1 Geometry of Points, Lines, and Planes

. . . and then geometry will become what
geometry ought to be.

Mr. Querulous
Ball's "*A Dynamical Parable*" (1887)

1.1 Introduction

Points, lines, and planes are the fundamental elements of spatial geometry. A point can be thought of as a location in 3D space, and its coordinates have units of length. A line can be considered to be an infinite collection of points defined by a direction (which is a dimensionless vector) that passes through some given point (which has units of meters). A plane is a two-dimensional set of points that can be defined, for example, by three points or by a line and one point. This chapter introduces the concept of homogeneous coordinates as applied to points, lines, and planes. The *homogeneous coordinates* of each will be defined together with the *equation* for each. The equation of a point, line, or plane will be shown to be a vector equation where any vector that satisfies that equation is a member of that point, line, or plane.

1.2 The Position Vector of a Point

The position vector to a point Q_1 from a reference point O will be referred to as $\boldsymbol{r}_1$ and can be expressed in the form

$$\boldsymbol{r}_1 = \frac{x_1\,\boldsymbol{i} + y_1\,\boldsymbol{j} + z_1\,\boldsymbol{k}}{w_1} \tag{1.1}$$

or

$$\boldsymbol{r}_1\,w_1 = \boldsymbol{S}_{O1}, \tag{1.2}$$

where $\boldsymbol{S}_{O1} = x_1\,\boldsymbol{i} + y_1\,\boldsymbol{j} + z_1\,\boldsymbol{k}$ and the components of the vector $\boldsymbol{S}_{O1}$ have units of length. The term w_1 is dimensionless. In Figure 1.1 it is assumed that $w_1 = 1$ and (x_1, y_1, z_1) are the usual Cartesian coordinates for the point Q_1. The coordinates $\boldsymbol{r}$ of some general point Q may be expressed as

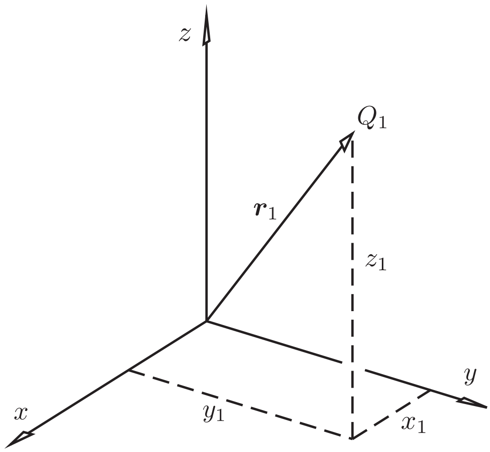

Figure 1.1 Coordinates of a point

$$\boldsymbol{r}\, w = \boldsymbol{S}_O, \tag{1.3}$$

where $\boldsymbol{S}_O = x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k}$. The subscript O has been introduced to signify that $\boldsymbol{S}_O$ is origin dependent. Clearly, if we choose some other reference point, the actual point Q would not change. However, the coordinates (x, y, z), which determine Q, would change. The ratios x/w, y/w, and z/w are three independent scalars and, therefore, there are ∞^3 points in space.

It is interesting to consider the cases where $\boldsymbol{S}_O = \boldsymbol{0}$ and where $w = 0$. From (1.3)

$$|\boldsymbol{r}| = \frac{|\boldsymbol{S}_O|}{|w|}, \tag{1.4}$$

where the notation $||$ denotes absolute magnitude. From (1.4), when $|\boldsymbol{S}_O| = 0$, $|\boldsymbol{r}| = 0$ and Q coincides with the reference point O. When $|w| = 0$, $|\boldsymbol{r}|$ is infinite and the point Q is said to be at infinity in the direction parallel to $\boldsymbol{S}_O$. The introduction of w makes it possible to designate a point by the array of four coordinates $(w; x, y, z)$[1] and is a means of introducing the concept of infinity, or more specifically infinite points, into the geometry without introducing the symbol ∞. Any point at infinity is designated by the coordinates $(0; x, y, z)$. It is important to recognize that $|\boldsymbol{S}_O|$ and $|w|$ cannot be zero simultaneously since $|\boldsymbol{r}|$ would be indeterminate. In other words, the array $(0; 0, 0, 0)$ is not permitted.

It is of interest to examine the geometry of points labeled with the four coordinates $(w; x, y, z)$. A number of readers will know that this geometry is called projective geometry, which is the subject of many texts (see Coxeter, Meserve, Semple and Kneebone, Faulkner, and Scott to name a few.[2]) Firstly, the four coordinates $(w; x, y, z)$ are homogeneous since from (1.3) $(\lambda w; \lambda \boldsymbol{S}_O)$, where λ is a non-zero scalar, determine the

[1] The semi-colon is introduced into the notation to signify that the dimension of w is different from that of x, y, and z.
[2] Coxeter (2003), Meserve (2010), Semple and Kneebone (1998), Faulkner (2006), and Scott (1894).

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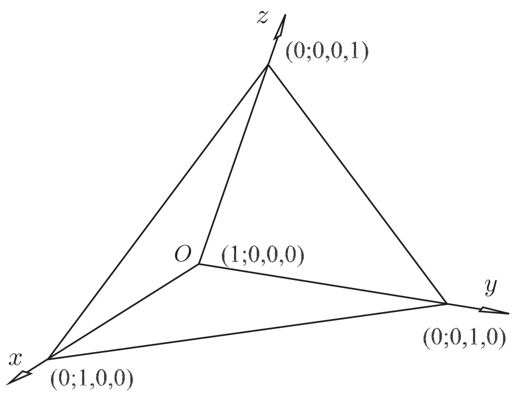

Figure 1.2 Tetrahedron of reference

same point. For instance, the coordinates $(1; 2, 3, 4)$, $(2; 4, 6, 8)$, and $(-2; -4, -6, -8)$, where the last three coordinates have the same unit of length, all determine the same point. Further, the homogeneous coordinates for the origin of this coordinate system are $(w; 0, 0, 0)$ or $(1; 0, 0, 0)$. The homogeneous coordinates for points on the x, y, and z axes are, respectively, $(w; x, 0, 0)$, $(w; 0, y, 0)$, and $(w; 0, 0, z)$. Therefore, the homogeneous coordinates for points at infinity on the x, y, and z axes are $(0; x, 0, 0)$, $(0; 0, y, 0)$, and $(0; 0, 0, z)$ or $(0; 1, 0, 0)$, $(0; 0, 1, 0)$, and $(0; 0, 0, 1)$ or $(0; -1, 0, 0)$, $(0; 0, -1, 0)$, and $(0; 0, 0, -1)$. These three points together with the origin form the four vertices of the so-called tetrahedron of reference, illustrated in Figure 1.2.

The projective space of Figure 1.2 is radically different from the Euclidean space labeled by the x, y, and z Cartesian coordinate frame in Figure 1.1. There can be no calibration of the x, y, and z axes in Figure 1.2, i.e., there is no concept of a unit length of measure or of angle. The four coordinates $(w; x, y, z)$ have no dimensions, and the semi-colon is somewhat redundant in projective space. It will, however, be retained simply to signify the order in which the coordinates are written. There are no parallel lines that do not meet. Lines that are parallel in the Euclidean sense meet in the projective space at points on the plane at infinity, which can be drawn through the three points $(0; 1, 0, 0)$, $(0; 0, 1, 0)$, and $(0; 0, 0, 1)$ at infinity on the x, y, and z axes. This brief discussion is sufficient for the purposes of this text. It remains to label the four planes of the tetrahedron of reference with their homogeneous plane coordinates and to label the six edges of the tetrahedron with their homogeneous line coordinates. The homogeneous coordinates of planes and lines are defined in the subsequent two sections of this chapter.

1.3 The Equation of a Plane

The equation of a plane through a point Q_1 with coordinates (x_1, y_1, z_1) and perpendicular to a vector $\boldsymbol{S} = A\,\boldsymbol{i} + B\,\boldsymbol{j} + C\,\boldsymbol{k}$ (see Figure 1.3) can be expressed in the form

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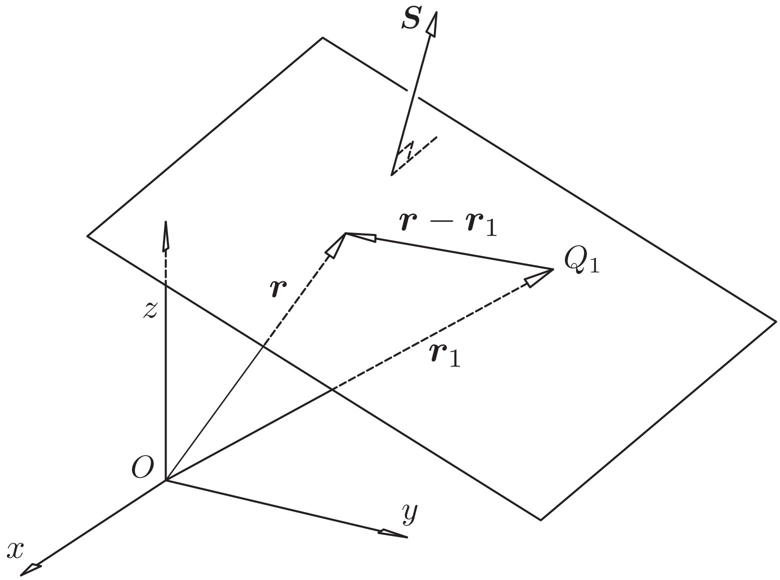

Figure 1.3 Determination of a plane

$$(\boldsymbol{r} - \boldsymbol{r}_1) \cdot \boldsymbol{S} = 0, \tag{1.5}$$

where $\boldsymbol{r} = x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k}$ is a vector from the origin to any general point on the plane, and $\boldsymbol{r}_1$ is the vector from the origin to the point Q_1. The components of the vectors $\boldsymbol{r}$ and $\boldsymbol{r}_1$ have units of length, while the components of $\boldsymbol{S}$ are dimensionless. Equation (1.5) may be written as

$$\boldsymbol{r} \cdot \boldsymbol{S} + D_O = A\,x + B\,y + C\,z + D_O = 0, \tag{1.6}$$

where

$$D_O = -r_1 \cdot \boldsymbol{S} = -(A\,x_1 + B\,y_1 + C\,z_1). \tag{1.7}$$

The scalar value D_O has units of length and is origin dependent.

The coordinate of a plane will be written as $[D_O; A, B, C]$, where square brackets are now used to distinguish these values from the coordinates of a point where parentheses were used. The array

$$[D_O; A, B, C]$$

represents the homogeneous coordinates of the plane since from (1.6) the coordinates of

$$[\lambda D_O; \lambda A, \lambda B, \lambda C],$$

where λ is any non-zero scalar, determine the same plane. From (1.6) it is apparent that the dimension of D_O is different from those of A, B, and C, and a semi-colon is introduced in the array to signify this.

Dividing (1.6) by D_O yields

$$\frac{A}{D_O}x + \frac{B}{D_O}y + \frac{C}{D_O}z + 1 = 0. \tag{1.8}$$

The ratios $\frac{A}{D_O}$, $\frac{B}{D_O}$, and $\frac{C}{D_O}$ are three independent scalars and, therefore, there are ∞^3 planes in space.

The distance of the plane from the reference point O is determined by the length of the vector $\boldsymbol{p}$, which extends from point O to a point on the plane such that $\boldsymbol{p}$ is perpendicular to the plane. Since the vectors $\boldsymbol{p}$ and $\boldsymbol{S}$ must be parallel,

$$\boldsymbol{p} \times \boldsymbol{S} = 0. \tag{1.9}$$

Further, since $\boldsymbol{p}$ is a vector to a point on the plane, it must satisfy (1.6) and

$$\boldsymbol{p} \cdot \boldsymbol{S} = -D_O. \tag{1.10}$$

Performing a cross product of $\boldsymbol{S}$ with (1.9) yields

$$\boldsymbol{S} \times (\boldsymbol{p} \times \boldsymbol{S}) = \boldsymbol{0}. \tag{1.11}$$

Expanding this expression[3] yields

$$\boldsymbol{p}\,(\boldsymbol{S} \cdot \boldsymbol{S}) - \boldsymbol{S}\,(\boldsymbol{p} \cdot \boldsymbol{S}) = \boldsymbol{0}. \tag{1.12}$$

Using (1.10) to substitute for $\boldsymbol{p} \cdot \boldsymbol{S}$ and then solving for $\boldsymbol{p}$ gives

$$\boldsymbol{p} = \frac{-D_O \boldsymbol{S}}{\boldsymbol{S} \cdot \boldsymbol{S}}. \tag{1.13}$$

The perpendicular distance of the plane from the origin, i.e., the magnitude of $\boldsymbol{p}$ is, therefore,

$$|\boldsymbol{p}| = \frac{|-D_O|\ \ |\boldsymbol{S}|}{|\boldsymbol{S}|\ \ |\boldsymbol{S}|} = \frac{|-D_O|}{|\boldsymbol{S}|}. \tag{1.14}$$

Clearly, when $|\boldsymbol{S}| = 1$, the triple (A, B, C) are the direction cosines of a vector normal to the plane, and $|D_O|$ is the perpendicular distance of the plane from the reference point O. Further, when $D_O = 0$, the equation of the plane, i.e., (1.6), becomes

$$A\,x + B\,y + C\,z = 0, \tag{1.15}$$

and the plane passes through O.

The yz plane passes through the origin, and its normal vector is parallel to the x axis. The coordinates for this plane are, thus, $[0; A, 0, 0]$ or $[0; 1, 0, 0]$. Similarly, the coordinates for the zx plane and the xy plane are $[0; 0, 1, 0]$ and $[0; 0, 0, 1]$, respectively. These three planes together with the plane at infinity for which $\boldsymbol{S} = \boldsymbol{0}$ and whose coordinates[4] are therefore $[D; 0, 0, 0]$ or $[1; 0, 0, 0]$ are labeled in Figure 1.4. All points at infinity lie on this plane.

[3] The product $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$ is equal to $(\boldsymbol{a} \cdot \boldsymbol{c})\,\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\,\boldsymbol{c}$.

[4] From here on the subscript O will be omitted from, D, although it should be remembered that the value of D is, indeed, origin dependent.

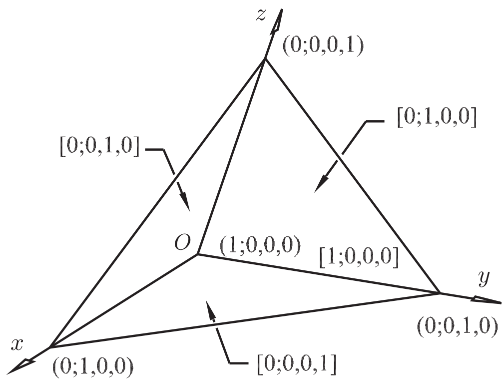

Figure 1.4 Planes and points on the tetrahedron of reference

The plane at infinity, $[1; 0, 0, 0]$, can also be thought of as the outer surface of a sphere of infinite radius centered at the origin of the reference system. This surface, with its infinite radius of curvature, would appear as a plane to an observer located at a point on this plane.

