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# THE ALGEBRAIC CLOSURE IN FUNCTION FIELDS OF QUADRATIC FORMS IN CHARACTERISTIC 2

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For a field $k$ of characteristic not two, it is known that $k$ is algebraically closed in the function field of any (non-degenerate) quadratic form in three or more variables. In this note we consider fields of characteristic two and decide when $k$ is algebraically closed in a function field of a quadratic $k$-form. For quadratic forms in three variables this has recently been done by Ohm.

## 1. INTRODUCTION

Let $k$ be a field of characteistic not two. Then $k$ is algebraically closed in the function field of any (non-degenerate) quadratic $k$-form in three or more variables. This is because such forms are absolutely irreducible (that is, they remain irreducible over the algebraic closure of $k$) and therefore their function fields are regular (see [**3**, p.18, Theorem 5]).

In this note we take $k$ to be a field of characteristic two, and $Q$ to be an irreducible quadratic $k$-form. We answer the following.

QUESTION: When is $k$ algebraically closed in the function field of $Q$?
For function fields of conics, the question has been answered by Ohm in [**2**, 2.8-2.12].

TERMINOLOGY AND PRELIMINARIES: By a quadratic $k$-form $Q(X)$ we mean a homogeneous polynomial of degree 2 in the variables $X = (X_1, \dots, X_n)$ with coefficients from $k$. If $Q$ is irreducible, then by the function field $k(Q)$ of $Q$ over $k$ we mean the field of fractions of the integral domain $k[X]/(Q)$, where $(Q)$ denotes the ideal in $k[X]$ generated by the polynomial $Q$. Therefore an extension $K/k$ is (isomorphic to) the function field of $Q(X_0, \dots, X_n)$ if and only if $K = k(x_0, \dots, x_n)$ such that $Q(x) = 0$ and the transcendence degree of $K/k$, abbreviated $\operatorname{dt}(K/k)$, equals $n$. If $Q'$ is obtained from $Q$ by means of an invertible linear change of variables, then $k(Q)$ and $k(Q')$ are $k$-isomorphic.

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Over a field $k$ of characteristic two, any quadratic $k$-form can be written (after an invertible linear change of variables) as

$$(*)\qquad Q(X, Y, Z) = \sum_{i=1}^{r} \left(a_i X_i^2 + X_i Y_i + b_i Y_i^2\right) + \sum_{i=1}^{s} c_i Z_i^2$$

where $a_i$, $b_i$, $c_i \in k$ and $r$, $s \geqslant 0$ (see [**1**]).

## 2. The Results

Remark 1. Let $Q$ be as in $(*)$ above. Direct calculation shows that

(1) $Q$ is reducible if and only if either
   i. $r = 0$ and $c_i/c_j \in k^2$ for $1 \leqslant i, j \leqslant s$ and $c_j \neq 0$, or
   ii. $s = 0$ and $r = 1$ and $a_1 T^2 + T + b_1$ is reducible in $k[T]$.

(2) $Q$ is absolutely irreducible if and only if either $r \geqslant 2$, or $r = 1$ and $c_i \neq 0$ for some $i$.

If $Q$ is absolutely irreducible, then its function field is regular and therefore $k$ is algebraically closed in $k(Q)$. If $r = 1$ and $c_i = 0$ for $1 \leqslant i \leqslant s$, then $Q = a_1 X^2 + XY + b_1 Y^2$ and therefore $k$ is not algebraically closed in $k(Q)$. It remains to discuss the question when $r = 0$; that is, when $Q$ is diagonal. This is done in our theorem below.

Remark 2. Let $k$ be a field of characteristic 2, and $a_1, \dots, a_n \in k$. Let $z_1, \dots, z_n$ be algebraically independent elements over $k$. Then the polynomial

$$Z^2 + \left(a_1 z_1^2 + \cdots + a_n z_n^2\right)$$

in one variable $Z$ over $L := k(z_1, \dots, z_n)$ is reducible if and only if $\sqrt{a_i} \in k$ for all $1 \leqslant i \leqslant n$.

