Composito Mathematica **122:** 23–111, 2000.

Cycles on Arithmetic Surfaces

AHMED ABBES
CNRS UMR 7539, Institut Galilée, Université Paris-Nord, 93430 Villetaneuse, France. e-mail: abbes@math.univ-paris13.fr

(Received: 19 February 1998; in final form: 27 November 1998)

Abstract. We develop a localized intersection theory for arithmetic schemes on the model of Fulton's intersection theory. We prove a Lefschetz fixed point formula for arithmetic surfaces, and give an application to a conjecture of Serre on the existence of Artin's representations for regular local rings of dimension 2 and unequal characteristic.

Mathematics Subject Classifications (2000): 14C17, 14G40, 14F20.

Key words: arithmetic surfaces, localized intersection theory, bivariant classes, Lefschetz fixed point formula, Artin's representations.

Notation

Let $S = \mathrm{Spec}(R)$ be the spectrum of a characteristic 0 complete discrete valuation ring R with algebraically closed residue field k of characteristic $p > 0$ and fraction field K. s, η and $\overline{\eta}$ denote, respectively, its closed point, its generic point and a geometric generic point, corresponding to an algebraic closure $\overline{K}$ of K. Let G be the Galois group of $\overline{K}$ over K.

We fix a prime number $l \neq p$ and let Λ be one of the rings $\mathbb{Z}/l^n\mathbb{Z}$, $\mathbb{Z}_l$ or $\mathbb{Q}_l$. To any finitely generated Λ-module M with a continuous action of G, we associate its Swan conductor $\underline{\mathrm{sw}}(M)$ which is a finitely generated Λ-module (see Section 2 for the definition). Any G-equivariant endomorphism of M induces an endomorphism of $\underline{\mathrm{sw}}(M)$.

We work in the category of separated schemes of finite type over S. The subscripts s and η, associated with an object in this category, denote, respectively, its closed and its generic fibers. Associated with a morphism, they denote the induced morphisms over the closed and the generic fibers.

Let X be a separated scheme of finite type over S. The group of cycles $Z(X) = \oplus_n Z_n(X)$ and the Chow group $A(X) = \oplus_n A_n(X)$ are graded by the absolute dimension over S. The latter is the sum of the relative dimension over S and the dimension of S. In this paper, dimension stands for the absolute dimension over S. Notice that if X is proper over S, the absolute dimension coincides with the Krull dimension.

The group of n-bivariant classes associated with a map $X \to Y$ is denoted $A^n(X \to Y)$ ([7] chapter 17).

If $i\colon X \to Y$ is a closed immersion with ideal sheaf $\mathcal{I}$, we denote $\mathcal{N}_X Y = \mathcal{I}/\mathcal{I}^2$ its conormal sheaf and $\mathcal{S}_X Y = \bigoplus_{n \geqslant 0} \mathcal{I}^n/\mathcal{I}^{n+1}$ the graded algebra giving its normal cone.

An arithmetic scheme over S stands for a regular integral scheme proper and flat over S. An arithmetic scheme of dimension 2 is an arithmetic surface. One such is semi–stable if its closed fiber is a reduced divisor with normal crossings.

If X is an arithmetic scheme and σ is an S-automorphism of X, we denote $H^*_{et}(X_z, \mathbb{Q}_\ell)$ for $z = s$ or $\overline{\eta}$ the ℓ–adic étale cohomology groups of respectively its closed and its geometric generic fibers. We use the short-hands $\mathrm{tr}(\sigma)|H^*_{et}(X_z, \mathbb{Q}_\ell)$ where $z = s$ or $\overline{\eta}$ and $\mathrm{tr}(\sigma)|\underline{\mathrm{sw}}(H^*_{et}(X_{\overline{\eta}}, \mathbb{Q}_\ell))$ for the alternating sum of the traces of σ.

1. Introduction

The study of cycles on arithmetic schemes was pioneered by S. Bloch. The idea behind his approach was to overcome the lack of a ground field for these schemes by providing functorial constructions of cycle classes over their special fibers. In [3], Bloch associates with an arithmetic scheme X over S of relative dimension d, a kind of Euler characteristic which measures its arithmetic complexity. The ℓ-adic étale cohomology of the geometric generic fiber of X can produce such an invariant. But a good candidate should also have an analogue on the cycle level. For instance, the Euler characteristic of a proper smooth variety over a field coincides with the degree of the top Chern class of its tangent bundle. In a first approach, one can consider the top Chern class of the sheaf of relative differentials $\Omega^1_{X/S}$. Since $A_0(S)$ is trivial, we cannot get any relevant information from this class.

Any zero-cycle class on X can be represented by a cycle on the closed fiber of X. In term of Chow groups, the push–forward map $A_0(X_s) \longrightarrow A_0(X)$ is surjective, but it is far from being injective. In the special case of $c_{d+1}(\Omega^1_{X/S}) \cap [X] \in A_0(X)$, the graph construction of Fulton and MacPherson [7] provides a canonical lifting. More precisely, this construction gives a bivariant class $c^X_{d+1,X_s}(\Omega^1_{X/S}) \in A^{d+1}(X_s \to X)$ which refines the usual Chern class $c_{d+1}(\Omega^1_{X/S})$. Bloch [3] defines the *localized Euler characteristic* of X to be the degree of $(-1)^{d+1} c^X_{d+1,X_s}(\Omega^1_{X/S}) \cap [X]$, which is a zero cycle class on the closed fiber X_s. He conjectured that the localized Euler characteristic is minus the Artin conductor of the arithmetic scheme. The main result of his paper is a proof of this conjecture for arithmetic surfaces. Later, many works have emphasized the importance of this invariant [4, 16, 17].

In this paper, we develop a general theory which includes Bloch's approach. His main result turns out to be a special case of a Lefschetz fixed point formula in this theory. The latter was conjectured by K. Kato, S. Saito and T. Saito for any relative regular curve over an excellent henselian discrete valuation ring ([10] conjecture (1.5)), and proved by them in the geometric case (i.e. the equal characteristic case). Finally, we give an application of our formula to a conjecture of Serre on the existence of Artin's representations for two-dimensional regular local rings in the unequal characteristic case.