It is important to recognize that in three-dimensional projective space a point and a plane are analogous, or, more specifically, they are *dual*. Sets of four homogeneous coordinates $(w; x, y, z)$ and $[D; A, B, C]$ define points and planes in projective space, and an ∞^3 of both points and planes fill space. A plane can be drawn through three non-collinear points. This statement can be rephrased for a point by making the appropriate grammatical changes in order for it to make sense. The dual statement is that three non-parallel planes meet or intersect at a point. It is always possible to formulate (prove) a proposition (theorem) for one dual element and to simply state a corresponding proposition (theorem) for the corresponding dual element. A further two examples are *a line is the join of two points* which is dual to *a line is the meet of two planes* and *a line intersects a plane (which does not contain the line) in a point* which is dual to *a line and any point not on the line determine a plane*.

Finally, Klein considered the point and the plane to be equally important (see Klein [1939]). One can write their incidence relationship in the form

$$D\,w + A\,x + B\,y + C\,z = 0. \tag{1.16}$$

The point coordinates $(w; x, y, z)$ and plane coordinates $[D; A, B, C]$ play equal roles in (1.16). When the coordinates $[D; A, B, C]$ are specified, (1.16) expresses the condition that ∞^2 points lie on a plane. When the coordinates $(w; x, y, z)$ are specified, (1.16) expresses the condition that an ∞^2 (a bundle) of planes passes through the point.

1.3.1 Sample Problem

The coordinates of a plane are given as $[-8\,;\,2, -3, 5]$, where the first term has units of meters and the last three are dimensionless. Determine the coordinates of a point on the plane.

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The point to be determined will be written as

$$\boldsymbol{r} = r_x\,\mathbf{i} + r_y\,\mathbf{j} + r_z\,\mathbf{k}. \tag{1.17}$$

This point must satisfy the equation of the plane and, thus,

$$\boldsymbol{r} \cdot \boldsymbol{S} + D_O = 0, \tag{1.18}$$

$$2\,r_x - 3\,r_y + 5\,r_z - 8 = 0. \tag{1.19}$$

Equation (1.19) is one equation in three unknowns. Free choices may be made for two of the point coordinate values, and the third may then determined from (1.19). For example, choosing $r_x = 0$ and $r_y = -1$ and solving for r_z gives $r_z = 1$ and, thus, the point [1; 0, −1, 1], where the first term is dimensionless and the last three have units of meters, is on the plane.

1.3.2 Sample Problem

(i) *Determine the coordinates of the plane that passes through the three points* $P = (1; 3, 4, 1)$*,* $Q = (1; -1, 2, 4)$*, and* $R = (1; 3, 2, 2)$*, where the Cartesian coordinates are give in units of meters.*

The direction vector from point P to point Q may be written as

$$\boldsymbol{S}_{pq} = -4\,\boldsymbol{i} - 2\,\boldsymbol{j} + 3\,\boldsymbol{k}, \tag{1.20}$$

and the direction vector form point P to point R may be written as

$$\boldsymbol{S}_{pr} = -2\,\boldsymbol{j} + 1\,\boldsymbol{k}, \tag{1.21}$$

where these direction vectors are dimensionless. The vector perpendicular to the plane, $\boldsymbol{S}$, may be calculated as

$$\boldsymbol{S} = \boldsymbol{S}_{pq} \times \boldsymbol{S}_{pr} = 4\,\boldsymbol{i} + 4\,\boldsymbol{j} + 8\,\boldsymbol{k}, \tag{1.22}$$

where again the direction vector is dimensionless. From (1.7)

$$D_O = -\boldsymbol{r}_1 \cdot \boldsymbol{S}, \tag{1.23}$$

where $\boldsymbol{r}_1$ can be any of the three points on the plane. Using point P gives

$$D_O = -(3\boldsymbol{i} + 4\boldsymbol{j} + 1\boldsymbol{k}) \cdot (4\boldsymbol{i} + 4\boldsymbol{j} + 8\boldsymbol{k}) = -36\ \text{m}. \tag{1.24}$$

Thus, the coordinates of the plane may be written as

$$[D_O; \boldsymbol{S}] = [-36; 4, 4, 8], \tag{1.25}$$

where the first component has units of meters and the remaining three are dimensionless.

(ii) *Determine the distance of this plane from the origin.*

The vector from the origin that is perpendicular to the plane can be determined from (1.13) as

$$\boldsymbol{p} = \frac{-D_O \boldsymbol{S}}{\boldsymbol{S} \cdot \boldsymbol{S}} = \frac{36\,(4\boldsymbol{i} + 4\boldsymbol{j} + 8\boldsymbol{k})}{4^2 + 4^2 + 8^2} = 1.5\boldsymbol{i} + 1.5\boldsymbol{j} + 3\boldsymbol{k}, \tag{1.26}$$

where the components of the vector have units of meters. The distance of the plane from the origin is equal to the magnitude of this vector, i.e.,

$$|\boldsymbol{p}| = 3.674 \text{ m}. \tag{1.27}$$

(iii) *Determine the Z coordinate of the point on the plane whose x and y coordinate values are 6 m and −10 m, respectively.*

The point on the plane can be written as

$$\boldsymbol{r}_4 = 6\boldsymbol{i} - 10\boldsymbol{j} + z_4\boldsymbol{k}. \tag{1.28}$$

Inserting this point into (1.6), the equation of the plane, yields

$$\boldsymbol{r}_4 \cdot \boldsymbol{S} + D_O = 0,$$

$$(6\boldsymbol{i} - 10\boldsymbol{j} + z_4\boldsymbol{k}) \cdot (4\boldsymbol{i} + 4\boldsymbol{j} + 8\boldsymbol{k}) - 36 = 0. \tag{1.29}$$

Solving for z_4 gives

$$z_4 = 6.5 \text{ m}. \tag{1.30}$$

1.4 Projection of a Point onto a Plane

Often it is desired to find the point on a plane that is closest to a given point. Figure 1.5 shows a plane whose coordinates are given as $[D_O;\ \boldsymbol{S}]$ and a given point Q_1 whose coordinates are defined by the vector $\boldsymbol{r}_1$. The objective is to determine the coordinates of the point Q_p, which is the point on the plane that is closest to the point Q_1. The coordinates of point Q_p are defined by the vector $\boldsymbol{r}_p$.

The vector from point Q_p to Q_1 is labeled as $\boldsymbol{r}_{p\to 1}$, and it is apparent that this vector must be parallel to $\boldsymbol{S}$. Thus, $\boldsymbol{r}_{p\to 1}$ is some scalar multiple of $\boldsymbol{S}$ and can be written as

$$\boldsymbol{r}_{p\to 1} = d\,\boldsymbol{S}. \tag{1.31}$$

Further it can be seen from the figure that

$$\boldsymbol{r}_p = \boldsymbol{r}_1 - \boldsymbol{r}_{p\to 1} = \boldsymbol{r}_1 - d\,\boldsymbol{S}. \tag{1.32}$$

Since point Q_p lies in the given plane, the vector $\boldsymbol{r}_p$ must satisfy the equation of the plane and, thus,

$$(\boldsymbol{r}_1 - d\,\boldsymbol{S}) \cdot \boldsymbol{S} + D_O = 0. \tag{1.33}$$

Solving for d gives

$$d = \frac{\boldsymbol{r}_1 \cdot \boldsymbol{S} + D_O}{\boldsymbol{S} \cdot \boldsymbol{S}}. \tag{1.34}$$

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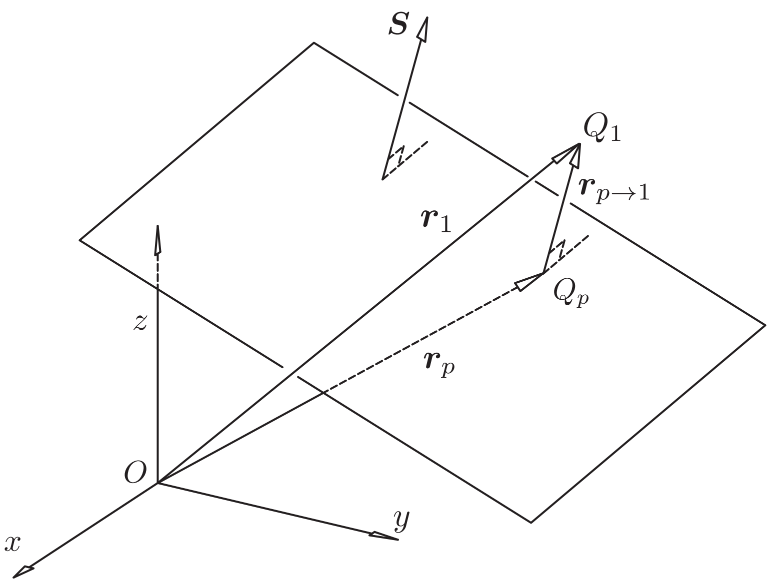

Figure 1.5 Projection of a point onto a plane

Substituting (1.34) into (1.32) gives the result

$$r_p = r_1 - \left(\frac{\boldsymbol{r}_1 \cdot \boldsymbol{S} + D_O}{\boldsymbol{S} \cdot \boldsymbol{S}} \right) \boldsymbol{S}. \tag{1.35}$$

It is of note that the distance from point Q_1 to the plane is obtained as the magnitude of (1.31) as

$$|\boldsymbol{r}_{p \to 1}| = d\,|\boldsymbol{S}|, \tag{1.36}$$

and a positive value for d indicates that the point lies on the side of the plane pointed to by the direction of $\boldsymbol{S}$.

1.5 The Equation of a Line

The join of two distinct points $\boldsymbol{r}_1$ (x_1, y_1, z_1) and $\boldsymbol{r}_2$ (x_2, y_2, z_2), where the elements of $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ have units of length, determine a line. The vector $\boldsymbol{S}$ whose direction is along the line may be written as

$$\boldsymbol{S} = \boldsymbol{r}_2 - \boldsymbol{r}_1. \tag{1.37}$$

Direction is a unitless concept and, thus, the elements of $\boldsymbol{S}$ are dimensionless. The vector $\boldsymbol{S}$ may alternatively be expressed as

$$\boldsymbol{S} = L\,\boldsymbol{i} + M\,\boldsymbol{j} + N\,\boldsymbol{k}, \tag{1.38}$$

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where $L = x_2 - x_1$, $M = y_2 - y_1$, and $N = z_2 - z_1$ are defined as the dimensionless direction ratios. From (1.38) the direction ratios (L, M, N) are related to $|\boldsymbol{S}|$ by

$$L^2 + M^2 + N^2 = |\boldsymbol{S}|^2. \tag{1.39}$$

Often, L, M, and N are expressed in the form

$$L = \frac{x_2 - x_1}{|\boldsymbol{S}|}, \qquad M = \frac{y_2 - y_1}{|\boldsymbol{S}|}, \qquad N = \frac{z_2 - z_1}{|\boldsymbol{S}|}, \tag{1.40}$$

which are unit direction ratios or direction cosines of the line. In this case, (1.39) reduces to

$$L^2 + M^2 + N^2 = 1. \tag{1.41}$$

Letting $\boldsymbol{r}$ designate a vector from the origin to any general point on the line (see Figure 1.6), it is apparent that the vector $\boldsymbol{r} - \boldsymbol{r}_1$ is parallel to $\boldsymbol{S}$. Thus, it may be written that

$$(\boldsymbol{r} - \boldsymbol{r}_1) \times \boldsymbol{S} = \boldsymbol{0}. \tag{1.42}$$

This can be expressed in the form

$$\boldsymbol{r} \times \boldsymbol{S} = \boldsymbol{S}_{OL}, \tag{1.43}$$

where

$$\boldsymbol{S}_{OL} = \boldsymbol{r}_1 \times \boldsymbol{S} \tag{1.44}$$

is the moment of the line about the origin O, which is clearly origin dependent. The elements of the vector $\boldsymbol{S}_{OL}$ have units of length. Further, since $\boldsymbol{S}_{OL} = \boldsymbol{r}_1 \times \boldsymbol{S}$, the vectors $\boldsymbol{S}$ and $\boldsymbol{S}_{OL}$ are perpendicular and, as such, satisfy the orthogonality condition

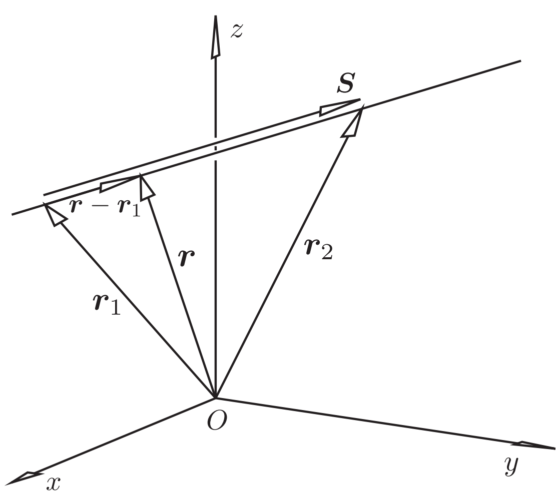

Figure 1.6 Determination of a line

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$$\boldsymbol{S} \cdot \boldsymbol{S}_{OL} = 0. \tag{1.45}$$

The coordinates of a line will be written as $\{\boldsymbol{S}; \boldsymbol{S}_{OL}\}$ and will be referred to as the Plücker coordinates[5] of the line. The semi-colon is introduced to signify that the dimensions differ between the first three coordinates and the last three. The coordinates $\{\boldsymbol{S}; \boldsymbol{S}_{OL}\}$ are homogeneous since from (1.43) the coordinates $\{\lambda \boldsymbol{S}; \lambda \boldsymbol{S}_{OL}\}$, where λ is a non-zero scalar, determine the same line.