Proof: The $L$-polynomial $Z^2 + \left(a_1 z_1^2 + \cdots + a_n z_n^2\right)$ is reducible over $L$ if and only if $a_1 z_1^2 + \cdots + a_n z_n^2$ is a square in $L$. That is, $\left(a_1 z_1^2 + \cdots + a_n z_n^2\right) g^2 = f^2$ where $f$ and $g$ and $k$-polynomials in (the algebraically independent elements) $z_1, \dots, z_n$. Comparing the leading coefficients of $z_i$ in the last equation, we have $a_i \in k^2$. ∎

The following lemma will serve as the inductive step for the proof of our theorem and is due to Ohm (see [**2**, 2.12]).

Lemma. *Let $k$ be a field of characteristic 2 and let $Q(X, Y, Z) = X^2 + aY^2 + bZ^2$ be an irreducible $k$-form such that $k$ is not algebraically closed in $k(Q)$. Then $\left[k\left(\sqrt{a}, \sqrt{b}\right) : k\right] = 2$.*

Proof: The function field $k(Q)$ equals $k(y, z)\left(\sqrt{(ay^2 + bz^2)}\right)$ with $y$, $z$ algebraically independent over $k$. Set $\alpha = \sqrt{(ay^2 + bz^2)}$. By hypothesis there exists

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$d \in k(Q)$ algebraic over $k$ and $d \notin k$. Since $k(y, z)$ is pure transcendental over $k$, $d \notin k(y, z)$. Therefore $k(Q) = k(y, z)(\alpha) = k(d)(y, z)$. In particular, $[k(d) : k] = 2$. Also, the polynomial $X^2 + (ay^2 + bz^2)$ is reducible over $k(d)(y, z)$. By Remark 2, we have $\sqrt{a}, \sqrt{b} \in k(d)$; hence $\left[k\left(\sqrt{a}, \sqrt{b}\right) : k\right] = [k(d) : k] = 2$. ∎

**THEOREM.** *Let $k$ be a field of characteristic 2 and let $Q(X) = X_0^2 + a_1X_1^2 + \cdots + a_nX_n^2$ be an irreducible quadratic $k$-form. Then $k$ is algebraically closed in $k(Q)$ if and only if $\left[k\left(\sqrt{a_1}, \sqrt{a_2}, \ldots, \sqrt{a_n}\right) : k\right] \geqslant 4$.*

PROOF: As in the introduction, $k(Q) = k(x_0, \ldots, x_n)$ such that

$$x_0^2 + a_1x_1^2 + \cdots + a_nx_n^2 = 0 \tag{1}$$

and the transcendence degree of $k(x_0, \ldots, x_n)/k$ equals $n$. For the rest of the proof let $L := k\left(\sqrt{a_1}, \sqrt{a_2}, \ldots, \sqrt{a_n}\right)$.

For $n = 1$, the $k$-polynomial $T^2 + a_1$ has a root in $k(Q)$, hence the theorem is true. Now, let $n > 1$. Assume first that $[L : k] = 2$. By symmetry, we may assume that $\sqrt{a_1} \notin k$, and for $i > 1$, $\sqrt{a_i} = \alpha_i + \beta_i\sqrt{a_1}$ where $\alpha_i, \beta_i \in k$. Substituting in (1) we get

$$0 = (x_0 + \alpha_2x_2 + \cdots + \alpha_{n-1}x_{n-1} + \alpha_nx_n) + \sqrt{a_1}(x_1 + \beta_2x_2 + \cdots + \beta_{n-1}x_{n-1} + \beta_nx_n).$$

If $x_1 + \beta_2x_2 + \beta_3x_3 + \cdots + \beta_nx_n = 0$, then $x_0 + a_2x_2 + \alpha_3x_3 + \cdots + \alpha_nx_n = 0$ and therefore $x_1, x_2 \in k(x_2, \ldots, x_n)$. Hence $\mathrm{dt}\,(k(Q)/k) \leqslant n - 1$; a contradiction. So, $x_1 + \beta_2x_2 + \beta_3x_3 + \cdots + \beta_nx_n \neq 0$. Then the last displayed equation implies that

$$\sqrt{a_1} = \frac{x_0 + \alpha_2x_2 + \cdots + \alpha_{n-1}x_{n-1} + \alpha_nx_n}{x_1 + \beta_2x_2 + \cdots + \beta_{n-1}x_{n-1} + \beta_nx_n} \in k(Q).$$

Thus $\sqrt{a_1} \in k(Q)$ and $\sqrt{a_1} \notin k$. Therefore $k$ is not algebraically closed in $k(Q)$.