In the following, we describe the results of this paper in more detail. In the spirit of Fulton's theory [7], we develop a *localized intersection theory* adapted to the localized Chern classes. Our theory works for general arithmetic schemes but the main theorems are proved only for arithmetic surfaces. For this reason, we restrict the introduction to the case of an arithmetic surface X over S. Let $\Delta_X \to X \times_S X$ be the diagonal closed immersion. We associate with any fiber square:

$$\begin{tikzcd} W \arrow[r] \arrow[d] & V \arrow[d] \\ \Delta_X \arrow[r] & X \times_S X \end{tikzcd}$$

where V is a scheme of pure dimension k, a $(k-2)$-cycle class in *the closed fiber* of W, called localized intersection product of Δ_X with V, and denoted $(\Delta_X.[V])_{loc} \in A_{k-2}(W_s)$. The formation of this class is compatible with proper push-forward and flat pull-back, and it satisfies an excess formula. Moreover, it is uniquely determined by these properties. In this theory, the localized Euler characteristic occurs as the self-intersection of the diagonal: $(\Delta_X.\Delta_X)_{loc} = c^X_{2,X_s}(\Omega^1_{X/S}) \cap [X] \in A_0(X_s)$.

1.1. LEFSCHETZ FIXED POINT FORMULA FOR ARITHMETIC SURFACES

Let X be an arithmetic surface over S, σ be an S-automorphism of X, and $\Gamma \subset X \times_S X$ be its graph. The localized intersection product $(\Delta_X.\Gamma)_{loc}$ is a 0-cycle class in the closed fiber of X. Therefore, we can take its degree which is equally denoted $(\Delta_X.\Gamma)_{loc}$.

THEOREM 1.1. *Let X be an arithmetic surface over S and σ be an S-automorphism of X. Then,*

$$(\Delta_X.\Gamma)_{loc} = -\mathrm{tr}(\sigma)|\underline{\mathrm{sw}}(H^*_{et}(X_{\overline{\eta}}, \mathbb{Q}_l)) + \mathrm{tr}(\sigma)|H^*_{et}(X_s, \mathbb{Q}_l) - \mathrm{tr}(\sigma)|H^*_{et}(X_{\overline{\eta}}, \mathbb{Q}_l). \quad (1)$$

The Swan conductors $\underline{\mathrm{sw}}(H^*_{et}(X_{\overline{\eta}}, \mathbb{Q}_l))$ vanish when the action of the Galois group G of $\overline{K}$ over K on $H^*_{et}(X_{\overline{\eta}}, \mathbb{Q}_l)$ is tame. Hence, for a semi-stable arithmetic surface X, the Lefschetz fixed point formula becomes

$$(\Delta_X.\Gamma)_{loc} = \mathrm{tr}(\sigma)|H^*(X_s, \mathbb{Q}_l) - \mathrm{tr}(\sigma)|H^*(X_{\overline{\eta}}, \mathbb{Q}_l). \quad (2)$$

Formula (1) for $\sigma \neq \mathrm{id}$ was conjectured, in a different formulation, by K. Kato, S. Saito and T. Saito for any relative regular curve over an excellent Henselian discrete valuation ring ([10] conjecture 1.5), and proved by them in the geometric case. We prove in remark 10.1 that equation (1) is equivalent to their formulation. The formula for $\sigma = \mathrm{id}$ was proved by Bloch [3]. Our proof closely follows his, even

https://doi.org/10.1023/A:1001822419774 Published online by Cambridge University Press

if the technical details are more involved. To avoid large intersections with Bloch's article, we will restrict the study to non-trivial automorphisms. The reader should consult this article for a proof of the theorem in the case $\sigma = \mathrm{id}$.

We outline the proof of Theorem 1.1. We proceed in two steps. In the first step, we prove the theorem for semi-stable surfaces. The vanishing cycles for these surfaces can be computed explicitly. In particular, the difference of the alternating traces of σ can be expressed in term of its action over the singular points in the special fiber. Then, we prove that $(\Delta_X.\Gamma)_{loc}$ is given by the same expression. For this purpose, we establish a residual formula for the localized intersection theory.

The second step is a reduction to the semi-stable case. By the semi-stable reduction theorem, there exists a finite flat totally ramified Galois extension T of S such that $X \times_S T$ has a semi-stable regular model V. The automorphism σ extends to a uniquely determined T-automorphism of V. Then, the main problem is to compare the Lefschetz numbers of σ over X and over V. Inspired by the classical intersection theory and Bloch's work [3], we solve this problem by a projection formula. This formula relates the Lefschetz number of σ over X to the sum of the Lefschetz numbers of $\sigma \circ \tau$ over V, where τ runs over the Galois group of T over S. The projection formula is the basic ingredient in the proof of Theorem 1.1. But its importance should be emphasized as an independent result.

1.2. PROJECTION FORMULA

Let $f: X \to Y$ be a morphism of finite degree n between arithmetic surfaces over S. Given two S-automorphisms σ of X and τ of Y such that $\tau \circ f = f \circ \sigma$, we would like to compare the Lefschetz numbers of σ and τ. For this purpose, we consider the Cartesian diagram

$$\begin{tikzcd} W \arrow[r] \arrow[d] & X \times_Y X \arrow[r] \arrow[d] & \Delta_Y \arrow[d, "j"] \\ \Gamma_\sigma \arrow[r, "i"] & X \times_S X \arrow[r] & Y \times_S Y \end{tikzcd}$$

where W is the intersection of Γ_σ with $X \times_Y X$. On the one hand, $X \times_Y X$ has pure dimension 2. Its localized intersection with Γ_σ is a cycle class of dimension 0 over W_s. On the other hand, the localized intersection of Γ_σ with Δ_Y is also a cycle class of dimension 0 over W_s. We expect that these cycle classes are the same:

$$(\Gamma_\sigma.[X \times_Y X])_{loc} = (\Delta_Y.\Gamma_\sigma)_{loc} \in A_0(W_s).$$

We call this formula a projection formula because we think of $X \times_Y X$ as the pull-back of Δ_Y. To prove the Lefschetz fixed point formula, we need to consider the projection formula only for morphisms f such that $f_\eta: X_\eta \to Y_\eta$ is étale. This case will be treated in Section 6 for non-trivial automorphisms. The trivial automorphism case was proved by Bloch [3]. We would like here to emphasize

https://doi.org/10.1023/A:1001822419774 Published online by Cambridge University Press

an other aspect of this formula, namely a relation with a *Hurwitz formula* for arithmetic surfaces.