Expanding (1.44) yields

$$\boldsymbol{S}_{OL} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x_1 & y_1 & z_1 \\ L & M & N \end{vmatrix}, \tag{1.46}$$

which can be expressed in the form

$$\boldsymbol{S}_{OL} = P\,\boldsymbol{i} + Q\,\boldsymbol{j} + R\,\boldsymbol{k}, \tag{1.47}$$

where

$$\begin{aligned} P &= y_1\,N - z_1\,M, \\ Q &= z_1\,L - x_1\,N, \\ R &= x_1\,M - y_1\,L. \end{aligned} \tag{1.48}$$

From (1.38) and (1.47) the orthogonality condition $\boldsymbol{S} \cdot \boldsymbol{S}_{OL} = 0$ can be expressed in the form

$$L\,P + M\,Q + N\,R = 0. \tag{1.49}$$

The six Plücker coordinates of the line $\{L, M, N; P, Q, R\}$ are illustrated in Figure 1.7. Note that for the case shown in the figure, L will have a negative value. Unitized coordinates for a line can be obtained by imposing the constraint that $|\boldsymbol{S}| = 1$. The Plücker coordinates must thus satisfy equations (1.41) and (1.49) and, therefore, only four of the six scalars L, M, N, P, Q, and R are independent. It follows that there are ∞^4 lines in space.[6]

Equations (1.37) and (1.44) can be used to obtain the Plücker coordinates of a line when given two points on the line. It is also important to be able to determine a point on the line when given the Plücker coordinates of the line. Suppose that the Plücker coordinates $\{L, M, N; P, Q, R\}$ of a line are given. Let $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ represent a vector to some point on the line. Thus, $\boldsymbol{r}$ must satisfy (1.43) and

$$(x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k}) \times (L\,\boldsymbol{i} + M\,\boldsymbol{j} + N\,\boldsymbol{k}) = P\,\boldsymbol{i} + Q\,\boldsymbol{j} + R\,\boldsymbol{k}. \tag{1.50}$$

Restating the cross product on the left side of this equation gives

$$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x & y & z \\ L & M & N \end{vmatrix} = P\,\boldsymbol{i} + Q\,\boldsymbol{j} + R\,\boldsymbol{k}. \tag{1.51}$$

[5] See Plücker (1865).

[6] Systems of lines and their properties are described in Hunt (1978) (pp. 310–330), which contains an extensive bibliography on the subject. A line series (∞^1), congruence (∞^2), and complex (∞^3) are discussed.

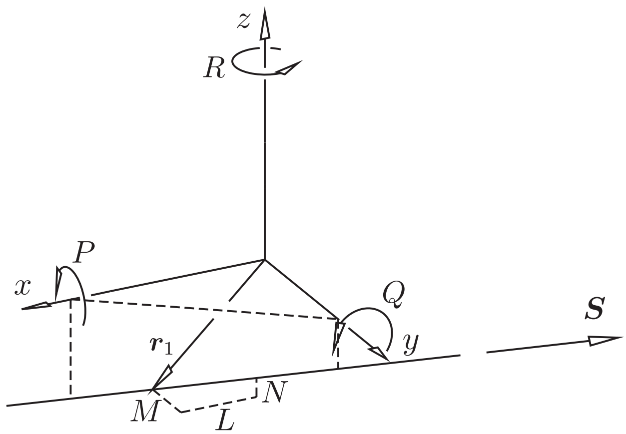

Figure 1.7 Plücker line coordinates

Equating the components of the $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ vectors yields the three scalar equations

$$\begin{aligned} yN - zM &= P, \\ zL - xN &= Q, \\ xM - yL &= R, \end{aligned} \tag{1.52}$$

which may be written in matrix format as

$$\begin{bmatrix} 0 & N & -M \\ -N & 0 & L \\ M & -L & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} P \\ Q \\ R \end{bmatrix}, \tag{1.53}$$

where the 3×3 coefficient matrix is skew symmetric. It can be shown that the rank of this coefficient matrix is two and the equations of (1.52) are, therefore, linearly dependent. As such, an infinite number of solutions exist for x, y, and z (corresponding to the infinite number of points on the line $\{L, M, N; P, Q, R\}$). Any arbitrary value for x may be selected, and corresponding values for y and z may be determined from the last two equations of (1.52) as

$$y = \frac{xM - R}{L}, \quad z = \frac{xN + Q}{L}. \tag{1.54}$$

Note that if $L = 0$, then x is a constant given by either the second or third equation of (1.52). The value for y or z could then be arbitrarily selected and the other calculated from the first equation of (1.52).

The distance of a line from the origin is determined by the length of the vector $\boldsymbol{p}$, which originates at O and terminates at a point on the line such that the direction of $\boldsymbol{p}$ is perpendicular to the direction of the line, $\boldsymbol{S}$. The vector $\boldsymbol{p}$ must also satisfy (1.43) and, therefore,

$$\boldsymbol{p} \cdot \boldsymbol{S} = 0, \tag{1.55}$$

$$\boldsymbol{p} \times \boldsymbol{S} = \boldsymbol{S}_{OL}. \tag{1.56}$$

Performing a cross product of $\boldsymbol{S}$ with (1.56) yields

$$\boldsymbol{S} \times (\boldsymbol{p} \times \boldsymbol{S}) = \boldsymbol{S} \times \boldsymbol{S}_{OL}. \tag{1.57}$$

Expanding the left side of (1.57) gives

$$(\boldsymbol{S} \cdot \boldsymbol{S})\boldsymbol{p} - (\boldsymbol{p} \cdot \boldsymbol{S})\boldsymbol{S} = \boldsymbol{S} \times \boldsymbol{S}_{OL}. \tag{1.58}$$

Since $\boldsymbol{p}$ is perpendicular to $\boldsymbol{S}$, $\boldsymbol{p} \cdot \boldsymbol{S} = 0$. Substituting this into (1.58) and solving for $\boldsymbol{p}$ gives

$$\boldsymbol{p} = \frac{\boldsymbol{S} \times \boldsymbol{S}_{OL}}{\boldsymbol{S} \cdot \boldsymbol{S}}. \tag{1.59}$$

A vector $\boldsymbol{e}$ is now defined as a unit vector perpendicular to $\boldsymbol{S}$ and $\boldsymbol{S}_{OL}$ and may be written as

$$\boldsymbol{e} = \frac{\boldsymbol{S} \times \boldsymbol{S}_{OL}}{|\boldsymbol{S} \times \boldsymbol{S}_{OL}|}. \tag{1.60}$$

Substituting $(\boldsymbol{S} \times \boldsymbol{S}_{OL}) = |\boldsymbol{S} \times \boldsymbol{S}_{OL}|\boldsymbol{e}$ in (1.59) yields

$$\boldsymbol{p} = \frac{|\boldsymbol{S} \times \boldsymbol{S}_{OL}|}{\boldsymbol{S} \cdot \boldsymbol{S}}\boldsymbol{e}. \tag{1.61}$$

The magnitude of the cross product in the numerator of (1.61) is simply $|\boldsymbol{S}||\boldsymbol{S}_{OL}| \sin \frac{\pi}{2}$. The scalar product in the denominator will equal the square of the magnitude of $\boldsymbol{S}$. Equation (1.61) may, thus, be written as

$$\boldsymbol{p} = \frac{|\boldsymbol{S}||\boldsymbol{S}_{OL}|}{|\boldsymbol{S}||\boldsymbol{S}|}\boldsymbol{e} = \frac{|\boldsymbol{S}_{OL}|}{|\boldsymbol{S}|}\boldsymbol{e}. \tag{1.62}$$

Therefore, the distance of the line from the origin may be determined as

$$|\boldsymbol{p}| = \frac{|\boldsymbol{S}_{OL}|}{|\boldsymbol{S}|}. \tag{1.63}$$

When $\boldsymbol{S}_{OL} = \boldsymbol{0}$, $|\boldsymbol{p}| = 0$ and the line passes through the origin and its coordinates are $\{\boldsymbol{S}; \boldsymbol{0}\}$. When $\boldsymbol{S} = \boldsymbol{0}$, $|\boldsymbol{p}| = \infty$ and the line is a line at infinity. In this case, the direction of the moment vector defines the line at infinity, and its coordinates are written as $\{\boldsymbol{0}; \boldsymbol{S}\}$, where $\boldsymbol{S}$ is dimensionless. Further, the coordinates of the line at infinity are origin independent. This line at infinity lies in the plane at infinity. If the plane at infinity is thought of as the surface of a sphere of infinite radius, then a line in the plane at infinity can be thought of as a circle on this sphere. For the line $\{\boldsymbol{0}; \boldsymbol{S}\}$ which lies at infinity, the direction of the moment, $\boldsymbol{S}$, must be perpendicular to the direction of the line, as is the case for all lines (see Figure 1.8). Thus, for the line $\{\boldsymbol{0}; \boldsymbol{S}\}$ the direction of the moment vector $\boldsymbol{S}$ can be thought of as being perpendicular to the "plane" defined by the "circle of infinite radius".

The Plücker coordinates for the line joining the points with coordinates $(1; x_1, y_1, z_1)$ and $(1; x_2, y_2, z_2)$ were elegantly expressed by Grassmann[7] by the six 2×2 determinants of the array

[7] On pg. 20, Klein (1939) gave immense credit to Grassmann (1862) regarding his extension theory.

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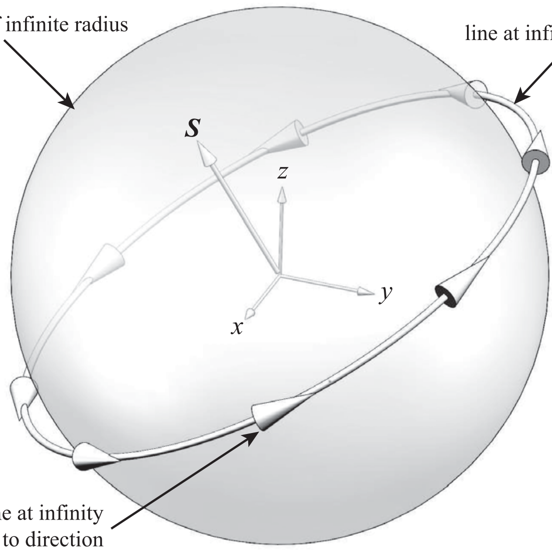

Figure 1.8 Conceptualization of the line at infinity, $\{\mathbf{0}; \boldsymbol{S}\}$.

$$\begin{bmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \end{bmatrix} \tag{1.64}$$

as

$$L = \begin{vmatrix} 1 & x_1 \\ 1 & x_2 \end{vmatrix}, \; M = \begin{vmatrix} 1 & y_1 \\ 1 & y_2 \end{vmatrix}, \; N = \begin{vmatrix} 1 & z_1 \\ 1 & z_2 \end{vmatrix},$$

$$P = \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}, \; Q = \begin{vmatrix} z_1 & x_1 \\ z_2 & x_2 \end{vmatrix}, \; R = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}. \tag{1.65}$$

It is a useful exercise to deduce the Plücker coordinates of the six edges of the tetrahedron of reference using (1.65), using the pairs of coordinates labeling the vertices of the tetrahedron of reference shown in Figure 1.2.

1.5.1 Sample Problem

(i) *Determine the Plücker coordinates of the line that passes through the points* $\boldsymbol{r}_1 = 3\boldsymbol{i} + 5\boldsymbol{j} - 6\boldsymbol{k}$ *and* $\boldsymbol{r}_2 = 6\boldsymbol{i} - 5\boldsymbol{j} + 2\boldsymbol{k}$, where the point coordinates are given in units of meters.

The direction of the line is obtained from (1.37) as

$$\boldsymbol{S} = \boldsymbol{r}_2 - \boldsymbol{r}_1 = 3\boldsymbol{i} - 10\boldsymbol{j} + 8\boldsymbol{k}, \tag{1.66}$$

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where the elements of the direction vector $\boldsymbol{S}$ are dimensionless. The moment of the line is calculated from (1.44) as

$$\boldsymbol{S}_{OL} = \boldsymbol{r}_1 \times \boldsymbol{S} = -20\boldsymbol{i} - 42\boldsymbol{j} - 45\boldsymbol{k}, \tag{1.67}$$

where the elements of $\boldsymbol{S}_{OL}$ have units of meters. The Plücker coordinates of the line may now be written as

$$\{3, -10, 8; -20 \text{ m}, -42 \text{ m}, -45 \text{ m}\}. \tag{1.68}$$

(ii) *Determine the perpendicular distance of this line from the origin.*

The coordinates of the point on the line that is closest to the origin can be determined from (1.59) as

$$\begin{aligned} \boldsymbol{p} = \frac{\boldsymbol{S} \times \boldsymbol{S}_{OL}}{\boldsymbol{S} \cdot \boldsymbol{S}} &= \frac{(3\boldsymbol{i} - 10\boldsymbol{j} + 8\boldsymbol{k}) \times (-20\boldsymbol{i} - 42\boldsymbol{j} - 45\boldsymbol{k})}{(3\boldsymbol{i} - 10\boldsymbol{j} + 8\boldsymbol{k}) \cdot (3\boldsymbol{i} - 10\boldsymbol{j} + 8\boldsymbol{k})} \\ &= 4.543\boldsymbol{i} - 0.145\boldsymbol{j} - 1.884\boldsymbol{k}, \end{aligned} \tag{1.69}$$

where the elements of the vector $\boldsymbol{p}$ have units of meters. The magnitude of $\boldsymbol{p}$ is the distance of the perpendicular distance of the line from the origin and is calculated as

$$|\boldsymbol{p}| = 4.921 \text{ m}. \tag{1.70}$$

(iii) *Determine the coordinates of another point on the line.*

An arbitrary point on the line will be referred to as $\boldsymbol{r}$, where

$$\boldsymbol{r} = x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k}. \tag{1.71}$$

The equation of the line was presented in (1.43) and may be written as

$$\begin{aligned} \boldsymbol{r} \times \boldsymbol{S} &= \boldsymbol{S}_{OL}, \\ (x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}) \times (3\boldsymbol{i} - 10\boldsymbol{j} + 8\boldsymbol{k}) &= -20\boldsymbol{i} - 42\boldsymbol{j} + 45\boldsymbol{k}, \end{aligned} \tag{1.72}$$

where the terms on the right side of the equation and the unknowns x, y, and z have units of meters. Expanding the cross product on the left hand side of this equation gives

$$(8y + 10z)\boldsymbol{i} + (3z - 8x)\boldsymbol{j} + (-10x - 3y)\boldsymbol{k} = -20\boldsymbol{i} - 42\boldsymbol{j} + 45\boldsymbol{k}. \tag{1.73}$$

Equating the $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ components of this equation yields the three scalar equations

$$\begin{aligned} 8y + 10z &= -20, \\ 3z - 8x &= -42, \\ -10x - 3y &= -45. \end{aligned} \tag{1.74}$$

Multiplying the first equation by 3 and the second equation by -10 and adding gives

$$24y + 80x = 360, \tag{1.75}$$

which, when divided by -8, is identical to the third equation of (1.74) and, thus, the three scalar equations are linearly dependent. This was to be expected, since there are

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an infinity of points on a line. A free choice can be made for one of the coordinates, say $x = 0$, and the other two parameters calculated. In this case, $y = 15$ m and $z = -14$ m.