Now assume that $[L : k] \geqslant 4$. We want to show that in this case $k$ is algebraically closed in $k(Q)$. We use induction on $n$. The case $n = 2$ follows from the Lemma above. So assume that $n > 2$.

We first treat the case $[L : k] = 4$. Without loss of generality, we may assume that $L = k\left(\sqrt{a_1}, \sqrt{a_2}\right)$. Since $n > 2$, we have $\sqrt{a_n} \in k\left(\sqrt{a_1}, \sqrt{a_2}\right)$. Therefore $\sqrt{a_n} = \alpha + \beta\sqrt{a_1} + \gamma\sqrt{a_2} + \delta\sqrt{a_1a_2}$, which implies that $a_n = \alpha^2 + \beta^2a_1 + \gamma^2a_2 + \delta^2a_1a_2$ where $\alpha, \beta, \gamma, \delta \in k$. Substituting in (1), we get

$$\begin{aligned} 0 &= (x_0 + \alpha x_n)^2 + a_1(x_1 + \beta x_n)^2 + a_2(x_2 + \gamma x_n)^2 + a_3x_3^2 \\ &\quad + \cdots + a_{n-1}x^2 + a_1a_2\delta^2x_n^2 \\ &= y_0^2 + a_1y_1^2 + a_2y_2^2 + a_3y_3^2 + \cdots + a_{n-1}y_{n-1}^2 + a_1a_2\delta^2y_n^2 \end{aligned} \tag{2}$$

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where $y_0 = x_0 + \alpha x_n$, $y_1 = x_1 + \beta x_n$, $y_2 = x_2 + \gamma x_n$, and $y_i = x_i$, $i \geqslant 3$. Note that $k(Q) = k(x_0, \dots, x_n) = k(y_0, \dots, y_n)$. In particular, the transcendence degree of the field $K := k(y_0, \dots, y_{n-1})/k \geqslant n-1$.

If $\delta = 0$, then from equation (2) we conclude that $\operatorname{dt}(K/k) = n-1$ and that $K$ is the function field of the quadratic form $Y_0^2 + a_1 Y_1^2 + a_2 Y_2^2 + a_3 Y_3^2 + \dots + a_{n-1} Y_{n-1}^2$. Therefore by the inductive hypothesis, $k$ is algebraically closed in $K$ since $\left[k\left(\sqrt{a_1}, \sqrt{a_2}, \cdots, \sqrt{a_{n-1}}\right) : k\right] = 4$. Now since $k(Q) = K(y_n)$ and $\operatorname{dt}(k(Q)/k) = 1 + dt(K/k)$, $y_n$ is transcentdental over $K$. Therefore $k$ is algebraically closed in $K(y_n) = k(Q)$.

On the other hand, if $\delta \neq 0$, then from (2) we have

$$0 = y_0^2 + a_1 y_1^2 + a_2\left(y_2^2 + a_1 \delta^2 y_n^2\right) + a_3 y_3^2 + \dots + a_{n-1} y_{n-1}^2. \tag{3}$$

If $y_2^2 + a_1\delta^2 y_n^2 = 0$, then $y_2$ is algebraic over $k(y_n)$, and equation (3) implies that $y_0$ is algebraic over $k(y_1, \dots, y_{n-1})$. Hence $y_0$ and $y_2$ are algebraic over $k(y_1, y_3, \dots, y_{n-1})$. This implies that $\operatorname{dt}(k(y_0, \dots, y_n)/k) \leqslant n-1$. But $k(Q) = k(y_0, \dots, y_n)$ has transcendence degree $n$, a contradiction. Therefore $y_2^2 + a_1\delta^2 y_n^2 \neq 0$. Now by setting $z_n = \delta y_n / y_2$, $z_0 = (y_0 + a_1 y_1 z_n)/\left(1 + a_1 z_n^2\right)$, $z_1 = (y_1 + y_0 z_n)/\left(1 + a_1 z_n^2\right)$, $z_2 = y_2$, and for $3 \leqslant i < n$, $z_i = y_i$ and $A_i = a_i/\left(1 + a_1 z_n^2\right) \in k(z_n)$, we have from equation (3)