Given a morphism $f\colon X \to Y$ of finite degree n between two arithmetic surfaces, we ask for a formula relating the localized Euler characteristics of X and Y and the ramification of f in the spirit of the usual Hurwitz formula. The equation

$$f_{s*}(\Delta_X.[X \times_Y X])_{loc} = n(\Delta_Y.\Delta_Y)_{loc} \in A_0(Y_s) \tag{3}$$

may be a good candidate for a Hurwitz formula. Indeed, we decompose $X \times_Y X$ into its irreducible components, $[X \times_Y X] = \sum_{i=1}^{r} n_i[V_i]$. The diagonal Δ_X occurs as one component with multiplicity one, put $V_1 = \Delta_X$. Then, formula (3) reads

$$f_{s*}(c^X_{2,X_s}(\Omega^1_{X/S}) \cap [X]) - nc^Y_{2,Y_s}(\Omega^1_{Y/S}) \cap [Y] = -\sum_{i=2}^{r} n_i f_{s*}(\Delta_X.V_i)_{loc} \in A_0(Y_s).$$

For $i > 1$, the scheme theoretic intersection $\Delta_X \cap V_i$ is contained in the ramification divisor of f, and $(\Delta_X.V_i)_{loc}$ should be understood as its contribution to the Hurwitz formula. We lack a general proof of this formula. But at least two reasons stand for it. First (3) holds if $f_\eta\colon X_\eta \to Y_\eta$ is étale, and second it holds in $A_0(Y)$.

1.3. ON SERRE'S CONJECTURE ON THE EXISTENCE OF ARTIN'S REPRESENTATIONS

Let A be a regular local ring with maximal ideal $\mathfrak{m}$ and G be a finite group of automorphisms of A. For $\sigma \in G$, let I_σ be the ideal of A generated by $\{a - \sigma(a), a \in A\}$. Assume

(i) $A^G = \{a \in A; a = \sigma(a) \ \ \forall \sigma \in G\}$ is a Noetherian ring and A is finitely generated A^G-module;
(ii) for each $\sigma \in G - \{1\}$, A/I_σ has finite length;
(iii) the map $A^G/(A^G \cap \mathfrak{m}) \to A/\mathfrak{m}$ is an isomorphism.

Then, define the function $a_G : G \to \mathbb{Z}$ by

$$a_G(\sigma) = -\mathrm{leng}_A(A/I_\sigma) \quad \text{if } \sigma \neq 1,$$
$$a_G(1) = -\sum_{\sigma \in G - \{1\}} a_G(\sigma).$$

Serre conjectured that a_G is the character of a $\mathbb{Q}_l$-rational representation of G for any prime number l which is invertible in A ([19], chapter 6). This conjecture was proved in dimension 1 by Artin, Arf and Serre [18, 19] and in dimension 2 and equal characteristic case by K. Kato, S. Saito and T. Saito [10]. As a corollary of the Lefschetz fixed point formula (1), we can prove Serre's conjecture for some 2-dimensional regular local rings in the unequal characteristic case. A proof in this case was announced by Kato [8, 9] but has not been published.

LEMMA 1.2. *Let X be an arithmetic surface over S and σ be an S-automorphism of X. Assume that the scheme of σ-fixed points over X consists of one closed point x, and let $\mathcal{I}_\sigma$ be the ideal of $\mathcal{O}_{X,x}$ generated by $\sigma(a) - a$ for $a \in \mathcal{O}_{X,x}$. Then,*

$$(\Delta_X.\Gamma_\sigma)_{loc} = \mathrm{Leng}(\mathcal{O}_{X,x}/\mathcal{I}_\sigma).$$

We deduce the following from Theorem 1.1 and Lemma 1.2 (the proof is given in Section 10):

COROLLARY 1.3. *Let X be an arithmetic surface over S and G be a finite group of S-automorphisms of X. Assume that there exists a closed point x in X which is the unique σ-fixed point of X for any $\sigma \in G - \{1\}$. Then, Serre's conjecture holds for the regular local ring $\mathcal{O}_{X,x}$.*

2. Swan Conductors

In this section, we recall the definition of Swan conductors [10, 18, 19]. Let L/K be a finite Galois extension of Galois group $G(L/K)$. Let π be a uniformizing element in L and v be the discrete valuation of L. For $\sigma \in G(L/K) - \{1\}$, put $i(\sigma) = v(\sigma(\pi) - \pi)$. Then, define the function $\mathrm{sw}_{L/K}\colon G(L/K) \to \mathbb{Z}$ by the following:

$$\mathrm{sw}_{L/K}(\sigma) = \begin{cases} 1 - i(\sigma) & \text{if } \sigma \neq 1, \\ \sum_{\tau\neq 1}(i(\tau) - 1) & \text{if } \sigma = 1. \end{cases}$$

This is clearly a central function over $G(L/K)$ and we have the fundamental result [18–20]:

THEOREM 2.1 (Artin, Arf, Serre). *For any prime number $l \neq p$, there exists a $\mathbb{Z}_l[G(L/K)]$–projective module $\mathrm{Sw}_{L/K}$ such that $\mathrm{Sw}_{L/K} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ has $\mathrm{sw}_{L/K}$ as a character. This module is unique up to isomorphism.*

Let G be the Galois group of $\overline{K}$ over K. Fix a prime number $l \neq p$ and let Λ be one of the rings $\mathbb{Z}/l^n\mathbb{Z}$, $\mathbb{Z}_l$ or $\mathbb{Q}_l$. Let M be a finitely generated Λ-module with a continuous action of G. We associate to M the Λ-module $\underline{\mathrm{sw}}(M)$ defined as follows:

(1) If $\Lambda = \mathbb{Z}/l^n\mathbb{Z}$, then the action of G over M factors through $G(L/K)$ for a finite Galois extension L over K. Put $\underline{\mathrm{sw}}(M) = \mathrm{Hom}_{G(L/K)}(\mathrm{Sw}_{L/K}, M)$. It is a finitely generated Λ-module which does not depend on L.

