_1\biggl(-\sqrt{\pi s} + \frac{K(sx_1)}{\sqrt{x_1}}\biggr) \\ &\qquad - \sum_{j=2}^r z_j\biggl(\frac{K(sx_j)}{\sqrt{x_j}} - K(sx_{j-1})\sqrt{x_{j-1}}\biggr).\end{aligned}$$

Changing the variable sx_r to s gives the assertion.

Theorem 3.1 gives the possibility of calculating any joint or marginal distribution of the longest excursions in the Brownian bridge. First, consider the special case where $r = 1$ and $N_1 = N$ is equal to the number of excursions longer than x, $0 \le x \le 1$. We obtain

$$\begin{aligned}\mathrm{E}(z^N) &= \frac{1}{2\mathrm{i}\sqrt{x}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{\mathrm{e}^{s/x}}{K(s) - z(K(s) - \sqrt{\pi s})}\,\mathrm{d}s \\ &= \sum_{k=0}^{\infty} z^k \frac{1}{2\mathrm{i}\sqrt{x}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{\mathrm{e}^{s/x}}{K(s)} \left(1 - \frac{\sqrt{\pi s}}{K(s)}\right)^k \mathrm{d}s.\end{aligned}$$

The event $\{N \le j-1\}$ means that the number of excursions that are longer than x cannot exceed $j-1$, i.e. the jth longest excursion is shorter than x. Thus, picking the appropriate coefficient in the sum above, we obtain the following result.

Proposition 3.2. *We have*

$$\mathrm{P}(V_j \le x) = \mathrm{P}(N \le j-1) = \sum_{k=0}^{j-1} \mathrm{P}(N = k),$$

where

$$\mathrm{P}(N = k) = \frac{1}{2\mathrm{i}\sqrt{x}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{\mathrm{e}^{s/x}(K(s) - \sqrt{\pi s})^k}{K(s)^{k+1}}\,\mathrm{d}s. \tag{3.4}$$

The integrals in (3.4) can be evaluated by calculus of residues. We will omit this straightforward but somewhat tedious exercise in what follows. However, we note here that it will be important to understand the behavior of the poles as well as being able to differentiate the Kummer function. We refer the reader to Appendix A and [5].

3.2. Marginal distributions of the three longest excursions

The following result is given in [7, p. 24].

Corollary 3.1. *For* $0 \le x \le 1$, *we have*

$$\mathrm{P}(V_1 \le x) = \frac{2\pi}{\sqrt{x}} \sum_{k=-\infty}^{\infty} -s_k \exp\left(s_k\left(1 + \frac{1}{x}\right)\right),$$

where the s_k*s are the* 0*s of* $K(s)$*; see Appendix A.*

Proof. Using Proposition 3.2, we have

$$\begin{aligned}\mathrm{P}(V_1 \le x) = \mathrm{P}(N = 0) &= \frac{1}{2\mathrm{i}\sqrt{x}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{\mathrm{e}^{s/x}}{K(s)}\,\mathrm{d}s \\ &= \frac{2\pi\mathrm{i}}{2\mathrm{i}\sqrt{x}} \sum_{k=-\infty}^{\infty} \frac{\mathrm{e}^{s_k/x}}{K'(s_k)} \\ &= \frac{\pi}{\sqrt{x}} \sum_{k=-\infty}^{\infty} \frac{\mathrm{e}^{s_k/x}}{-\mathrm{e}^{-s_k}/2s_k}\end{aligned}$$

by calculus of residues; see Appendix A.