1.6 Two Planes Determine a Line

The coordinates of two planes are given as $[D_{O1}; A_1, B_1, C_1]$ and $[D_{O2}; A_2, B_2, C_2]$. From (1.6), the equations of each plane may be written as

$$\boldsymbol{r}_1 \cdot \boldsymbol{S}_1 + D_{O1} = 0, \tag{1.76}$$

$$\boldsymbol{r}_2 \cdot \boldsymbol{S}_2 + D_{O2} = 0. \tag{1.77}$$

where $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are vectors to any point on the first and second plane, respectively, and

$$\boldsymbol{S}_1 = A_1\,\boldsymbol{i} + B_1\,\boldsymbol{j} + C_1\,\boldsymbol{k}, \tag{1.78}$$

$$\boldsymbol{S}_2 = A_2\,\boldsymbol{i} + B_2\,\boldsymbol{j} + C_2\,\boldsymbol{k}. \tag{1.79}$$

The equation of the line of intersection of these two planes (see Figure 1.9) will now be determined.

The line of intersection is perpendicular to each of the vectors $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ and it is, therefore, parallel to $\boldsymbol{S}_1 \times \boldsymbol{S}_2$. Expanding the triple vector cross product $\boldsymbol{r} \times (\boldsymbol{S}_1 \times \boldsymbol{S}_2)$, where $\boldsymbol{r}$ is a vector to any point on the line of intersection, yields the vector equation for the line and is written as

$$\boldsymbol{r} \times (\boldsymbol{S}_1 \times \boldsymbol{S}_2) = (\boldsymbol{r} \cdot \boldsymbol{S}_2)\boldsymbol{S}_1 - (\boldsymbol{r} \cdot \boldsymbol{S}_1)\boldsymbol{S}_2. \tag{1.80}$$

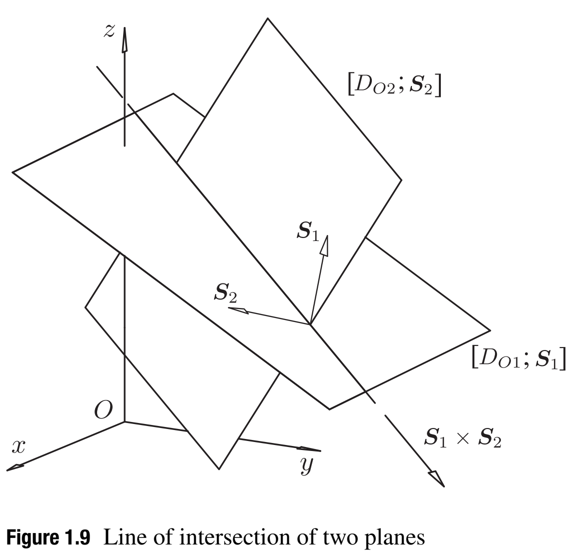

Figure 1.9 Line of intersection of two planes

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Since $\boldsymbol{r}$ must lie on both of the planes, (1.76) and (1.77) may be substituted into (1.80) to give

$$\boldsymbol{r} \times (\boldsymbol{S}_1 \times \boldsymbol{S}_2) = (-D_{O2})\boldsymbol{S}_1 - (-D_{O1})\boldsymbol{S}_2. \tag{1.81}$$

The coordinates of the line of intersection of the two planes are obtained from (1.81) as $\{\boldsymbol{S}_1 \times \boldsymbol{S}_2; D_{O1}\, \boldsymbol{S}_2 - D_{O2}\, \boldsymbol{S}_1\}$. It is apparent that these coordinates do, indeed, represent a line, as the direction of the line, $\boldsymbol{S}_1 \times \boldsymbol{S}_2$, is perpendicular to the moment of the line, $D_{O1}\, \boldsymbol{S}_2 - D_{O2}\, \boldsymbol{S}_1$. When $\boldsymbol{S}_1 \times \boldsymbol{S}_2 = \boldsymbol{0}$, the planes are parallel, and they intersect in a line in the plane at infinity with coordinates $\{\boldsymbol{0}; D_{O1}\, \boldsymbol{S}_2 - D_{O2}\, \boldsymbol{S}_1\}$. Since in this case $\mathbf{S}_1 \parallel \mathbf{S}_2$, the line at infinity may simply be written as $\{\boldsymbol{0}; \mathbf{S}_1\}$.

By substituting (1.78) and (1.79) into the Plücker coordinates of the line of intersection and evaluating the cross product $\boldsymbol{S}_1 \times \boldsymbol{S}_2$, these coordinates may be written as $\{L, M, N; P, Q, R\}$, where

$$\begin{aligned} &L = B_1\, C_2 - B_2\, C_1, \quad M = C_1\, A_2 - C_2\, A_1, \quad N = A_1\, B_2 - A_2\, B_1, \\ &P = D_{O1}\, A_2 - D_{O2}\, A_1, \quad Q = D_{O1}\, B_2 - D_{O2}\, B_1, \quad R = D_{O1}\, C_2 - D_{O2}\, C_1. \end{aligned} \tag{1.82}$$

The Plücker coordinates in (1.82) may be obtained directly using Grassmann's determinant principle by expressing the coordinates of the planes in the 2×4 array

$$\begin{bmatrix} D_{O1} & A_1 & B_1 & C_1 \\ D_{O2} & A_2 & B_2 & C_2 \end{bmatrix} \tag{1.83}$$

and by expanding the sequence of determinants

$$\begin{aligned} P &= \begin{vmatrix} D_{O1} & A_1 \\ D_{O2} & A_2 \end{vmatrix}, \quad Q = \begin{vmatrix} D_{O1} & B_1 \\ D_{O2} & B_2 \end{vmatrix}, \quad R = \begin{vmatrix} D_{O1} & C_1 \\ D_{O2} & C_2 \end{vmatrix}, \\ L &= \begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix}, \quad M = \begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix}, \quad N = \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}. \end{aligned} \tag{1.84}$$

The array $\{P, Q, R; L, M, N\}$ is known as the *axis coordinates* for the line determined by the meet of two planes. Clearly, the line can be considered as the *axis* of a pencil of planes. On the other hand, the array $\{L, M, N; P, Q, R\}$ is known as the *ray coordinates* for a line. Clearly, the line can be considered as a *ray* of light joining any two distinct points on the line. The line can thus be formed by pairs of dual elements, i.e., points or planes. Because of this, a line in three-dimensional space is considered to be dual with itself or self-dual.

A simple notation will now be introduced that will be used to distinguish between the ray and axis coordinates in later chapters. The ray and axis coordinates will be designated by lower and upper case symbols $\hat{s} = \{\boldsymbol{S}; \boldsymbol{S}_{OL}\}$ and $\hat{\boldsymbol{S}} = \{\boldsymbol{S}_{OL}; \boldsymbol{S}\}$, respectively. Also, a line, whether written in ray or axis coordinates, will be designated as **\$**. When the same line **\$** is determined by the meet of two planes $[D_{O1}; A_1, B_1, C_1]$ and $[D_{O2}; A_2, B_2, C_2]$ and by the join of two points $(1; x_1, y_1, z_1)$ and $(1; x_2, y_2, z_2)$, as shown in Figure 1.10, then axis coordinates $\{\boldsymbol{S}_{OL}; \boldsymbol{S}\}$ are obtained by counting the 2×2 determinants of

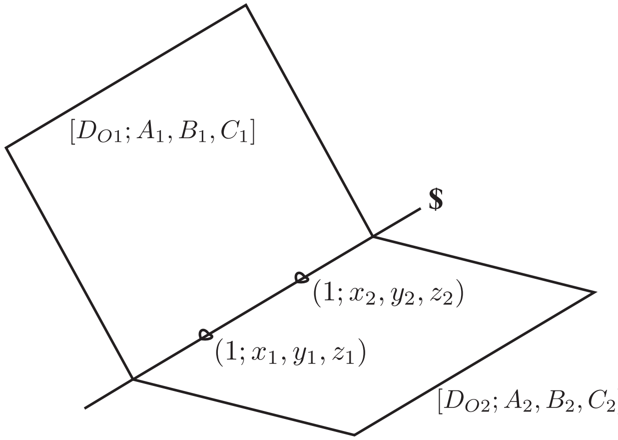

Figure 1.10 A line determined by two planes or two points

$$\begin{bmatrix} D_{O1} & A_1 & B_1 & C_1 \\ D_{O2} & A_2 & B_2 & C_2 \end{bmatrix}, \tag{1.85}$$

whereas the ray coordinates $\{\boldsymbol{S}; \boldsymbol{S}_{OL}\}$ are obtained by counting the 2×2 determinants of

$$\begin{bmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \end{bmatrix}. \tag{1.86}$$

The two sets of coordinates are identical to a scalar multiple σ and

$$\hat{\boldsymbol{s}} = \sigma \boldsymbol{\Delta} \hat{\boldsymbol{S}}, \tag{1.87}$$

where $\boldsymbol{\Delta}$ is a 6×6 matrix that converts axis coordinates to ray coordinates and vice versa and is given by

$$\boldsymbol{\Delta} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}. \tag{1.88}$$

The matrix $\boldsymbol{\Delta}$ essentially exchanges the first three elements with the last three elements of the column vector $\hat{\boldsymbol{S}}$ and is often written as

$$\boldsymbol{\Delta} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I}_3 \\ \boldsymbol{I}_3 & \boldsymbol{0} \end{bmatrix}, \tag{1.89}$$

where, in this case, $\boldsymbol{0}$ represents a 3×3 matrix with all terms equal to zero, and $\boldsymbol{I}_3$ represents a 3×3 identity matrix.

As an example, consider a pair of planes with the coordinates $[1; -4, 12, 3]$ and $[1; 3, 4, -12]$, where the first component has units of meters and the last three

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components are dimensionless. The axis coordinates for the line of intersection **\$** are obtained from the 2×2 determinants of the array

$$\begin{bmatrix} 1 & -4 & 12 & 3 \\ 1 & 3 & 4 & -12 \end{bmatrix} \tag{1.90}$$

and

$$\begin{aligned} P &= \begin{vmatrix} 1 & -4 \\ 1 & 3 \end{vmatrix} = 7 \text{ m}, \quad Q = \begin{vmatrix} 1 & 12 \\ 1 & 4 \end{vmatrix} = -8 \text{ m}, \\ R &= \begin{vmatrix} 1 & 3 \\ 1 & -12 \end{vmatrix} = -15 \text{ m}, \quad L = \begin{vmatrix} 12 & 3 \\ 4 & -12 \end{vmatrix} = -156, \\ M &= \begin{vmatrix} 3 & -4 \\ -12 & 3 \end{vmatrix} = -39, \quad N = \begin{vmatrix} -4 & 12 \\ 3 & 4 \end{vmatrix} = -52. \end{aligned} \tag{1.91}$$

The numerical results can be verified by using the orthogonality condition of (1.49). The axis coordinates of the line of intersection may thus be written as $\hat{\boldsymbol{S}} = \{\boldsymbol{S}_{OL}; \boldsymbol{S}\}$, where

$$\begin{aligned} \boldsymbol{S}_{OL} &= (7\boldsymbol{i} - 8\boldsymbol{j} - 15\boldsymbol{k}) \text{ m} \\ \boldsymbol{S} &= -156\boldsymbol{i} - 39\boldsymbol{j} - 52\boldsymbol{k}. \end{aligned} \tag{1.92}$$

Two points on this line will now be determined. Writing a point on the line as $\boldsymbol{r} = x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k}$, y and z may be determined for an arbitrary value of x from (1.54). For $x = 1$ m, y and z are evaluated as $\frac{2}{13}$ m and $\frac{5}{13}$ m. For $x = -1$ m, y and z are evaluated as $-\frac{9}{26}$ m and $-\frac{11}{39}$ m. The ray coordinates for the line **\$** can now be evaluated from the 2×2 determinants of the array

$$\begin{bmatrix} 1 & 1 & \frac{2}{13} & \frac{5}{13} \\ 1 & -1 & -\frac{9}{26} & -\frac{11}{39} \end{bmatrix} \tag{1.93}$$

and

$$\begin{aligned} L &= \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2, \quad M = \begin{vmatrix} 1 & \frac{2}{13} \\ 1 & -\frac{9}{26} \end{vmatrix} = -\frac{1}{2}, \\ N &= \begin{vmatrix} 1 & \frac{5}{13} \\ 1 & -\frac{11}{39} \end{vmatrix} = -\frac{2}{3}, \quad P = \begin{vmatrix} \frac{2}{13} & \frac{5}{13} \\ -\frac{9}{26} & -\frac{11}{39} \end{vmatrix} = \frac{7}{78} \text{ m}, \\ Q &= \begin{vmatrix} \frac{5}{13} & 1 \\ -\frac{11}{39} & -1 \end{vmatrix} = -\frac{4}{39} \text{ m}, \quad R = \begin{vmatrix} 1 & \frac{2}{13} \\ -1 & -\frac{9}{26} \end{vmatrix} = -\frac{5}{26} \text{ m}. \end{aligned} \tag{1.94}$$

These numerical results satisfy the orthogonality condition of (1.49). The ray coordinates of the line of intersection **\$** may now be written as $\hat{s} = \{\boldsymbol{S}; \boldsymbol{S}_{OL}\}$, where

$$\begin{aligned} \boldsymbol{S} &= -2\boldsymbol{i} - \frac{1}{2}\boldsymbol{j} - \frac{2}{3}\boldsymbol{k}, \\ \boldsymbol{S}_{OL} &= \left(\frac{7}{78}\boldsymbol{i} - \frac{4}{39}\boldsymbol{j} - \frac{5}{26}\boldsymbol{k}\right) \text{ m}. \end{aligned} \tag{1.95}$$

Comparing (1.92) and (1.95), it is apparent that $\hat{s} = \sigma\Delta\hat{S}$, where for this example $\sigma = \frac{1}{78}$.