(2) If $\Lambda = \mathbb{Z}_l$, put $\underline{\mathrm{sw}}(M) = \varprojlim_n \underline{\mathrm{sw}}(M/l^nM)$.

(3) If $\Lambda = \mathbb{Q}_l$, take a $\mathbb{Z}_l$-lattice B of M stable under the action of G (which exists by the compactness of G), and put $\underline{\mathrm{sw}}(M) = \underline{\mathrm{sw}}(B) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$.

Hence, $\underline{\mathrm{sw}}(M)$ is finitely generated over Λ, and $\underline{\mathrm{sw}}$ is an exact functor which sends free modules to free modules. If the action of G on M is tame then $\underline{\mathrm{sw}}(M) = 0$.

We now consider M a finite dimensional $\mathbb{Q}_l$-vector space on which G operates continuously and σ a $\mathbb{Q}_l[G]$-linear map on M. Let P be the p-Sylow subgroup of G. By compactness of G, the group P acts on M through a finite quotient. Let L be a finite Galois extension of K contained in $\overline{K}$ such that P acts on M by its quotient $P(L/K)$ in the Galois group $G(L/K)$ of L over K (i.e. $P \cap G_L$ acts trivially on M where G_L denotes the Galois group of $\overline{K}$ over L). Then,

$$\operatorname{tr}(\sigma)|\underline{\mathrm{sw}}(M) = \frac{1}{\#G(L/K)} \sum_{\tau \in P(L/K)} \mathrm{sw}_{L/K}(\tau)\operatorname{tr}(\sigma\tau)|M \tag{4}$$

$$= \frac{1}{\#G(L/K)} \sum_{\tau \in G(L/K)} \mathrm{sw}_{L/K}(\tau)\operatorname{tr}(\sigma\tau)|M. \tag{5}$$

The group $G(L/K)$ does not act on M. The meaning of (5) is that $\mathrm{sw}_{L/K}(\tau)$ vanishes if $\tau \notin P(L/K)$.

3. Localized Chern Classes

The construction of these bivariant classes is based on the graph construction of Fulton and MacPherson ([7], chapter 18). We recall in the following a variant introduced by Bloch [3].

3.1. THE GRAPH CONSTRUCTION

For the beginning of this section, we work in the category of separated schemes of finite type over an arbitrary regular Noetherian base scheme S. Let X be a closed subscheme of a scheme Y and $\mathcal{E}.$ be a bounded complex of locally free $\mathcal{O}_Y$-modules of finite ranks:

$$0 = \mathcal{E}_{n+1} \xrightarrow{d_{n+1}} \mathcal{E}_n \xrightarrow{d_n} \mathcal{E}_{n-1} \longrightarrow \cdots\cdots \longrightarrow \mathcal{E}_1 \xrightarrow{d_1} \mathcal{E}_0 \xrightarrow{d_0} \mathcal{E}_{-1} = 0.$$

Let $\mathcal{H}.(\mathcal{E}.)$ be the homology of this complex and assume that

$$(\mathrm{P}) \begin{cases} \text{(i)} & \mathcal{E}_i = 0 \text{ for } i < 0, \\ \text{(ii)} & \mathcal{H}_i(\mathcal{E}.) \text{ is supported on } X \text{ for } i > 0, \\ \text{(iii)} & \mathcal{H}_0(\mathcal{E}.)_{|Y-X} \text{ is locally free of rank } e \geqslant 0. \end{cases}$$

We associate with $\mathcal{E}.$ localized Chern classes $c^Y_{p,X}(\mathcal{E}.) \in A^p(X \to Y)$ for all $p \geqslant e+1$. We first construct a map

$$c^Y_{p,X}(\mathcal{E}.) \cap \_ : Z_*(Y) \to A_{*-p}(X).$$

Let e_i be the rank of $\mathcal{E}_i$, $G_i = \mathrm{Grass}_{e_i}(\mathcal{E}_i \oplus \mathcal{E}_{i-1})$ and set $G =$

$G_n \times_Y G_{n-1} \times_Y \ldots \times_Y G_0$. Let ξ_i be the tautological bundle of rank e_i on G_i, and set

$$\xi = \sum_{i=0}^{n} (-1)^i [\mathrm{pr}_i^* \xi_i] \in K^0(G),$$

where $\mathrm{pr}_i : G \to G_i$ is the projection. There is a natural closed immersion:

$$Y \times_S \mathbb{A}^1 \xrightarrow{\varphi} G \times_S \mathbb{A}^1, \qquad (y, \lambda) \mapsto \left(\prod_i \Gamma(\lambda d_i(y)), \lambda \right),$$

where $\Gamma(\lambda d_i(y)) \subset \mathcal{E}_i(y) \oplus \mathcal{E}_{i-1}(y)$ is the graph of $\lambda d_i(y)$.

Define integers k_i by $k_n = 0$ and by requiring $k_i + k_{i-1} = e_i$ for $0 \leqslant i \leqslant n$. Assume that $Y - X$ is not empty. Then, $k_i \geqslant 0$ for all $0 \leqslant i \leqslant n$ and $k_0 = e_0 - e$. Denote $H_i = \mathrm{Grass}_{k_i}(\mathcal{E}_i)$, and set $H = H_n \times_Y H_{n-1} \times_Y \ldots \times_Y H_0$. There is a canonical closed immersion $\tau \colon H \to G$ defined by

$$(L_n, L_{n-1}, \ldots, L_1, L_0) \mapsto (L_n \oplus L_{n-1}, L_{n-1} \oplus L_{n-2}, \ldots, L_1 \oplus L_0, y),$$

where y is the projection of L_0 to Y (remember that $G_0 = Y$). The pull-back of ξ to H is $\tau^*(\xi) = \mathrm{pr}_0^* \theta_0$, where $\mathrm{pr}_0 : H \to H_0$ is the projection, and θ_0 is the canonical quotient bundle of rank e defined over $H_0 = \mathrm{Grass}_{e_0 - e}(\mathcal{E}_0)$. Let H^0 be the restriction of H to $Y - X$. There is a natural section of H^0 over $Y - X$. It determines a canonical closed immersion $\psi \colon (Y - X) \to H^0$ given by

$$y \mapsto (\ker d_n(y), \ker d_{n-1}(y), \ldots, \ker d_1(y), \mathrm{im}\, d_1(y)).$$