Corollary 3.2. *For* $0 \le x \le \frac{1}{2}$, *we have*

$$\mathrm{P}(V_2 \le x) = 2\,\mathrm{P}(V_1 \le x) + 2\pi \sum_{k=-\infty}^{\infty} -s_k \mathrm{e}^{2s_k}\left(2\mathrm{e}^{s_k/x}\left(1 + s_k + \frac{s_k}{x}\right)K\left(\frac{s_k}{x}\right) - 1\right).$$

Proof. Using Proposition 3.2 and identifying previously calculated integrals, we have

$$\begin{aligned}
\mathrm{P}(V_2 \le x) &= \mathrm{P}(N = 0) + \mathrm{P}(N = 1) \\
&= \mathrm{P}(V_1 \le x) + \mathrm{P}(N = 1) \\
&= \mathrm{P}(V_1 \le x) + \frac{1}{2\mathrm{i}\sqrt{x}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{\mathrm{e}^{s/x}(K(s) - \sqrt{\pi s})}{K(s)^2}\,\mathrm{d}s \\
&= 2\,\mathrm{P}(V_1 \le x) - \frac{1}{2\mathrm{i}\sqrt{x}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{\mathrm{e}^{s/x}\sqrt{\pi s}}{K(s)^2}\,\mathrm{d}s.
\end{aligned}$$

We do not evaluate the last integral here. Instead, we refer the reader to the special case for $x_1 = 1$ of the joint distribution in Corollary 3.4, below.

Corollary 3.3. *For* $0 \le x \le \frac{1}{3}$, *we have*

$$\begin{aligned}
\mathrm{P}(V_3 \le x) &= 3(\mathrm{P}(V_2 \le x) - \mathrm{P}(V_1 \le x)) \\
&\quad - \frac{2\pi^2}{\sqrt{x}} \sum_{k=-\infty}^{\infty} s_k^2 \mathrm{e}^{s_k(1/x+3)}\left(s_k^2\left(\frac{2}{x^2} + \frac{6}{x} + 4\right) + s_k\left(\frac{7}{x} + 9\right) + 3\right).
\end{aligned}$$

Proof. Again, using Proposition 3.2, we have

$$\mathrm{P}(V_3 \le x) = \sum_{k=0}^{2} \mathrm{P}(N = k).$$

Identifying previously calculated integrals, we obtain

$$\begin{aligned}
\mathrm{P}(N = 2) &= \frac{1}{2\mathrm{i}\sqrt{x}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{\mathrm{e}^{s/x}(K(s) - \sqrt{\pi s})^2}{K(s)^3}\,\mathrm{d}s \\
&= \mathrm{P}(V_1 \le x) - \frac{2}{2\mathrm{i}\sqrt{x}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{\mathrm{e}^{s/x}\sqrt{\pi s}}{K(s)^2}\,\mathrm{d}s + \frac{\pi}{2\mathrm{i}\sqrt{x}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{\mathrm{e}^{s/x}s}{K(s)^3}\,\mathrm{d}s.
\end{aligned}$$

By calculus of residues we obtain

$$\frac{\pi}{2\mathrm{i}\sqrt{x}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{\mathrm{e}^{s/x}s}{K(s)^3}\,\mathrm{d}s = -\frac{2\pi^2}{\sqrt{x}} \sum_{k=-\infty}^{\infty} s_k^2 \mathrm{e}^{s_k(1/x+3)}\left(s_k^2\left(\frac{2}{x^2} + \frac{6}{x} + 4\right) + s_k\left(\frac{7}{x} + 9\right) + 3\right),$$

and the assertion follows.

Any marginal distribution can be obtained in an analogous way.

3.3. Joint distribution of the longest and second longest excursions

Consider the joint distribution of (V_1, V_2). We have, for $0 \leq x_2 \leq x_1 \leq 1$,

$$\begin{aligned} \mathrm{P}(V_2 \leq x_2, V_1 \leq x_1) &= \mathrm{P}(V_1 \leq x_2) + \mathrm{P}(V_2 \leq x_2 < V_1 \leq x_1) \\ &= \mathrm{P}(V_1 \leq x_2) + \mathrm{P}(N_1 = 0,\ N_2 = 1). \end{aligned}$$