$$0 = \left(1 + a_1 z_n^2\right)\left(z_0^2 + a_1 z_1^2 + a_2 z_2^2 + A_3 z_3^2 + \dots + A_{n-1} z_{n-1}^2\right),$$

and therefore

$$0 = z_0^2 + a_1 z_1^2 + a_2 z_2^2 + A_3 z_3^2 + \dots + A_{n-1} z_{n-1}^2.$$

Also note that $k(Q) = k(z_n)(z_0, \dots, z_{n-1})$ and the transcendence degree of $k(Q)/k(z_n)$ equals $n-1$. Therefore $k(Q)/k(z_n)$ is the function field of the $k(z_n)$-quadratic form

$$Z_0^2 + a_1 Z_1^2 + a_2 Z_2^2 + A_3 Z_3^2 + \dots + A_{n-1} Z_{n-1}^2.$$

Since $\left[k(z_n)\left(\sqrt{a_1}, \sqrt{a_2}, \sqrt{A_3}, \dots, \sqrt{A_{n-1}}\right) : k(z_n)\right] \geqslant 4$, the inductive hypothesis implies that $k(z_n)$ is algebraically closed in $k(z_n)(z_0, \dots, z_{n-1}) = k(Q)$. In particular, $k$ is algebraically closed in $k(Q)$. This concludes the case $[L : k] = 4$.

To finish the proof of the theorem, it is left to show that $K$ is algebraically closed in $k(Q)$ when $[L : k] \geqslant 8$. In this case we may assume, without loss of generality, that $\sqrt{a_2} \notin k\left(\sqrt{a_1}\right) \neq k$. Let $x = x_0 - \sqrt{a_1} x_1 \in k(Q)\left(\sqrt{a_1}\right)$. Then from equation (1) we have

$$0 = x^2 + a_2 x_2^2 + \dots + a_n x_n^2,$$

and $\operatorname{dt}\left(k\left(\sqrt{a_1}\right)(x, x_2, \dots, x_n)/k\left(\sqrt{a_1}\right)\right) = n-1$. Thus $k\left(\sqrt{a_1}\right)(x, x_2, \dots, x_n)$ is the function field of the $k\left(\sqrt{a_1}\right)$-form $X^2 + a_2 X_2^2 + \dots + a_n X_n^2$. By the inductive hypothesis, $k\left(\sqrt{a_1}\right)$ is algebraically closed in $k\left(\sqrt{a_1}\right)(x, x_2, \dots, x_n)$ because

$\left[k(\sqrt{a_1}, \sqrt{a_2}, \ldots, \sqrt{a_n}) : k(\sqrt{a_1})\right] \geqslant 4$. Since $k(\sqrt{a_1})(x, x_2, \ldots, x_n)(x_1) = k(\sqrt{a_1})(x_0, \ldots, x_n)$ and $x_1$ is transcendental over $k(\sqrt{a_1})(x, x_2, \ldots, x_n)$, it follows that $k(\sqrt{a_1})$ is algebraically closed in $k(\sqrt{a_1})(x_0, \ldots, x_n)$. By symmetry, $k(\sqrt{a_2})$ is also algebraically closed in $k(\sqrt{a_2})(x_0, \ldots, x_n)$. Now if $d \in k(x_0, \ldots, x_n)$ is algebraic over $k$ (and thus algebraic over $k(\sqrt{a_1})$ and $k(\sqrt{a_2})$), then $d \in k(\sqrt{a_1})$ and $d \in k(\sqrt{a_2})$. Thus $d \in k$. Thus $k$ is algebraically closed in $k(Q)$. □

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