Consider now the following non-commutative diagram

$$\begin{tikzcd}
Y \times_S \mathbb{A}^1 \arrow[rr, "\varphi"] & & G \times_S \mathbb{A}^1 \arrow[r] & G \times_S \mathbb{P}^1 \\
(Y - X) \times_S \mathbb{A}^1 \arrow[r] \arrow[u] & (Y - X) \times_S \mathbb{P}^1 \arrow[r, "\psi \times 1"] & H^0 \times_S \mathbb{P}^1 \arrow[r] & H \times_S \mathbb{P}^1 \arrow[u, "\tau \times 1"']
\end{tikzcd} \tag{6}$$

Let α be a cycle on Y and denote by α^0 its restriction to $Y - X$. Choose two cycles:

(i) α' on $G \times_S \mathbb{P}^1$ which restricts to $\varphi_*(\alpha \times_S [\mathbb{A}^1])$ on $G \times_S \mathbb{A}^1$,
(ii) α'' on $H \times_S \mathbb{P}^1$ which restricts to $\psi_*(\alpha^0) \times_S [\mathbb{P}^1]$ on $H^0 \times_S \mathbb{P}^1$.

Let $\gamma = i_\infty^*(\alpha' - \alpha'')$, where i_∞^* is the Gysin homomorphism relatively to the regular embedding of the section ∞ in $\mathbb{P}^1$. Then γ is a well defined cycle on G that does not depend on the choice of α' and that changes by a cycle on $H_X = H \times_Y X$ for another choice of α''. As proved in [7] lemma 18.1, γ is a cycle on $G \times_Y X$.

https://doi.org/10.1023/A:1001822419774 Published online by Cambridge University Press

Let $\pi\colon G\times_Y X\to X$ be the projection, and for any $p\geqslant e+1$ set

$$c_{p,X}^Y(\mathcal{E}.)\cap\alpha=\pi_*(c_p(\xi)\cap\gamma)\in A_*(X).$$

Since ξ restricts to a locally free sheaf of rank e on H_X, this definition is independent of the choice of γ.

For any $h: Y'\to Y$ and $X'=X\times_Y Y'$, the complex $h^*\mathcal{E}.$ satisfies the conditions (P) relatively to the closed immersion $X'\to Y'$. The same construction gives a map which will be denoted simply:

$$c_{p,X}^Y(\mathcal{E}.)\cap\_: Z_*(Y')\to A_{*-p}(X').$$

These maps pass to rational equivalence. They are compatible with proper push-forward, flat pull-back and intersection product:

(C_1) if h is proper, let $h': X'\to X$ be the induced morphism, then for all $\alpha\in A_k(Y')$,

$$c_{p,X}^Y(\mathcal{E}.)\cap(h_*\alpha)=h'_*(c_{p,X}^Y(\mathcal{E}.)\cap\alpha)\in A_{k-p}(X),$$

(C_2) if h is flat of relative dimension d, then for all $\alpha\in A_k(Y)$,

$$c_{p,X}^Y(\mathcal{E}.)\cap(h^*\alpha)=h'^*(c_{p,X}^Y(\mathcal{E}.)\cap\alpha)\in A_{k-p+d}(X'),$$

(C_3) if we have a Cartesian diagram

$$\begin{array}{ccccc} X' & \longrightarrow & Y' & \longrightarrow & Z' \\ i''\downarrow & & \downarrow i' & & \downarrow i \\ X & \longrightarrow & Y & \longrightarrow & Z \end{array}$$

such that i is a regular embedding of codimension d. Then for all $\alpha\in A_k(Y)$,

$$i^!(c_{p,X}^Y(\mathcal{E}.)\cap\alpha)=c_{p,X}^Y(\mathcal{E}.)\cap(i^!\alpha)\in A_{k-p-d}(X'),$$

where $i^!$ is the refined Gysin morphism.

It follows from [7] chapter 17 that for any locally free $\mathcal{O}_Y$-module of finite rank F, for any integer $m\geqslant 0$, and for any $\alpha\in A_k(Y)$,

$$c_{p,X}^Y(\mathcal{E}.)\cap(c_m(F)\cap\alpha)=c_m(i^*F)\cap(c_{p,X}^Y(\mathcal{E}.)\cap\alpha)\in A_{k-p-m}(X),$$

where c_m is the mth Chern class of F and $i\colon X\to Y$ is the closed immersion. We will use the following propreties of localized Chern classes ([7] proposition 18.1 and example 18.1.3).

PROPOSITION 3.1. *Let $i\colon X\to Y$ be a closed immersion.*

(a) *Let $j: Y\to Z$ be a closed immersion and let $\mathcal{E}.$ be a complex of locally free $\mathcal{O}_Z$-modules satisfying (P) relatively to the closed immersion $X\to Z$. Then, for*

https://doi.org/10.1023/A:1001822419774 Published online by Cambridge University Press

all $\alpha \in A_k(Z)$ and $p \geqslant e+1$,

$$i_*(c^Z_{p,X}(\mathcal{E}.) \cap \alpha) = c^Z_{p,Y}(\mathcal{E}.) \cap \alpha \in A_{k-p}(Y).$$

(b) *Let $0 \to \mathcal{E}.^{(1)} \to \mathcal{E}.^{(2)} \to \mathcal{E}.^{(3)} \to 0$ be an exact sequence of complexes of locally free $\mathcal{O}_Y$-modules satisfying (P), and denote by e_i the rank of $\mathcal{H}_0(\mathcal{E}^{(i)}.)_{|Y-X}$ (so $e_2 = e_1 + e_3$). Then, for any $p \geqslant e_2 + 1$,*

$$c^Y_{p,X}(\mathcal{E}.^{(2)}) = \sum_{j=0}^{p} c'_j(\mathcal{E}.^{(1)}) c'_{p-j}(\mathcal{E}.^{(3)}),$$

where $c'_j(\mathcal{E}.^{(i)})$ is the localized Chern class $c^Y_{j,X}(\mathcal{E}.^{(i)})$ if $j \geqslant e_i + 1$, and the usual Chern class if $j \leqslant e_i$.