1.7 The Pencil of Planes through a Line

1.7.1 The Plane Defined by a Line and a Point

The plane containing the line $\{\boldsymbol{S}_1; \boldsymbol{S}_{OL1}\}$ can be rotated about the line, and, in this way, a pencil or single infinity of planes is generated. Imposing the constraint that the plane passes through a point A with position vector $\boldsymbol{r}_0$ yields a unique plane (see Figure 1.11). The direction vectors $\boldsymbol{S}_1$, $(\boldsymbol{r}_1 - \boldsymbol{r}_0)$, and $(\boldsymbol{r} - \boldsymbol{r}_0)$, where $\boldsymbol{r}_1$ is a vector to some point on the line and $\boldsymbol{r}$ is a vector to any point on the plane, are clearly coplanar and, therefore,

$$(\boldsymbol{r} - \boldsymbol{r}_0) \cdot (\boldsymbol{r}_1 - \boldsymbol{r}_0) \times \boldsymbol{S}_1 = 0. \tag{1.96}$$

There is no ambiguity as to the order of operations in the above equation, since a meaningful result occurs only if the cross product is performed prior to the scalar product. Expanding (1.96), regrouping terms, and making the substitution $\boldsymbol{r}_1 \times \boldsymbol{S}_1 = \boldsymbol{S}_{OL1}$ gives

$$\boldsymbol{r} \cdot (\boldsymbol{S}_{OL1} - \boldsymbol{r}_0 \times \boldsymbol{S}_1) - \boldsymbol{r}_0 \cdot (\boldsymbol{S}_{OL1} - \boldsymbol{r}_0 \times \boldsymbol{S}_1) = 0, \tag{1.97}$$

which reduces to

$$\boldsymbol{r} \cdot (\boldsymbol{S}_{OL1} - \boldsymbol{r}_0 \times \boldsymbol{S}_1) - \boldsymbol{r}_0 \cdot \boldsymbol{S}_{OL1} = 0. \tag{1.98}$$

From (1.98), the homogeneous coordinates of the plane are, thus, $[-\boldsymbol{r}_0 \cdot \boldsymbol{S}_{OL1}; \boldsymbol{S}_{OL1} - \boldsymbol{r}_0 \times \boldsymbol{S}_1]$. If point A happened to lie on the line, then $\boldsymbol{r}_0 \times \boldsymbol{S}_1 = \boldsymbol{S}_{OL1}$, and the point and line do not define a unique plane. For this case, substituting $\boldsymbol{S}_{OL1} = \boldsymbol{r}_0 \times \boldsymbol{S}_1$ into (1.98) causes the equation to vanish for all $\boldsymbol{r}$. When $\boldsymbol{r}_0 = 0$, (1.98) reduces to

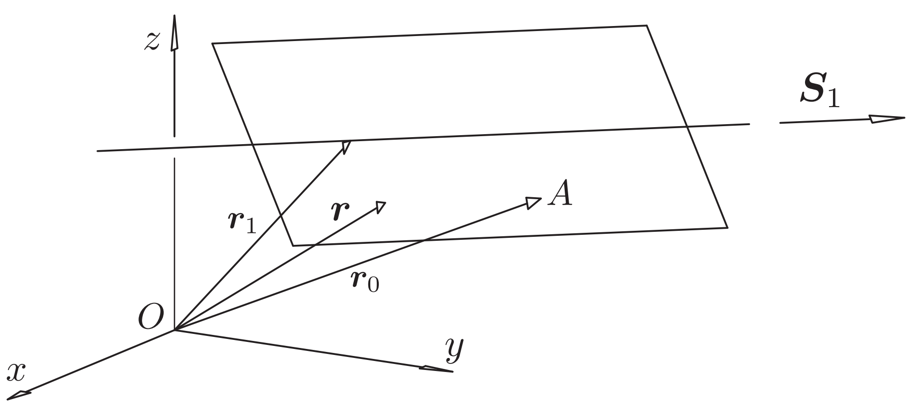

Figure 1.11 Plane determined by a point and a line

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$$\boldsymbol{r} \cdot \boldsymbol{S}_{OL1} = 0. \tag{1.99}$$

This simple result is the equation of a plane that passes through the origin and that contains the line $\{\boldsymbol{S}_1; \boldsymbol{S}_{OL1}\}$.

1.7.2 The Plane That Contains a Line and Is Parallel to a Second Line

A unique plane containing the line with coordinates $\{\boldsymbol{S}_1; \boldsymbol{S}_{OL1}\}$ can also be determined by imposing the constraint that the plane be parallel to or contain a second line $\{\boldsymbol{S}_2; \boldsymbol{S}_{OL2}\}$ (see Figure 1.12). In this case, the vector $(\boldsymbol{r} - \boldsymbol{r}_1)$ lies in the plane, and the vector $\boldsymbol{S}_1 \times \boldsymbol{S}_2$ is normal to the plane, where $\boldsymbol{r}$ is a vector to any general point on the plane. It may, therefore, be written that

$$(\boldsymbol{r} - \boldsymbol{r}_1) \cdot (\boldsymbol{S}_1 \times \boldsymbol{S}_2) = 0. \tag{1.100}$$

Expanding (1.100) yields

$$\boldsymbol{r} \cdot (\boldsymbol{S}_1 \times \boldsymbol{S}_2) - \boldsymbol{r}_1 \cdot (\boldsymbol{S}_1 \times \boldsymbol{S}_2) = 0. \tag{1.101}$$

Rearranging the second vector triple product[8] yields

$$\boldsymbol{r} \cdot (\boldsymbol{S}_1 \times \boldsymbol{S}_2) - (\boldsymbol{r}_1 \times \boldsymbol{S}_1) \cdot \boldsymbol{S}_2 = 0. \tag{1.102}$$

Substituting $\boldsymbol{r}_1 \times \boldsymbol{S}_1 = \boldsymbol{S}_{OL1}$ gives

$$\boldsymbol{r} \cdot (\boldsymbol{S}_1 \times \boldsymbol{S}_2) - \boldsymbol{S}_{OL1} \cdot \boldsymbol{S}_2 = 0. \tag{1.103}$$

The homogeneous coordinates for the plane are, therefore, $[-\boldsymbol{S}_{OL1} \cdot \boldsymbol{S}_2; \boldsymbol{S}_1 \times \boldsymbol{S}_2]$.

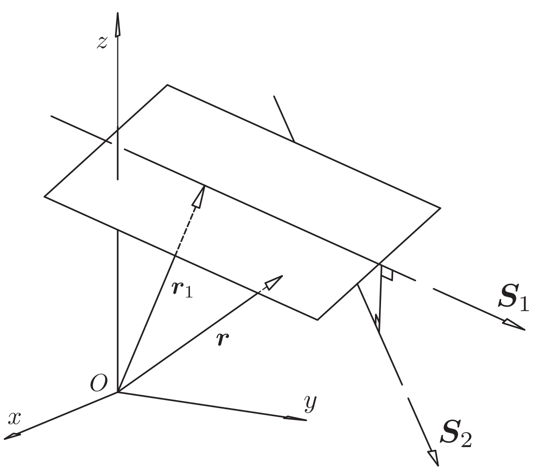

Figure 1.12 The plane through a line and parallel to a second line

[8] It can be proven that $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}$. This is also equal to the 3×3 determinant $|\boldsymbol{abc}|$, where $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ are the first, second, and third rows of the determinant, respectively.

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A special case needs to be considered. When the two given lines are parallel, $\boldsymbol{S}_1 \times \boldsymbol{S}_2 = \mathbf{0}$, and

$$\begin{aligned} \boldsymbol{S}_{OL1} \cdot \boldsymbol{S}_2 &= (\boldsymbol{r}_1 \times \boldsymbol{S}_1) \cdot \boldsymbol{S}_2 \\ &= \boldsymbol{r}_1 \cdot (\boldsymbol{S}_1 \times \boldsymbol{S}_2) \\ &= 0. \end{aligned}$$

Thus, (1.103) vanishes identically for all $\boldsymbol{r}$. In this case, there is a pencil of planes containing the line $\{\boldsymbol{S}_1; \boldsymbol{S}_{OL1}\}$, each of which is parallel to the line $\{\boldsymbol{S}_2; \boldsymbol{S}_{OL2}\}$. The next section focuses on the unique one of these planes that contains both lines.

1.7.3 The Plane Defined by a Pair of Parallel Lines

The equation for the plane through a pair of parallel lines for which $\boldsymbol{S}_1 \times \boldsymbol{S}_2 = \mathbf{0}$ can be most conveniently determined using the vectors $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$, which represent the unique point on each line such that the direction of $\boldsymbol{p}_1$ is perpendicular to $\boldsymbol{S}_1$ and $\boldsymbol{p}_2$ is perpendicular to $\boldsymbol{S}_2$ (see Figure 1.13). Assuming that $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ are unit vectors, then (1.59) may be used to write $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ as

$$\begin{aligned} \boldsymbol{p}_1 &= \boldsymbol{S}_1 \times \boldsymbol{S}_{OL1}, \\ \boldsymbol{p}_2 &= \boldsymbol{S}_2 \times \boldsymbol{S}_{OL2}. \end{aligned} \tag{1.104}$$

The perpendicular distance between the two lines may be written as $|\boldsymbol{p}_2 - \boldsymbol{p}_1|$. Clearly $(\boldsymbol{r} - \boldsymbol{p}_1)$, the direction of $(\boldsymbol{p}_1 - \boldsymbol{p}_2)$, and $\boldsymbol{S}_1$ are coplanar, where $\boldsymbol{r}$ is any vector from point O to a point on the plane. Also the vector $\boldsymbol{S}_1 \times (\boldsymbol{p}_1 - \boldsymbol{p}_2)$ must be perpendicular to the plane. The equation for the plane is, therefore,

$$(\boldsymbol{r} - \boldsymbol{p}_1) \cdot (\boldsymbol{S}_1 \times (\boldsymbol{p}_1 - \boldsymbol{p}_2)) = 0. \tag{1.105}$$

Note that in this equation, the expression $\boldsymbol{p}_1 - \boldsymbol{p}_2$ defines a direction that is perpendicular to the plane. As such, it can be considered to be dimensionless. Rearranging

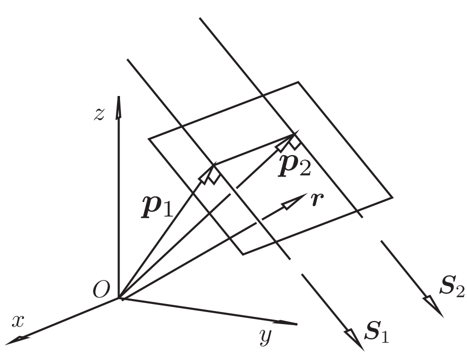

Figure 1.13 The plane through a pair of parallel lines

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this equation and substituting $\boldsymbol{S}_1 \times \boldsymbol{p}_1 = -\boldsymbol{S}_{OL1}$ and $\boldsymbol{S}_1 \times \boldsymbol{p}_2 = \boldsymbol{S}_2 \times \boldsymbol{p}_2 = -\boldsymbol{S}_{OL2}$ gives

$$(\boldsymbol{r} - \boldsymbol{p}_1) \cdot (\boldsymbol{S}_{OL2} - \boldsymbol{S}_{OL1}) = 0. \tag{1.106}$$

Substituting $\boldsymbol{p}_1 = \boldsymbol{S}_1 \times \boldsymbol{S}_{OL1}$ and rearranging gives

$$\boldsymbol{r} \cdot (\boldsymbol{S}_{OL2} - \boldsymbol{S}_{OL1}) + (\boldsymbol{S}_1 \times \boldsymbol{S}_{OL1}) \cdot (\boldsymbol{S}_{OL1} - \boldsymbol{S}_{OL2}) = 0, \tag{1.107}$$

which reduces to

$$\boldsymbol{r} \cdot (\boldsymbol{S}_{OL2} - \boldsymbol{S}_{OL1}) - (\boldsymbol{S}_1 \times \boldsymbol{S}_{OL1}) \cdot \boldsymbol{S}_{OL2} = 0. \tag{1.108}$$

The homogeneous coordinates for the plane are, therefore, $[-(\boldsymbol{S}_1 \times \boldsymbol{S}_{OL1}) \cdot \boldsymbol{S}_{OL2}; \boldsymbol{S}_{OL2} - \boldsymbol{S}_{OL1}]$. The coordinates as written should be divided by a unit of length so that the vector part will be dimensionless and the scalar part will have units of length. When $\boldsymbol{S}_{OL1} = \boldsymbol{S}_{OL2}$, the lines are the same and (1.108) vanishes identically.

The perpendicular distance between the two lines can be determined as $|\boldsymbol{p}_2 - \boldsymbol{p}_1|$. The vector $\boldsymbol{p}_2 - \boldsymbol{p}_1$ is given by

$$\boldsymbol{p}_2 - \boldsymbol{p}_1 = \boldsymbol{S}_2 \times \boldsymbol{S}_{OL2} - \boldsymbol{S}_1 \times \boldsymbol{S}_{OL1}. \tag{1.109}$$

The magnitude of this vector, remembering that $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ are unit vectors, is

$$|\boldsymbol{p}_2 - \boldsymbol{p}_1| = |\boldsymbol{S}_{OL2} - \boldsymbol{S}_{OL1}|. \tag{1.110}$$

1.8 A Line and a Plane Determine a Point

The coordinates of a line and a plane are given as $\{\boldsymbol{S}_1; \boldsymbol{S}_{OL1}\}$ and $[D_{O2}; \boldsymbol{S}_2]$, as shown in Figure 1.14, and their equations may be written as

$$\boldsymbol{r}_1 \times \boldsymbol{S}_1 = \boldsymbol{S}_{OL1}, \tag{1.111}$$

$$\boldsymbol{r}_2 \cdot \boldsymbol{S}_2 + D_{O2} = 0, \tag{1.112}$$

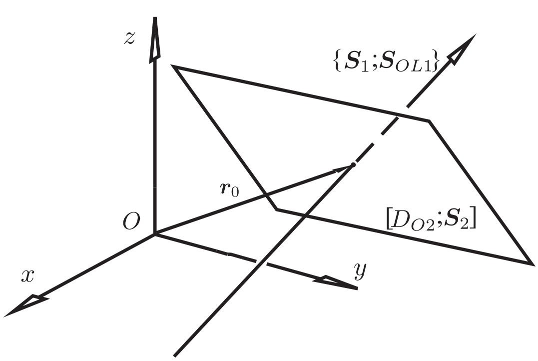

Figure 1.14 A line and a plane determine a point

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where $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are vectors to any point on the line and plane, respectively. The point of intersection of the line and the plane will be designated by the vector $\boldsymbol{r}_0$, and this vector must satisfy both (1.111) and (1.112) and, therefore,

$$\boldsymbol{r}_0 \times \boldsymbol{S}_1 = \boldsymbol{S}_{OL1}, \tag{1.113}$$

$$\boldsymbol{r}_0 \cdot \boldsymbol{S}_2 + D_{O2} = 0. \tag{1.114}$$

Forming a vector product of $\boldsymbol{S}_2$ with (1.113) gives

$$\boldsymbol{S}_2 \times (\boldsymbol{r}_0 \times \boldsymbol{S}_1) = \boldsymbol{S}_2 \times \boldsymbol{S}_{OL1}. \tag{1.115}$$

Expanding the left side of (1.115) gives

$$\boldsymbol{r}_0\, (\boldsymbol{S}_2 \cdot \boldsymbol{S}_1) - \boldsymbol{S}_1\, (\boldsymbol{S}_2 \cdot \boldsymbol{r}_0) = \boldsymbol{S}_2 \times \boldsymbol{S}_{OL1}, \tag{1.116}$$

and substituting (1.114) gives

$$\boldsymbol{r}_0\, (\boldsymbol{S}_2 \cdot \boldsymbol{S}_1) = \boldsymbol{S}_2 \times \boldsymbol{S}_{OL1} - D_{O2}\boldsymbol{S}_1 \tag{1.117}$$

and, thus,

$$\boldsymbol{r}_0 = \frac{\boldsymbol{S}_2 \times \boldsymbol{S}_{OL1} - D_{O2}\boldsymbol{S}_1}{\boldsymbol{S}_2 \cdot \boldsymbol{S}_1}. \tag{1.118}$$

The homogeneous coordinates of the point of intersection are $(\boldsymbol{S}_2 \cdot \boldsymbol{S}_1; \boldsymbol{S}_2 \times \boldsymbol{S}_{OL1} - D_{O2}\,\boldsymbol{S}_1)$. When $\boldsymbol{S}_2 \cdot \boldsymbol{S}_1 = 0$, then the line is parallel to the plane and the point of intersection is at infinity with coordinates $\{\boldsymbol{0}; \boldsymbol{S}_1\}$ unless the line lies in the plane and there is no unique point of intersection.