Using the appropriate coefficient of the generating function $\mathrm{E}(z_1^{N_1} z_2^{N_2})$, where $r = 2$, in Theorem 3.1, leads to

$$\begin{aligned} \mathrm{P}(N_1 = 0,\ N_2 = 1) &= \frac{1}{2\mathrm{i}\sqrt{x_2}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{\mathrm{e}^{s/x_2}}{(K(s))^2} \left(K(s) - K\left(\frac{s x_1}{x_2}\right) \sqrt{\frac{x_2}{x_1}} \right) \mathrm{d}s \\ &= \mathrm{P}(V_1 \leq x_2) - \frac{1}{2\mathrm{i}\sqrt{x_1}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{\mathrm{e}^{s/x_2} K(s x_1/x_2)}{(K(s))^2}\, \mathrm{d}s. \end{aligned}$$

By calculus of residues, the last integral can be written as

$$\frac{\pi}{\sqrt{x_1}} \sum_{k=-\infty}^{\infty} \mathrm{e}^{s_k/x_2} \left(-K\left(\frac{s_k x_1}{x_2}\right) \frac{K''(s_k)}{K'(s_k)} + K\left(\frac{s_k x_1}{x_2}\right) \Big/ x_2 + K'\left(\frac{s_k x_1}{x_2}\right) x_1 \Big/ x_2 \right) \Big/ (K'(s_k))^2.$$

Differentiating the Kummer function (see Appendix A) and using the fact that $K(s_k) = 0$, we obtain

$$-\frac{2\pi}{\sqrt{x_1}} \sum_{k=-\infty}^{\infty} -s_k \mathrm{e}^{s_k((1-x_1)/x_2+2)} \left(2\mathrm{e}^{s_k x_1/x_2} K\left(\frac{s_k x_1}{x_2}\right) \left(1 + s_k + \frac{s_k}{x_2}\right) - 1 \right).$$

Hence, we have obtained the following result.

Corollary 3.4. *For* $0 \leq x_2 \leq x_1 \leq 1$ *and* $0 \leq x_2 \leq \frac{1}{2}$, *we have*

$$\begin{aligned} &\mathrm{P}(V_1 \leq x_1,\ V_2 \leq x_2) \\ &\quad = 2\,\mathrm{P}(V_1 \leq x_2) \\ &\qquad + \frac{2\pi}{\sqrt{x_1}} \sum_{k=-\infty}^{\infty} -s_k \exp\left(s_k \left(\frac{1 - x_1}{x_2} + 2 \right) \right) \left(2\mathrm{e}^{s_k x_1/x_2} K\left(\frac{s_k x_1}{x_2}\right) \left(1 + s_k + \frac{s_k}{x_2}\right) - 1 \right). \end{aligned}$$

4. Numerical illustrations of the results

In this section we illustrate the results in the previous sections by plotting distribution and density functions. Good approximations are obtained using only the first term of the sums in the expressions for the distributions or densities. For a detailed analysis of the accuracy of the approximations, we refer the reader to [5]. The approximations are accurate for $x \leq \frac{1}{2}$. As only the longest excursion can take values larger than $\frac{1}{2}$, we obtain a good grasp of all distributions.

First, we study the probability functions of the longest excursion in a tied down random walk. Using Proposition 2.1, we obtain the plots in Figure 1.

Next we evaluate the distributions of the longest excursions in the Brownian bridge.

The series in the distribution for the longest excursion in Corollary 3.1 converges rapidly for $x \leq 0.5$. Recall that, for $0.5 \leq x \leq 1$, we have