Remark 3.2. Proposition 3.1 implies that $c^Y_{p,X}(\mathcal{E}.)$ depends only on the quasi-isomorphism class of $(\mathcal{E}.)$. In particular, if E is a coherent sheaf of finite homological dimension on Y such that $E_{|Y-X}$ is locally free of rank e, then $c^Y_{p,X}(E)$ can be defined for $p \geqslant e+1$ by choosing any resolution of E by locally free $\mathcal{O}_Y$-modules.

From now on, we assume that $S = \mathrm{Spec}(R)$ is the spectrum of the discrete valuation ring R fixed at the beginning of the article. The closed immersions which play an important role in our theory are of the type $X_s \to X$ where X is a scheme of finite type over S and X_s is its closed fiber. For instance, let X be an arithmetic scheme over S of relative dimension d. The sheaf of relative differentials $\Omega^1_{X/S}$ has finite homological dimension and is locally free of rank d on the generic fiber X_η. Hence, one can compute $c^X_{d+1,X_s}(\Omega^1_{X/S}) \cap [X]$ as a zero cycle class over the closed fiber X_s.

DEFINITION 3.3. The localized Euler characteristic of X is

$$c^{loc}_{d+1}(X) = \deg((-1)^{d+1} c^X_{d+1,X_s}(\Omega^1_{X/S}) \cap [X]).$$

3.2. RATIONAL MAPS

Let $g\colon W \to S$ be a separated scheme of finite type over S and U and V be two invertible sheaves over W. A rational map $U \dashrightarrow V$ is an isomorphism $U_\eta \xrightarrow{\sim} V_\eta$ over the generic fiber of W. Let $\mathfrak{m}$ be the maximal ideal of R and $L = g^*\mathfrak{m}$ be its pull-back.

LEMMA 3.4. *Let $\varphi\colon U \dashrightarrow V$ be a rational map over W. Then, there exist a positive integer n and a morphism $\psi\colon L^{\otimes n} \otimes U \to V$ extending the isomorphism $\varphi\colon U_\eta \to V_\eta$.*

Proof. It follows from [5], proposition 4. □

https://doi.org/10.1023/A:1001822419774 Published online by Cambridge University Press

Following Saito [16], we use Lemma 3.4 to associate to any rational map $\varphi\colon U \dashrightarrow V$, localized Chern classes $c_{i,W_s}^{W}(U \dashrightarrow V) \in A^i(W_s \to W)$. Let n be an integer as in Lemma 3.4 and $\psi\colon L^{\otimes n} \otimes U \to V$ be a morphism extending the isomorphism φ on the generic fiber. Put

$$\begin{aligned} &c_{i,W_s}^{W}(U \dashrightarrow V) \\ &\quad = c_{i,W_s}^{W}(L^{\otimes n} \otimes U \to V) - \sum_{k=0}^{i-1} c_k(U \dashrightarrow V) c_{i-k,W_s}^{W}(L^{\otimes n} \otimes U \to U), \end{aligned}$$

where $c_k(U \dashrightarrow V)$ is the usual Chern class $(c(V)c(U)^{-1})_{\dim = k}$. This definition does not depend on the integer n and the morphism ψ.

Remark 3.5. Let $U \dashrightarrow V$ be a rational map and define the inverse rational map $V \dashrightarrow U$ to be the inverse isomorphism $V_\eta \xrightarrow{\sim} U_\eta$. One can prove easily that

$$c_{1,W_s}^{W}(U \dashrightarrow V) = -c_{1,W_s}^{W}(V \dashrightarrow U) \in A^1(W_s \to W).$$

PROPOSITION 3.6. *Let $\mathcal{E}.$ be a perfect complex of locally free $\mathcal{O}_W$-modules and* $\det(\mathcal{E}.)$ *be its determinant line bundle. Assume that $\mathcal{E}.$ is exact off W_s. Then, there exists a canonical rational section $\mathcal{O}_W \dashrightarrow \det(\mathcal{E}.)$, and we have*

$$c_{1,W_s}^{W}(\mathcal{E}.) = c_{1,W_s}^{W}(\mathcal{O}_W \dashrightarrow \det(\mathcal{E}.)) \in A^1(W_s \to W).$$

Proof. It is enough to prove that for any irreducible scheme X of dimension n and any perfect complex $\mathcal{E}.$ of locally free $\mathcal{O}_X$-modules which is exact off X_s, we have

$$c_{1,X_s}^{X}(\mathcal{O}_X \dashrightarrow \det(\mathcal{E}.)) \cap [X] = c_{1,X_s}^{X}(\mathcal{E}.) \cap [X] \in A_{n-1}(X_s). \tag{7}$$

This relation is clearly satisfied if $X_\eta = \emptyset$. So, we can assume that $X_\eta \neq \emptyset$.

Let $0 \to \mathcal{E}.^{(1)} \to \mathcal{E}.^{(2)} \to \mathcal{E}.^{(3)} \to 0$ be an exact sequence of perfect complexes over X which are exact off X_s. It easily follows from the definitions and the additivity of localized Chern classes that if (7) holds for $\mathcal{E}.^{(1)}$ and $\mathcal{E}.^{(3)}$, then it holds for $\mathcal{E}.^{(2)}$.

Fulton ([7], example 18.3.12, see also [15], chapter 4) proved the following splitting principle: there exists a proper birational map $f\colon \widetilde{X} \to X$ such that $f^*\mathcal{E}.$ has a filtration by perfect complexes exact off $\widetilde{X}_s$, with quotients of the form $0 \to L_i \to L_{i-1} \to 0$ where L_i and L_{i-1} are invertible sheaves. For such complexes, relation (7) is obvious.