1.9 Determination of the Point on a Line That Is Closest to a Given Point

Figure 1.15 shows a line whose coordinates are given as $\{\boldsymbol{S}\,;\, \boldsymbol{S}_{OL}\}$. The coordinates of a point are given by $\boldsymbol{p}_1$, and the objective is to find the point on the line that is closest to this point. This closest point is denoted by $\boldsymbol{p}_2$, and the vector from $\boldsymbol{p}_1$ to $\boldsymbol{p}_2$ is shown as $\boldsymbol{d}$. It is apparent that $\boldsymbol{d} \perp \boldsymbol{S}$.

The point $\boldsymbol{p}_2$ must satisfy the equation of the line and, thus,

$$\boldsymbol{p}_2 \times \boldsymbol{S} = \boldsymbol{S}_{OL}. \tag{1.119}$$

Since $\boldsymbol{d} \perp \boldsymbol{S}$,

$$\boldsymbol{d} \cdot \boldsymbol{S} = 0. \tag{1.120}$$

It is apparent from 1.15 that

$$\boldsymbol{p}_2 = \boldsymbol{p}_1 + \boldsymbol{d}, \tag{1.121}$$

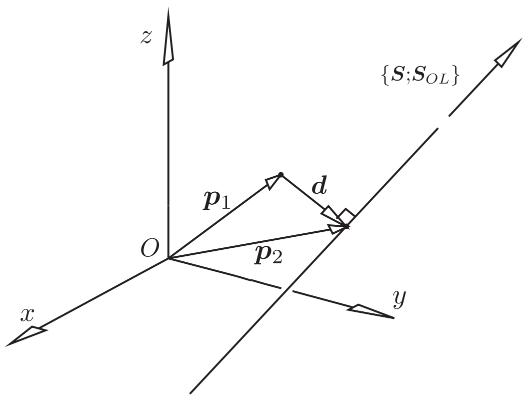

Figure 1.15 Closest point on a line to a given point

and substituting (1.121) into (1.119) gives

$$(\boldsymbol{p}_1 + \boldsymbol{d}) \times \boldsymbol{S} = \boldsymbol{S}_{OL}. \tag{1.122}$$

Expanding (1.122) and rearranging gives

$$\boldsymbol{d} \times \boldsymbol{S} = \boldsymbol{S}_{OL} - \boldsymbol{p}_1 \times \boldsymbol{S}. \tag{1.123}$$

Applying a cross product of $\boldsymbol{S}$ to both sides of (1.123) gives

$$\boldsymbol{S} \times (\boldsymbol{d} \times \boldsymbol{S}) = \boldsymbol{S} \times (\boldsymbol{S}_{OL} - \boldsymbol{p}_1 \times \boldsymbol{S}). \tag{1.124}$$

Expanding the left side of (1.124) gives

$$(\boldsymbol{S} \cdot \boldsymbol{S})\, \boldsymbol{d} - (\boldsymbol{S} \cdot \boldsymbol{d})\, \boldsymbol{S} = \boldsymbol{S} \times (\boldsymbol{S}_{OL} - \boldsymbol{p}_1 \times \boldsymbol{S}). \tag{1.125}$$

Substituting (1.120) into (1.125) gives

$$\boldsymbol{d} = \frac{\boldsymbol{S} \times (\boldsymbol{S}_{OL} - \boldsymbol{p}_1 \times \boldsymbol{S})}{\boldsymbol{S} \cdot \boldsymbol{S}}. \tag{1.126}$$

The coordinates of $\boldsymbol{p}_2$ are determined by substituting (1.126) into (1.121) to yield

$$\boldsymbol{p}_2 = \boldsymbol{p}_1 + \frac{\boldsymbol{S} \times (\boldsymbol{S}_{OL} - \boldsymbol{p}_1 \times \boldsymbol{S})}{\boldsymbol{S} \cdot \boldsymbol{S}}. \tag{1.127}$$

1.10 The Mutual Moment of Two Lines

The Plücker coordinates of two skew lines in space are given as $\{\boldsymbol{S}_1; \boldsymbol{S}_{OL1}\}$ and $\{\boldsymbol{S}_2; \boldsymbol{S}_{OL2}\}$, where $\mathbf{S}_1$ and $\mathbf{S}_2$ are unit vectors (see Figure 1.16) and their vector equations may be written as

$$\boldsymbol{r}_1 \times \boldsymbol{S}_1 = \boldsymbol{S}_{OL1}, \tag{1.128}$$

$$\boldsymbol{r}_2 \times \boldsymbol{S}_2 = \boldsymbol{S}_{OL2}, \tag{1.129}$$

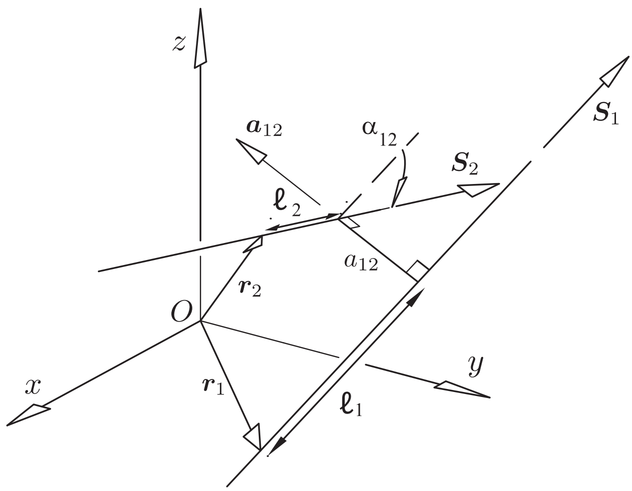

Figure 1.16 A pair of skew lines

where $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are vectors to any point on the first and second line, respectively. The perpendicular distance between the lines is labeled as a_{12}, and the twist angle between the lines is labeled α_{12} and is measured in a right-hand sense about the vector $\boldsymbol{a}_{12}$, where $\boldsymbol{a}_{12}$ is a unit vector whose direction is parallel, or anti-parallel, to $\boldsymbol{S}_1 \times \boldsymbol{S}_2$.

The projection of any moment vector $(\boldsymbol{r}_2 - \boldsymbol{r}_1) \times \boldsymbol{S}_2$ on $\boldsymbol{S}_1$ is the scalar quantity $[(\boldsymbol{r}_2 - \boldsymbol{r}_1) \times \boldsymbol{S}_2] \cdot \boldsymbol{S}_1$, which will be shown to be invariant with the position vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$. This quantity, which can also be obtained by projecting the moment vector $(\boldsymbol{r}_1 - \boldsymbol{r}_2) \times \boldsymbol{S}_1$ on $\boldsymbol{S}_2$, is defined as the mutual moment of two lines. Now the mutual moment may be written as

$$(\boldsymbol{r}_2 - \boldsymbol{r}_1) \times \boldsymbol{S}_2 \cdot \boldsymbol{S}_1 = \boldsymbol{r}_2 \times \boldsymbol{S}_2 \cdot \boldsymbol{S}_1 + \boldsymbol{r}_1 \times \boldsymbol{S}_1 \cdot \boldsymbol{S}_2. \tag{1.130}$$

Substituting (1.128) and (1.129) into the right side of (1.130) yields

$$(\boldsymbol{r}_2 - \boldsymbol{r}_1) \times \boldsymbol{S}_2 \cdot \boldsymbol{S}_1 = \boldsymbol{S}_1 \cdot \boldsymbol{S}_{OL2} + \boldsymbol{S}_2 \cdot \boldsymbol{S}_{OL1}. \tag{1.131}$$

Note that the mutual moment is readily calculated from the coordinates of the two lines and that this quantity has units of length.

The mutual moment will now be calculated in a different manner based on the geometry. From Figure 1.16,

$$(\boldsymbol{r}_2 - \boldsymbol{r}_1) + \ell_2 \boldsymbol{S}_2 - a_{12} \boldsymbol{a}_{12} - \ell_1 \boldsymbol{S}_1 = \boldsymbol{0}. \tag{1.132}$$

Therefore, the mutual moment may now be calculated as

$$(\boldsymbol{r}_2 - \boldsymbol{r}_1) \times \boldsymbol{S}_2 \cdot \boldsymbol{S}_1 = (\ell_1 \boldsymbol{S}_1 + a_{12} \boldsymbol{a}_{12} - \ell_2 \boldsymbol{S}_2) \times \boldsymbol{S}_2 \cdot \boldsymbol{S}_1. \tag{1.133}$$

Rearranging this equation and recognizing that $\boldsymbol{S}_1 \times \boldsymbol{S}_2 \cdot \boldsymbol{S}_1 = 0$ and $\boldsymbol{S}_2 \times \boldsymbol{S}_2 \cdot \boldsymbol{S}_1 = 0$ gives

$$(\boldsymbol{r}_2 - \boldsymbol{r}_1) \times \boldsymbol{S}_2 \cdot \boldsymbol{S}_1 = a_{12} \boldsymbol{a}_{12} \times \boldsymbol{S}_2 \cdot \boldsymbol{S}_1. \tag{1.134}$$

Equation (1.134) may be rewritten as

$$(\boldsymbol{r}_2 - \boldsymbol{r}_1) \times \boldsymbol{S}_2 \cdot \boldsymbol{S}_1 = -a_{12}\, \boldsymbol{a}_{12} \cdot \boldsymbol{S}_1 \times \boldsymbol{S}_2. \tag{1.135}$$

Recall that $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ are unit vectors and that the unit vector $\boldsymbol{a}_{12}$ is either parallel or anti-parallel to $\boldsymbol{S}_1 \times \boldsymbol{S}_2$. Thus,

$$\boldsymbol{a}_{12} = \pm \frac{\boldsymbol{S}_1 \times \boldsymbol{S}_2}{|\boldsymbol{S}_1 \times \boldsymbol{S}_2|} \tag{1.136}$$

and

$$\boldsymbol{S}_1 \times \boldsymbol{S}_2 = \sin\alpha_{12}\, \boldsymbol{a}_{12}. \tag{1.137}$$

Once the direction for the unit vector $\boldsymbol{a}_{12}$ is chosen, the angle α_{12} is defined as the angle between $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$, measured in a right-hand sense about $\boldsymbol{a}_{12}$. The sine and cosine of this angle may be determined from

$$\boldsymbol{S}_1 \cdot \boldsymbol{S}_2 = \cos\alpha_{12}, \tag{1.138}$$

$$\boldsymbol{S}_1 \times \boldsymbol{S}_2 \cdot \boldsymbol{a}_{12} = \sin\alpha_{12}. \tag{1.139}$$

Substituting (1.139) into (1.135) gives

$$(\boldsymbol{r}_2 - \boldsymbol{r}_1) \times \boldsymbol{S}_2 \cdot \boldsymbol{S}_1 = -a_{12} \sin\alpha_{12}. \tag{1.140}$$

The mutual moment is, thus, a function of the perpendicular distance between the lines and the angle between the directions of the lines and is invariant with the choice of the position vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$.

Equating the right sides of (1.131) and (1.140) gives[9]

$$-a_{12} \sin\alpha_{12} = \boldsymbol{S}_1 \cdot \boldsymbol{S}_{OL2} + \boldsymbol{S}_2 \cdot \boldsymbol{S}_{OL1} \tag{1.141}$$

and further

$$-a_{12} \sin\alpha_{12} = L_1 P_2 + M_1 Q_2 + N_1 R_2 + L_2 P_1 + M_2 Q_1 + N_2 R_1. \tag{1.142}$$

When $a_{12} = 0$, the lines intersect at a finite point, and when $sin\alpha_{12} = 0$, they intersect at a point at infinity, and in either case the mutual moment of the lines is zero. For the general case, the vector $\boldsymbol{a}_{12}$ is chosen as shown in (1.136). The angle α_{12} may then be determined from (1.138) and (1.139) and then the distance a_{12} is determined from (1.141). A negative value for a_{12} will result if the "direction of travel" along the mutual perpendicular from the first line to the second line is opposite to the direction of the vector $\boldsymbol{a}_{12}$.

As an additional example, consider a line with coordinates $\{L, M, N; P, Q, R\}$. The coordinates of the lines along the x, y, and z axes are $\{1, 0, 0; 0, 0, 0\}$, $\{0, 1, 0; 0, 0, 0\}$, and $\{0, 0, 1; 0, 0, 0\}$, respectively and it can be easily shown that the mutual moment of the line with these three coordinate axes are P, Q, and R, respectively.

[9] Recall that during the derivation of this expression the Plücker coordinates of the two lines were written such that $|\boldsymbol{S}_1| = |\boldsymbol{S}_2| = 1$.

1.10.1 Numerical Example

The coordinates of two lines are given as

$$\{\boldsymbol{S}_1; \boldsymbol{S}_{OL1}\} = \{1, 2, 1; -2, 1, 0\}, \tag{1.143}$$

$$\{\boldsymbol{S}_2; \boldsymbol{S}_{OL2}\} = \{-3, 1, 0; 1, 3, 5\}, \tag{1.144}$$

where the direction vectors $\boldsymbol{S}_i$ are dimensionless, and the moment terms $\boldsymbol{S}_{OLi}$ have units of meters, $i = 1, 2$. It is desired to determine the angle between the line directions, α_{12}, and the perpendicular distance between the lines, a_{12}, based on the choice for the direction of the vector $\boldsymbol{a}_{12}$.