$$\mathrm{P}(V_1 \leq x) = 2 - \frac{1}{\sqrt{x}}.$$

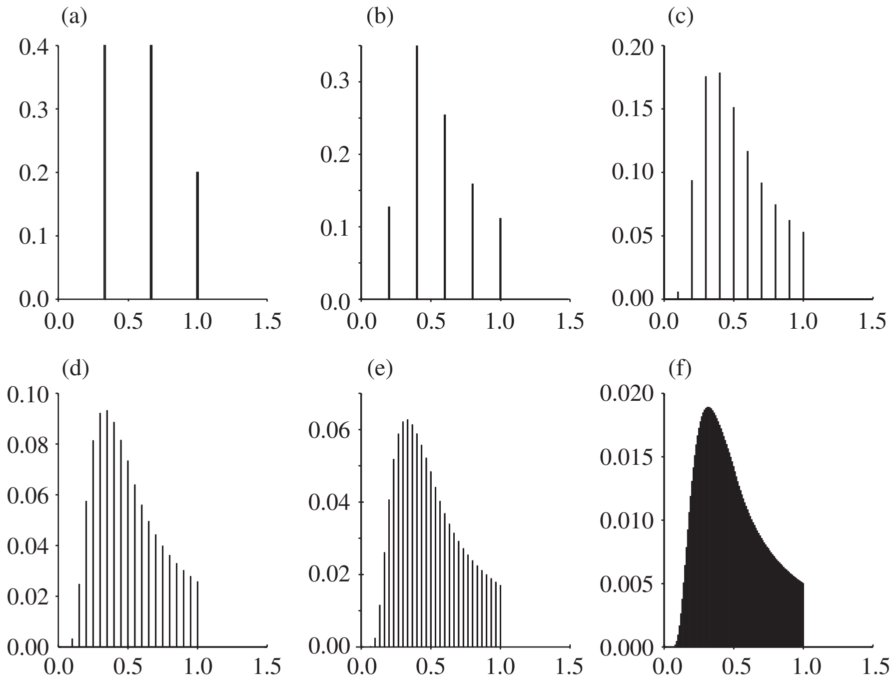

FIGURE 1: Probability functions of the longest excursion in tied down random walks. The plots show the density for (a) $n = 3$, (b) $n = 5$, (c) $n = 10$, (d) $n = 20$, (e) $n = 30$, and (f) $n = 100$.

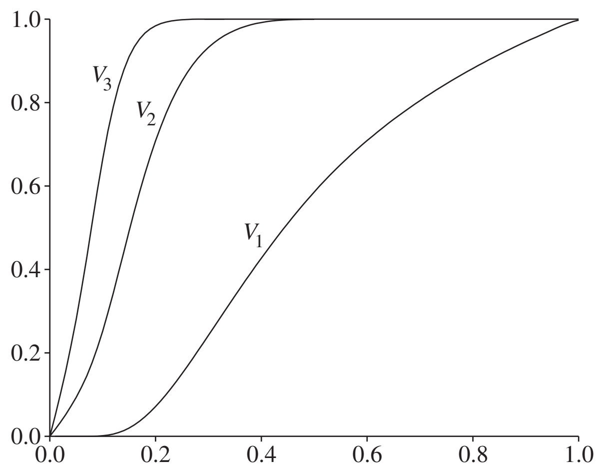

FIGURE 2: Distribution functions of the three longest excursions.

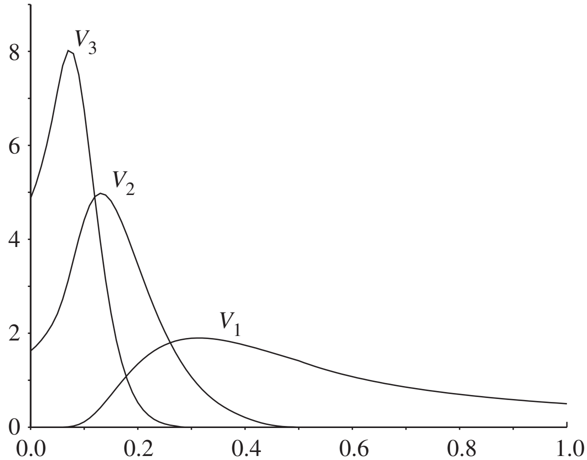

FIGURE 3: Density functions of the three longest excursions.