The referee pointed out a simpler devissage. By normalization, we may assume X normal. Since the dimension of X_s is $n-1$, equation (7) for X is equivalent to the analogue equation for any open neighborhood of the generic points of X_s. We choose a neighborhood U such that $\mathcal{E}.|_U$ admits a filtration by perfect complexes exact off U_s, with quotients of the form $\mathcal{O}_U \xrightarrow{.\pi} \mathcal{O}_U$. For such complexes, relation (7) is obvious. $\square$

https://doi.org/10.1023/A:1001822419774 Published online by Cambridge University Press

COROLLARY 3.7. *Let* $u\colon \mathcal{E}. \to \mathcal{G}.$ *be a surjective map of perfect complexes of locally free* $\mathcal{O}_W$*-modules which induces a quasi-isomorphism over the generic fiber* W_η. *Let* $\mathcal{F}.$ *be the kernel of* u *and* $t\colon \det(\mathcal{E}.) \dashrightarrow \det(\mathcal{G}.)$ *be the rational map induced by* u *on the generic fibers of the determinant line bundles of* $\mathcal{E}.$ *and* $\mathcal{G}.$. *Then,*

$$c^W_{1,W_s}(\mathcal{F}.) = -c^W_{1,W_s}(\det(\mathcal{E}.) \dashrightarrow \det(\mathcal{G}.)) \in A^1(W_s \to W).$$

Proof. By Proposition 3.6, we have

$$c^W_{1,W_s}(\mathcal{F}.) = c^W_{1,W_s}(\mathcal{O}_W \dashrightarrow \det(\mathcal{F}.)) \in A^1(W_s \to W).$$

The rational section $\mathcal{O}_W \dashrightarrow \det(\mathcal{F}.)$ can be obtained from the rational map t. Indeed, there exists a canonical isomorphism $\det(\mathcal{E}.) \simeq \det(\mathcal{F}.) \otimes \det(\mathcal{G}.)$. Combined with the rational map t, this gives a rational map $\det(\mathcal{F}.) \dashrightarrow \mathcal{O}_W$. Its inverse is the rational section we started with. By the above relation and remark 3.5, we have

$$\begin{aligned} c^W_{1,W_s}(\det(\mathcal{E}.) \dashrightarrow \det(\mathcal{G}.)) &= c^W_{1,W_s}(\det(\mathcal{F}.) \dashrightarrow \mathcal{O}_W) \\ &= -c^W_{1,W_s}(\mathcal{O}_W \dashrightarrow \det(\mathcal{F}.)) \\ &= -c^W_{1,W_s}(\mathcal{F}.) \in A^1(W_s \to W). \end{aligned}$$ □

EXAMPLE 3.8 ([16], Lemma 2). Assume that $W = \mathrm{Spec}(A)$ is the spectrum of a discrete valuation ring which is finite and flat over R. Let M_1 and M_2 be two invertible A-modules and $t\colon M_1 \dashrightarrow M_2$ be a rational map. Let π be a uniformizing element of R. There exist an integer $j \geqslant 0$ and a map $\pi^j M_1 \to M_2$ extending the isomorphism on the generic fibers. It is injective with a finite length cokernel $\mathcal{C}$. Define the order of t to be $\mathrm{ord}(t) = \mathrm{Leng}_R(\mathcal{C}) - dj$, where d is the degree of A over R. This definition does not depend on the integer j. Indeed, $\mathrm{ord}(t) = \deg c^W_{1,W_s}(M_1 \dashrightarrow M_2) \cap [W]$.

4. Localized Intersection Product

Let X be a separated scheme of finite type over S, and let $\mathcal{S}^\cdot = \oplus_{n \geqslant 0} \mathcal{S}^n$ be a graded $\mathcal{O}_X$-algebra such that $\mathcal{S}^0 = \mathcal{O}_X$ and $\mathcal{S}^1$ is coherent and generates $\mathcal{S}^\cdot$ over $\mathcal{O}_X$. We assume that $\mathcal{S}^1$ has finite homological dimension over X and is locally free of rank d over the generic fiber X_η, which is assumed to be non-empty. Let $Y = \mathrm{Spec}(\mathcal{S}^\cdot)$ be the cone of $\mathcal{S}^\cdot$, $\mathbb{P} = \mathrm{Proj}(\mathcal{S}^\cdot[z])$ be its projective completion, and q be the projection $\mathbb{P} \to X$. For any $h\colon X' \to X$, we will construct a map:

$$\psi_{X'}: A_*(\mathrm{Proj}(h^*\mathcal{S}^\cdot[z])) \to A_{*-d-1}(X'_s).$$

We start by defining ψ_X. Let ξ be the kernel of the canonical surjection $\delta\colon q^*(\mathcal{S}^1 \oplus \mathcal{O}_X) \to \mathcal{O}_{\mathbb{P}}(1)$. Let $\mathcal{E}.$ be a resolution of $\mathcal{S}^1$ by locally free $\mathcal{O}_X$-modules of finite ranks:

$$0 = \mathcal{E}_{n+1} \to \mathcal{E}_n \to \cdots\cdots \to \mathcal{E}_1 \to \mathcal{E}_0 \xrightarrow{\epsilon} \mathcal{S}^1 \to 0,$$

and let L be the kernel of the surjective morphism $\delta \circ (\epsilon \oplus 1)\colon q^*(\mathcal{E}_0 \oplus \mathcal{O}_X) \to \mathcal{O}_{\mathbb{P}}(1)$. We denote by $\mathcal{F}.$ the complex of locally free $\mathcal{O}_{\mathbb{P}}$-modules

$$\mathcal{F}. := 0 \to q^*\mathcal{E}_n \to q^*\mathcal{E}_{n-1} \to \cdots\cdots \to q^*\mathcal{E}_1 \to L \to 0.$$

LEMMA 4.1. *The complex $\mathcal{F}.$ satisfies the conditions* (*P*) *relatively to the closed immersion $\mathbb{P}_s \to \mathbb{P}$.*

Proof. The sheaf $\mathcal{S}^1$ is locally free over X_η. Therefore, the complex

$$0 \to q^*\mathcal{E}_n \to \cdots\cdots \to q^*\mathcal{E}_1 \to q^*(\mathcal{E}_0 \oplus \mathcal{O}_X) \to q^*(\mathcal{S}^1 \oplus \mathcal{O}_X) \to 0$$

is exact over the generic fiber of $\mathbb{P}$. It follows that for $i > 0$, $\mathcal{H}_i(\mathcal{F}.)$ is supported on the closed fiber $\mathbb{P}_s$, and that $\mathcal{H}_0(\mathcal{F}.)_{|\mathbb{P}_\eta} = \xi_{|\mathbb{P}_\eta}$ is locally free of rank d over the generic fiber of $\mathbb{P}$. □