The first step will be to scale the line coordinates such that the direction vector is a unit vector. Dividing all terms of the first line by $\sqrt{6}$ and the second line by $\sqrt{10}$ gives

$$\{\boldsymbol{S}_{1u}; \boldsymbol{S}_{OL1u}\} = \{0.4082, 0.8165, 0.4082; -0.8165, 0.4082, 0\}, \tag{1.145}$$

$$\{\boldsymbol{S}_{2u}; \boldsymbol{S}_{OL2u}\} = \{-0.9487, 0.3162, 0; 0.3162, 0.9487, 1.5811\}. \tag{1.146}$$

The direction of the unit vector that is perpendicular to both lines is selected as

$$\boldsymbol{a}_{12} = \frac{\boldsymbol{S}_{1u} \times \boldsymbol{S}_{2u}}{|\boldsymbol{S}_{1u} \times \boldsymbol{S}_{2u}|} = \begin{bmatrix} -0.1302 \\ -0.3906 \\ 0.9113 \end{bmatrix}. \tag{1.147}$$

The angle α_{12} is defined as the angle swept from the direction of the first line to the direction of the second as measured in a right-hand sense about $\boldsymbol{a}_{12}$. The sine and cosine of $\boldsymbol{a}_{12}$ are calculated as

$$\sin \alpha_{12} = (\boldsymbol{S}_{1u} \times \boldsymbol{S}_{2u}) \cdot \boldsymbol{a}_{12} = 0.9916, \tag{1.148}$$

$$\cos\alpha_{12} = \boldsymbol{S}_{1u} \cdot \boldsymbol{S}_{2u} = -0.1291, \tag{1.149}$$

and the angle α_{12} is calculated as

$$\alpha_{12} = 1.700 \text{ radians} = 97.4°. \tag{1.150}$$

The mutual moment of the two lines is calculated as

$$MM = \boldsymbol{S}_{1u} \cdot \boldsymbol{S}_{OL2u} + \boldsymbol{S}_{2u} \cdot \boldsymbol{S}_{OL1u} = 2.4529 \text{ m}. \tag{1.151}$$

From (1.141),

$$-a_{12} \sin \alpha_{12} = 2.4529. \tag{1.152}$$

Solving for a_{12} gives

$$a_{12} = -2.4736 \text{ m}. \tag{1.153}$$

1.11 Determination of the Unique Perpendicular Line to Two Given Lines

In many instances, it is necessary to determine the Plücker coordinates of the line that is mutually perpendicular to two given lines $\{\boldsymbol{S}_1; \boldsymbol{S}_{OL1}\}$ and $\{\boldsymbol{S}_2; \boldsymbol{S}_{OL2}\}$, where

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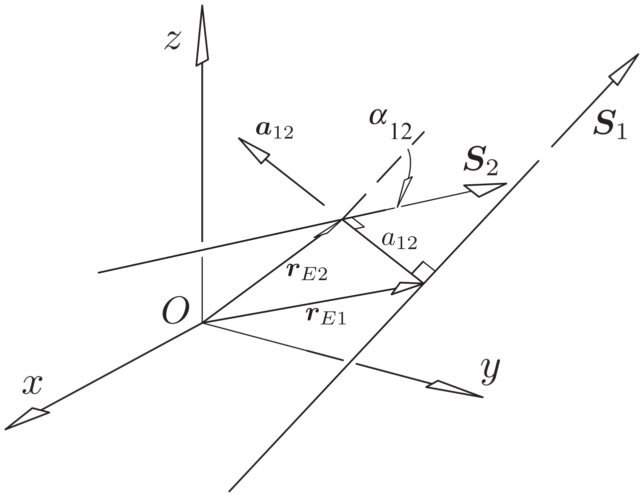

Figure 1.17 Definition of the points $\boldsymbol{r}_{E1}$ and $\boldsymbol{r}_{E2}$

again for this case $|\boldsymbol{S}_1| = |\boldsymbol{S}_2| = 1$. One method to determine the coordinates of this line will be developed here. A second method, utilizing a concept called the motor product, will be introduced in Section 3.12.

The direction of this line has already been determined as $\boldsymbol{a}_{12}$, which is a unit vector either parallel or anti-parallel to $\{\boldsymbol{S}_1 \times \boldsymbol{S}_2\}$. Two points on this line are $\boldsymbol{r}_{E1}$ and $\boldsymbol{r}_{E2}$, which are the intersections of the mutual perpendicular with the two given lines, as shown in Figure 1.17. The determination of either of these points will allow for the determination of the moment of the mutual perpendicular line, which is all that remains to be found in order to define its Plücker coordinates.

Since $\boldsymbol{r}_{E1}$ and $\boldsymbol{r}_{E2}$ lie on the first and second line, respectively,

$$\boldsymbol{r}_{E1} \times \boldsymbol{S}_1 = \boldsymbol{S}_{OL1}, \tag{1.154}$$

$$\boldsymbol{r}_{E2} \times \boldsymbol{S}_2 = \boldsymbol{S}_{OL2}. \tag{1.155}$$

Since $\boldsymbol{r}_{E2} = \boldsymbol{r}_{E1} + a_{12}\boldsymbol{a}_{12}$, (1.155) may be written as

$$(\boldsymbol{r}_{E1} + a_{12}\,\boldsymbol{a}_{12}) \times \boldsymbol{S}_2 = \boldsymbol{S}_{OL2}. \tag{1.156}$$

Expanding the cross product and rearranging terms gives

$$\boldsymbol{r}_{E1} \times \boldsymbol{S}_2 = \boldsymbol{S}_{OL2} - a_{12}\,\boldsymbol{a}_{12} \times \boldsymbol{S}_2. \tag{1.157}$$

Forming the cross product of (1.157) with $\boldsymbol{S}_{OL1}$ yields

$$S_{OL1} \times (\boldsymbol{r}_{E1} \times \boldsymbol{S}_2) = \boldsymbol{S}_{OL1} \times (\boldsymbol{S}_{OL2} - a_{12}\,\boldsymbol{a}_{12} \times \boldsymbol{S}_2), \tag{1.158}$$

and expanding the left side of this equation gives

$$\boldsymbol{r}_{E1}(\boldsymbol{S}_{OL1} \cdot \boldsymbol{S}_2) - \boldsymbol{S}_2(\boldsymbol{S}_{OL1} \cdot \boldsymbol{r}_{E1}) = \boldsymbol{S}_{OL1} \times (\boldsymbol{S}_{OL2} - a_{12}\,\boldsymbol{a}_{12} \times \boldsymbol{S}_2). \tag{1.159}$$

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Since $\boldsymbol{r}_{E1}$ lies on the first line, $\boldsymbol{S}_{OL1} \cdot \boldsymbol{r}_{E1} = 0$, and (1.159) can be solved for $\boldsymbol{r}_{E1}$ as

$$\boldsymbol{r}_{E1} = \frac{\boldsymbol{S}_{OL1} \times (\boldsymbol{S}_{OL2} - a_{12}\,\boldsymbol{a}_{12} \times \boldsymbol{S}_2)}{\boldsymbol{S}_{OL1} \cdot \boldsymbol{S}_2}. \tag{1.160}$$

The Plücker coordinates of the line that is mutually perpendicular to the two given lines can, thus, be written as $\{\boldsymbol{a}_{12}; \boldsymbol{r}_{E1} \times \boldsymbol{a}_{12}\}$.

A special case exists for the solution of $\boldsymbol{r}_{E1}$ if $\boldsymbol{S}_{OL1} \cdot \boldsymbol{S}_2 = 0$, which can occur if $\boldsymbol{S}_{OL1} = 0$, if $\boldsymbol{S}_2 = 0$, or if $\boldsymbol{S}_{OL1}$ is perpendicular to $\boldsymbol{S}_2$. If $\boldsymbol{S}_{OL1} = 0$, then the first line passes through the origin, and the point of intersection of the two lines may be written as

$$\boldsymbol{r}_{E1} = r\,\boldsymbol{S}_1. \tag{1.161}$$

Substituting (1.161) into (1.157) gives

$$r\,\boldsymbol{S}_1 \times \boldsymbol{S}_2 = \boldsymbol{S}_{OL2} - a_{12}\,\boldsymbol{a}_{12} \times \boldsymbol{S}_2. \tag{1.162}$$

Substituting (1.137) into (1.162) gives

$$r\,(\boldsymbol{a}_{12} \sin\alpha_{12}) = \boldsymbol{S}_{OL2} - a_{12}\,\boldsymbol{a}_{12} \times \boldsymbol{S}_2. \tag{1.163}$$

Performing a scalar product of both sides of (1.163) with $\boldsymbol{a}_{12}$ and solving for r gives

$$r = \frac{\boldsymbol{S}_{OL2} \cdot \boldsymbol{a}_{12}}{\sin\alpha_{12}}. \tag{1.164}$$

It is apparent from (1.164) that a further special case exists if $\sin\alpha_{12} = 0$, that is, if the directions of the two lines are parallel. In this special case (of a special case), let $\boldsymbol{r}_{E1}$ be the origin and let $\boldsymbol{r}_{E2} = a_{12}\,\boldsymbol{a}_{12} = \frac{\boldsymbol{S}_2 \times \boldsymbol{S}_{OL2}}{\boldsymbol{S}_2 \cdot \boldsymbol{S}_2}$ be the vector from the origin that is perpendicular to the second line.

If $\boldsymbol{S}_2 = \boldsymbol{0}$, then the second line is at infinity. The mutual moment of a line with a line at infinity will not be addressed here. The third special case occurs if $\boldsymbol{S}_{OL1}$ and $\boldsymbol{S}_2$ are perpendicular. Substituting $\boldsymbol{r}_{E1} \times \boldsymbol{S}_1 = \boldsymbol{S}_{OL1}$ into the scalar product $\boldsymbol{S}_{OL1} \cdot \boldsymbol{S}_2 = 0$ gives

$$\boldsymbol{r}_{E1} \times \boldsymbol{S}_1 \cdot \boldsymbol{S}_2 = 0. \tag{1.165}$$

This expression may be written as

$$\boldsymbol{r}_{E1} \cdot \boldsymbol{S}_1 \times \boldsymbol{S}_2 = \boldsymbol{r}_{E1} \cdot \boldsymbol{a}_{12} \sin\alpha_{12} = 0. \tag{1.166}$$

From this last expression it is apparent that the vector $\boldsymbol{r}_{E1}$ is perpendicular to $\boldsymbol{a}_{12}$ and, as such, the vector $\boldsymbol{r}_{E1}$ may be written as a linear combination of the vectors $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$, since both of these vectors are also perpendicular to $\boldsymbol{a}_{12}$. Thus, $\boldsymbol{r}_{E1}$ may be written as

$$\boldsymbol{r}_{E1} = r_1\,\boldsymbol{S}_1 + r_2\,\boldsymbol{S}_2. \tag{1.167}$$

Since $\boldsymbol{r}_{E1}$ lies on the first line,

$$\boldsymbol{r}_{E1} \times \boldsymbol{S}_1 = \boldsymbol{S}_{OL1}. \tag{1.168}$$

Substituting (1.167) into (1.168) gives

$$(r_1\,\boldsymbol{S}_1 + r_2\,\boldsymbol{S}_2) \times \boldsymbol{S}_1 = \boldsymbol{S}_{OL1}, \tag{1.169}$$

which simplifies to

$$r_2\,\boldsymbol{S}_2 \times \boldsymbol{S}_1 = \boldsymbol{S}_{OL1}. \tag{1.170}$$

Substituting (1.137) into (1.170) gives

$$-\,r_2\,\boldsymbol{a}_{12} \sin\alpha_{12} = \boldsymbol{S}_{OL1}. \tag{1.171}$$

Performing a scalar product of both sides of (1.171) with $\boldsymbol{a}_{12}$ and solving for r_2 gives

$$r_2 = \frac{-\boldsymbol{S}_{OL1} \cdot \boldsymbol{a}_{12}}{\sin\alpha_{12}}. \tag{1.172}$$

The value for r_1 can be obtained by substituting (1.167) into (1.157). The resulting expression for r_1 is written as

$$r_1 = \frac{\boldsymbol{S}_{OL2} \cdot \boldsymbol{a}_{12}}{\sin\alpha_{12}}. \tag{1.173}$$

1.12 A Pair of Intersecting Lines

Two intersecting lines $\{\boldsymbol{S}_1; \boldsymbol{S}_{OL1}\}$ and $\{\boldsymbol{S}_2; \boldsymbol{S}_{OL2}\}$ are shown in Figure 1.18, where it is assumed that $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ are unit vectors. Two approaches to determine the intersection point of these two given lines will now be presented.

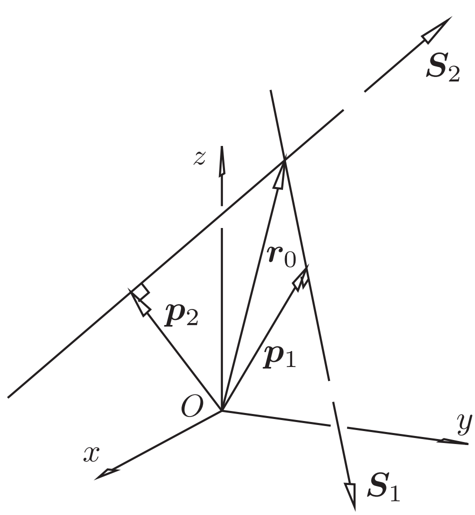

Figure 1.18 A pair of intersecting lines

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1.12.1 Approach 1

The vector equations for these two lines are

$$\boldsymbol{r}_1 \times \boldsymbol{S}_1 = \boldsymbol{S}_{OL1}, \tag{1.174}$$

$$\boldsymbol{r}_2 \times \boldsymbol{S}_2 = \boldsymbol{S}_{OL2}, \tag{1.175}$$

where $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are vectors to any point on the first and second line, respectively. The position vector $\boldsymbol{r}_0$ indicates the point of intersection of the two lines and, as such, it must satisfy both equations for the lines as

$$\boldsymbol{r}_0 \times \boldsymbol{S}_1 = \boldsymbol{S}_{OL1}, \tag{1.176}$$

$$\boldsymbol{r}_0 \times \boldsymbol{S}_2 = \boldsymbol{S}_{OL2}. \tag{1.177}$$

The vector $\boldsymbol{r}_0$ will be determined by first forming a cross product of (1.176) with $\boldsymbol{S}_2$, which yields

$$\boldsymbol{S}_2 \times (\boldsymbol{r}_0 \times \boldsymbol{S}_1) = \boldsymbol{S}_2 \times \boldsymbol{S}_{OL1}. \tag{1.178}$$

Expanding the left side of (1.178) gives

$$\boldsymbol{r}_0(\boldsymbol{S}_2 \cdot \boldsymbol{S}_1) - \boldsymbol{S}_1(\boldsymbol{r}_0 \cdot \boldsymbol{S}_2) = \boldsymbol{S}_2 \times \boldsymbol{S}_{OL1}. \tag{1.179}$$

Forming a scalar product of (1.179) with $\boldsymbol{S}_1$ yields

$$(\boldsymbol{r}_0 \cdot \boldsymbol{S}_1)(\boldsymbol{S}_2 \cdot \boldsymbol{S}_1) - (\boldsymbol{S}_1 \cdot \boldsymbol{S}_1)(\boldsymbol{r}_0 \cdot \boldsymbol{S}_2) = \boldsymbol{S}_1 \cdot (\boldsymbol{S}_2 \times \boldsymbol{S}_{OL1}). \tag{1.180}$$