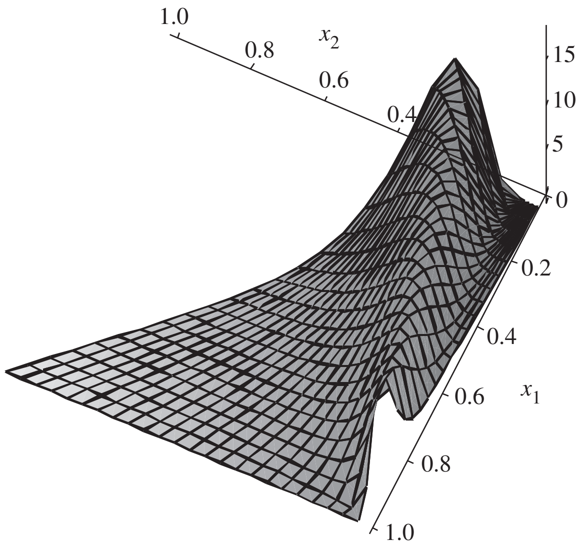

FIGURE 4: Three-dimensional plot of the joint density function of the two longest excursions.

Taking the first term in the series, with $s_0 = -0.8540\ldots$ (see Appendix A), gives $\mathrm{P}(V_1 \le x) \approx (2\pi/\sqrt{x})(-s_0)\mathrm{e}^{s_0(1+1/x)}$. Thus, $\mathrm{P}(V_1 \le 0.5) \approx 0.5854$ compared to the exact value, $2 - \sqrt{2} \approx 0.5858$.

Figures 2 and 3 show plots of marginal distributions and densities. Only the first term in the respective series is included (for $x \ge 0.5$ exact expressions).

Finally, using Corollary 3.4, we plot the joint density function of the two longest excursions in Figure 4. Only the first term in the series is included.

Appendix A. Properties of the Kummer function

The general Kummer (or confluent hypergeometric) function (see Abramovitz and Stegun [1, p. 304]) is defined as

$$M(a, b, s) = 1 + \sum_{k=1}^{\infty} \frac{a(a+1)\cdots(a+k-1)}{b(b+1)\cdots(b+k-1)} \frac{s^k}{k!}.$$

Note that $M(a, b, s) = \mathrm{e}^s M(b-a, b, -s)$. We consider, in particular,

$$\begin{aligned} K(s) &= M(-\tfrac{1}{2}, \tfrac{1}{2}, -s) \\ &= 1 - \sum_{k=1}^{\infty} \frac{1}{2k-1} \frac{(-s)^k}{k!} \\ &= \mathrm{e}^{-s} M(1, \tfrac{1}{2}, s) \\ &= \mathrm{e}^{-s}\bigg(1 + \sum_{k=1}^{\infty} \frac{(2s)^k}{(2k-1)!!}\bigg) \end{aligned}$$

with the derivatives

$$K'(s) = \frac{K(s) - \mathrm{e}^{-s}}{2s}, \qquad K''(s) = \frac{\mathrm{e}^{-s} - K'(s)}{2s}.$$

Further derivatives are easy to obtain recursively.

All zeros $\{s_k\}_{-\infty}^{\infty}$ of $K(s)$ lie in the left complex plane. In fact,

$$\begin{gathered} \bar{s}_{-k} = s_k = -a_k + \mathrm{i} b_k, \qquad k = 0, 1, 2, \ldots, \\ a_0 < a_1 < a_2 < \cdots \nearrow +\infty, \qquad b_0 < b_1 < b_2 < \cdots \nearrow +\infty. \end{gathered}$$

The first five 0s are given in Table 1. The numbers have been calculated using MAPLE®; see [5].

TABLE 1: Coefficients for the first five 0s in the Kummer function $K(s)$.

k	$a_{\pm k}$	b_k	b_{-k}
0	0.854 032 66 . . .	0	0
1	4.248 920 78 . . .	6.383 124 29 . . .	−6.383 124 29 . . .
2	5.184 114 73 . . .	12.885 305 17 . . .	−12.885 305 17 . . .

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