Substituting $\boldsymbol{S}_1 \cdot \boldsymbol{S}_1 = 1$ and rearranging gives

$$\boldsymbol{r}_0 \cdot [\boldsymbol{S}_2 - (\boldsymbol{S}_2 \cdot \boldsymbol{S}_1)\boldsymbol{S}_1] = (\boldsymbol{S}_1 \times \boldsymbol{S}_{OL1}) \cdot \boldsymbol{S}_2. \tag{1.181}$$

It remains to solve (1.177) and (1.181) for $\boldsymbol{r}_0$. Forming the cross product of (1.177) with $[\boldsymbol{S}_2 - (\boldsymbol{S}_2 \cdot \boldsymbol{S}_1)\boldsymbol{S}_1]$ yields

$$[\boldsymbol{S}_2 - (\boldsymbol{S}_2 \cdot \boldsymbol{S}_1)\boldsymbol{S}_1] \times [\boldsymbol{r}_0 \times \boldsymbol{S}_2] = [\boldsymbol{S}_2 - (\boldsymbol{S}_2 \cdot \boldsymbol{S}_1)\boldsymbol{S}_1] \times \boldsymbol{S}_{OL2}. \tag{1.182}$$

Expanding the left side of (1.182) and substituting (1.181) gives

$$\boldsymbol{r}_0\{[\boldsymbol{S}_2 - (\boldsymbol{S}_2 \cdot \boldsymbol{S}_1)\boldsymbol{S}_1] \cdot \boldsymbol{S}_2\} - \boldsymbol{S}_2\{(\boldsymbol{S}_1 \times \boldsymbol{S}_{OL1}) \cdot \boldsymbol{S}_2\} = [\boldsymbol{S}_2 - (\boldsymbol{S}_2 \cdot \boldsymbol{S}_1)\boldsymbol{S}_1] \times \boldsymbol{S}_{OL2}. \tag{1.183}$$

Rearranging this equation gives

$$\boldsymbol{r}_0[1 - (\boldsymbol{S}_1 \cdot \boldsymbol{S}_2)^2] = \boldsymbol{S}_2 \times \boldsymbol{S}_{OL2} - (\boldsymbol{S}_1 \cdot \boldsymbol{S}_2)\boldsymbol{S}_1 \times \boldsymbol{S}_{OL2} + (\boldsymbol{S}_1 \times \boldsymbol{S}_{OL1} \cdot \boldsymbol{S}_2)\boldsymbol{S}_2, \tag{1.184}$$

and solving for $\boldsymbol{r}_0$ gives

$$\boldsymbol{r}_0 = \frac{\boldsymbol{S}_2 \times \boldsymbol{S}_{OL2} - (\boldsymbol{S}_1 \cdot \boldsymbol{S}_2)\boldsymbol{S}_1 \times \boldsymbol{S}_{OL2} + (\boldsymbol{S}_1 \times \boldsymbol{S}_{OL1} \cdot \boldsymbol{S}_2)\boldsymbol{S}_2}{1 - (\boldsymbol{S}_1 \cdot \boldsymbol{S}_2)^2}. \tag{1.185}$$

The homogeneous coordinates of the point of intersection of the pair of lines are, thus, $(1 - (\boldsymbol{S}_1 \cdot \boldsymbol{S}_2)^2; \boldsymbol{S}_2 \times \boldsymbol{S}_{OL2} - (\boldsymbol{S}_1 \cdot \boldsymbol{S}_2)\boldsymbol{S}_1 \times \boldsymbol{S}_{OL2} + (\boldsymbol{S}_1 \times \boldsymbol{S}_{OL1} \cdot \boldsymbol{S}_2)\boldsymbol{S}_2)$.

1.12.2 Approach 2

A second solution approach to this problem was developed by Dr. David Dooner at the University of Puerto Rico, Mayaguez. Performing a cross product of the left and right sides of (1.176) and (1.177) yields

$$(\boldsymbol{r}_0 \times \boldsymbol{S}_1) \times (\boldsymbol{r}_0 \times \boldsymbol{S}_2) = \boldsymbol{S}_{OL1} \times \boldsymbol{S}_{OL2}. \tag{1.186}$$

Expanding this expression gives

$$[(\boldsymbol{r}_0 \times \boldsymbol{S}_1) \cdot \boldsymbol{S}_2]\boldsymbol{r}_0 - [(\boldsymbol{r}_0 \times \boldsymbol{S}_1) \cdot \boldsymbol{r}_0]\boldsymbol{S}_2 = \boldsymbol{S}_{OL1} \times \boldsymbol{S}_{OL2}. \tag{1.187}$$

Substituting $[(\boldsymbol{r}_0 \times \boldsymbol{S}_1) \cdot \boldsymbol{r}_0] = 0$ and $\boldsymbol{r}_0 \times \boldsymbol{S}_1 = \boldsymbol{S}_{OL1}$ and then solving for $\boldsymbol{r}_0$ gives

$$\boldsymbol{r}_0 = \frac{\boldsymbol{S}_{OL1} \times \boldsymbol{S}_{OL2}}{\boldsymbol{S}_{OL1} \cdot \boldsymbol{S}_2}. \tag{1.188}$$

The result for the intersection point obtained in (1.188) is much simpler than that in (1.185). However, for the cases where the first line passes through the origin ($S_{OL1} = 0$) or where the two lines lie in a plane that passes through the origin ($\mathbf{S}_1$ and $\mathbf{S}_2$ lie in the plane and, thus, $\mathbf{S}_{OL1}$ and $\mathbf{S}_{OL2}$ are parallel and perpendicular to the plane), equation (1.188) will reduce to an indeterminate state of $\frac{0}{0}$. Equation (1.185) will yield the correct intersection point for these cases.

1.13 Summary

This chapter introduced representations and notation for points, lines, and planes. It was shown that a point is the dual of a plane and that a line is dual with itself or self-dual. Several geometric problems were solved, such as determining the coordinates of the line that is the intersection of two non-parallel planes. The concept of the mutual moment of two lines was introduced, and a geometric interpretation was presented. Lastly, it was shown how to determine the Plücker coordinates of the line that intersects and is perpendicular to two given lines.

The material presented in this chapter should give the user a firm background in the geometry of points, lines, and planes. Subsequent chapters will expand upon this material to develop an analytic approach for solving velocity, acceleration, and static force balance problems for serial and planar spatial manipulators and mechanisms.

1.14 Problems

1. The equation of a plane is $3x - 4y - 12z - 1 = 0$. Here, the coefficients 3, 4, and 12 are dimensionless, while the coefficient 1 has units of meters. Determine:
   (a) The direction cosines of the unit vector normal to the plane.
   (b) The perpendicular distance of the plane from the origin.

(c) The equation of the plane parallel to the given plane that passes through the origin.

2. Draw the lines with the following Plücker coordinates and determine the perpendicular distance of each line from the origin. The first three coordinates are dimensionless, and the last three coordinates have units of meters.
   (a) $\{-4, 12, 3; -24, -6, -8\}$
   (b) $\{3, 4, -12; 16, 27, 13\}$
   (c) $\{2, 4, -1; -7, 4, 2\}$
   (d) $\{4, 1, 5; 5, 15, -7\}$
3. The equations of two planes are

$$\begin{aligned} -4x + 12y + 3z + 1 &= 0, \\ 3x + 4y - 12z + 1 &= 0. \end{aligned}$$

   Here, the coefficients multiplying the terms x, y, and z are dimensionless, and the remaining coefficient has units of meters. Determine the angle between the planes and the Plücker coordinates of the line of intersection. What is the condition that two planes are perpendicular?
4. Derive the equation for the plane that contains the line $\boldsymbol{r} \times \boldsymbol{S} = \boldsymbol{S}_{OL}$ and the origin where $\boldsymbol{S} = (-4, 12, 3)$ and $\boldsymbol{S}_{OL} = (-24, -6, -8)$. The vector $\mathbf{S}$ is dimensionless, and $\mathbf{S}_{OL}$ has units of meters.
5. Determine the point of intersection of the line $\{3, 4, -12; 16, 27, 13\}$ and the plane $4x + y + 5z + 1 = 0$. The first three components of the line coordinates are dimensionless, and the last three have units of meters. The coefficients of the equation of the plane that multiply the terms x, y, and z are dimensionless, while the remaining coefficient has units of meters.
6. Find the shortest distance between the straight lines AB and CD when the coordinates of points A, B, C, and D are given in meters as follows:
   (a) $A(-2, 4, 3), B(2, -8, 0), C(1, -3, 5), D(4, 1, -7)$
   (b) $A(2, 3, 1), B(0, -1, 2), C(1, 2, 5), D(-3, 1, 0)$
7. Prove that if the non-parallel lines $\{\boldsymbol{S}_1; \boldsymbol{S}_{OL1}\}$ and $\{\boldsymbol{S}_2; \boldsymbol{S}_{OL2}\}$ are coplanar, then $\boldsymbol{S}_1 \cdot \boldsymbol{S}_{OL2} + \boldsymbol{S}_2 \cdot \boldsymbol{S}_{OL1} = 0$. Show that they lie in the plane $[\boldsymbol{S}_{OL1} \cdot \boldsymbol{S}_2; \boldsymbol{S}_1 \times \boldsymbol{S}_2]$ and that they intersect at the point $(\boldsymbol{S}_{OL1} \cdot \boldsymbol{S}_2; \boldsymbol{S}_{OL1} \times \boldsymbol{S}_{OL2})$ provided $\boldsymbol{S}_{OL1} \cdot \boldsymbol{S}_2 \neq 0$.
8. A pair of lines $\{\boldsymbol{S}_1; \boldsymbol{S}_{OL1}\}$ and $\{\boldsymbol{S}_2; \boldsymbol{S}_{OL2}\}$ intersect at right angles, where $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ are unit vectors. Derive an equation for the plane that contains the first line and that is perpendicular to the second line. Hence, obtain the expression $\boldsymbol{r} = \boldsymbol{p}_2 + (\boldsymbol{p}_1 \cdot \boldsymbol{S}_2)\boldsymbol{S}_2$ for the position vector of the point of intersection of the lines. Verify this result by simple projection and also deduce that $\boldsymbol{r} = \boldsymbol{p}_1 + (\boldsymbol{p}_2 \cdot \boldsymbol{S}_1)\boldsymbol{S}_1$.
9. Show that the lines AB and CD are coplanar, and find their point of intersection. The coordinates of the four points are given in units of meters as $A(-2, -3, 4)$, $B(2, 3, 0)$, $C(-2, 3, 2)$, $D(2, 0, 1)$. Determine the angle between the lines.

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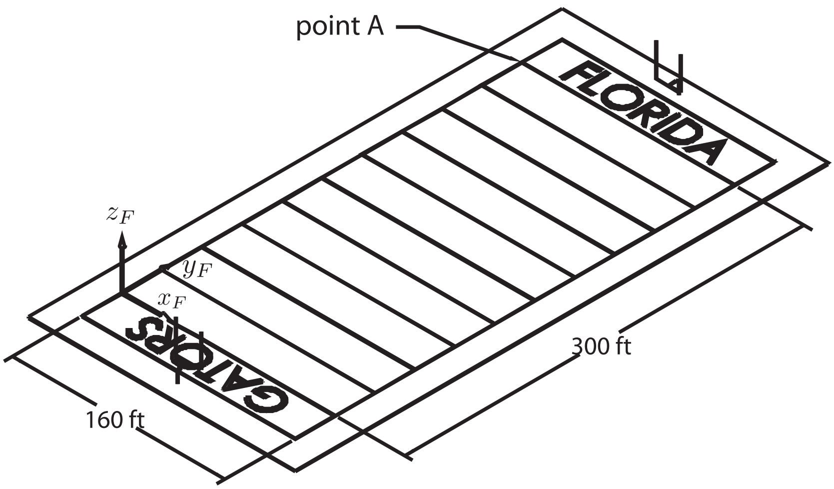

Figure 1.19 Football stadium

10. (a) Show that the three points $(w_1; \boldsymbol{S}_1)$, $(w_2; \boldsymbol{S}_2)$, and $(w_3; \boldsymbol{S}_3)$ determine the plane $[\boldsymbol{S}_1 \cdot \boldsymbol{S}_2 \times \boldsymbol{S}_3; w_1\boldsymbol{S}_2 \times \boldsymbol{S}_3 + w_2\boldsymbol{S}_3 \times \boldsymbol{S}_1 + w_3\boldsymbol{S}_1 \times \boldsymbol{S}_2]$ provided $(\boldsymbol{S}_1 \cdot \boldsymbol{S}_2 \times \boldsymbol{S}_3) \neq 0$, i.e., the points are not linearly dependent (colinear). Assume $w_i \neq 0, i = 1 \ldots 3$. (b) Show that the three planes $[D_1; \boldsymbol{S}_1]$, $[D_2; \boldsymbol{S}_2]$, and $[D_3; \boldsymbol{S}_3]$ meet in the point $(\boldsymbol{S}_1 \cdot \boldsymbol{S}_2 \times \boldsymbol{S}_3; D_1\boldsymbol{S}_2 \times \boldsymbol{S}_3 + D_2\boldsymbol{S}_3 \times \boldsymbol{S}_1 + D_3\boldsymbol{S}_1 \times \boldsymbol{S}_2)$ provided $(\boldsymbol{S}_1 \cdot \boldsymbol{S}_2 \times \boldsymbol{S}_3) \neq 0$.
11. Show that the equation of the plane through the origin that contains the line $\{\boldsymbol{S}_1; \boldsymbol{S}_{OL1}\}$ can be expressed in the form $P_1x + Q_1y + R_1z = 0$, where P_1, Q_1, R_1 are the components of the vector $\boldsymbol{S}_{OL1}$, and x, y, z are the components of $\mathbf{S}_1$.
12. Show that the equation of the plane that contains the line $\{\boldsymbol{S}_1; \boldsymbol{S}_{OL1}\} = \{L_1, M_1, N_1; P_1, Q_1, R_1\}$ and that is parallel to the x axis can be written as $N_1\,y - M_1\,z - P_1 = 0$.
13. A television camera is located within a stadium (see Figure 1.19). The objective is to determine the position of the camera as measured in terms of the coordinate system shown in the figure.

    The camera is aimed at point A and the unit direction vector from the camera to point A is measured as $[0.45339, 0.84633, -0.27959]^T$. The camera is then pointed at the origin of the reference coordinate system, and the unit direction vector from the camera to the origin is measured as $[0.29892, -0.93611, -0.18533]^T$.

    (a) Determine the Plücker coordinates of the line from the camera to point A and the line from the camera to the origin point.
    (b) Determine the perpendicular distance between the lines.
    (c) If the lines intersect, determine the point of intersection. If the lines do not intersect, determine the midpoint of the line segment that is perpendicular to the two lines.

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