Put

$$\begin{aligned}\psi\colon\ A_k(\mathbb{P}) &\longrightarrow A_{k-d-1}(X_s)\\ \alpha &\mapsto q_{s*}((-1)^{d+1} c^{\mathbb{P}}_{d+1,\mathbb{P}_s}(\mathcal{F}.) \cap \alpha).\end{aligned}$$

LEMMA 4.2. *The map ψ does not depend on the resolution $\mathcal{E}.$ of $\mathcal{S}^1$ over X.*

Proof. Let $\mathcal{E}.^{(1)}$ and $\mathcal{E}.^{(2)}$ be two resolutions of $\mathcal{S}^1$. Without loss of generality, we assume that $\mathcal{E}.^{(1)}$ dominates $\mathcal{E}.^{(2)}$. Let $\mathcal{G}.$ be the kernel of $\mathcal{E}.^{(1)} \to \mathcal{E}.^{(2)}$, it is an exact complex of locally free $\mathcal{O}_X$-modules. We denote by $\mathcal{F}.^{(1)}$ and $\mathcal{F}.^{(2)}$ the complexes over $\mathbb{P}$ deduced respectively from $\mathcal{E}.^{(1)}$ and $\mathcal{E}.^{(2)}$ as before. The following sequence of complexes over $\mathbb{P}$

$$0 \longrightarrow q^*(\mathcal{G}.) \longrightarrow \mathcal{F}.^{(1)} \longrightarrow \mathcal{F}.^{(2)} \longrightarrow 0$$

is exact. Hence $\mathcal{F}.^{(1)}$ and $\mathcal{F}.^{(2)}$ are quasi-isomorphic and define the same localized Chern classes. □

For any $h\colon X' \to X$, we define $\psi_{X'}$ by

$$\begin{aligned}\psi = \psi_{X'}\colon A_k(\mathrm{Proj}(h^*\mathcal{S}^{\cdot}[z])) &\longrightarrow A_{k-d-1}(X'_s)\\ \alpha &\mapsto q'_{s*}((-1)^{d+1} c^{\mathbb{P}}_{d+1,\mathbb{P}_s}(\mathcal{F}.) \cap \alpha),\end{aligned}$$

where $q'\colon \mathrm{Proj}(h^*\mathcal{S}^{\cdot}[z]) \to X'$ is the projection.

DEFINITION 4.3. A closed immersion $i\colon X \to Y$, defined by an ideal sheaf $\mathcal{I}$ on Y, is said to be a $*$-closed immersion of codimension d if the conormal sheaf $\mathcal{N}_X Y = \mathcal{I}/\mathcal{I}^2$ has finite homological dimension over X and is locally free of rank d over the generic fiber X_η, which is assumed to be non empty.

Let $i\colon X \to Y$ be a $*$-closed immersion of codimension d with conormal sheaf $\mathcal{N}_X Y = \mathcal{I}/\mathcal{I}^2$, where $\mathcal{I}$ is the ideal sheaf of X in Y. Denote $\mathcal{S}_X Y = \oplus_{n \geqslant 0} \mathcal{I}^n/\mathcal{I}^{n+1}$ and let $C = C_X Y = \mathrm{Spec}(\mathcal{S}_X Y)$ be the normal cone to X in Y and $\mathbb{P} = \mathrm{Proj}(\mathcal{S}_X Y[z])$ be its projective completion. Let $\mathcal{E}.$ be a resolution of $\mathcal{N}_X Y$ by locally free $\mathcal{O}_X$-modules. We construct $\mathcal{F}.$ the complex of locally free modules over $\mathbb{P}$ from the complex $\mathcal{E}.$ as before. Let V be a purely k-dimensional scheme and $f\colon V \to Y$ be a morphism. Form the fiber square

$$\begin{tikzcd} W \arrow[r] \arrow[d, "g"'] & V \arrow[d, "f"] \\ X \arrow[r, "i"] & Y \end{tikzcd} \tag{8}$$

so $W = X \times_Y V$. Let $\mathcal{J}$ be the ideal sheaf of W in V. There is a surjection over W

$$\oplus_{n \geqslant 0}\, g^*\mathcal{I}^n/\mathcal{I}^{n+1} \to \mathcal{S}_W V = \oplus_{n \geqslant 0}\, \mathcal{J}^n/\mathcal{J}^{n+1} \to 0.$$

It determines a closed immersion j which fits in the diagram:

$$\begin{tikzcd} \mathrm{Proj}(\mathcal{S}_W V[z]) \arrow[r, "j"] & \mathrm{Proj}(g^*\mathcal{S}_X Y[z]) \arrow[r, "l"] \arrow[d, "p"'] & \mathrm{Proj}(\mathcal{S}_X Y[z]) \arrow[d, "q"] \\ & W \arrow[r, "g"] & X \end{tikzcd} \tag{9}$$

Since $\mathrm{Proj}(\mathcal{S}_W V[z])$ is a purely k-dimensional scheme, it gives a k-cycle on $\mathrm{Proj}(g^*\mathcal{S}_X Y[z])$. Define the *localized intersection product* $(X.V)_{loc}$ to be the image of this cycle by ψ_W

$$(X.V)_{loc} = p_{s*}((-1)^{d+1} c^{\mathbb{P}}_{d+1,\mathbb{P}_s}(\mathcal{F}.) \cap [\mathrm{Proj}(\mathcal{S}_W V[z])]) \in A_{k-d-1}(W_s).$$

Let $V_1 \ldots V_r$ be the irreducible components of V and n_i be the multiplicity of V_i in V. Put $W_i = V_i \times_X Y$, then $[\mathrm{Proj}(\mathcal{S}_W V[z])] = \sum_i n_i[\mathrm{Proj}(\mathcal{S}_{W_i} V_i[z])]$. It follows that $(X.V)_{loc} = \sum_i n_i (X.V_i)_{loc}$.

DEFINITION 4.4. For any $Y' \to Y$ and $X' = X \times_Y Y'$, define the localized Gysin homomorphism to be

$$\begin{aligned} i^!_{loc} : Z_k(Y') &\to A_{k-d-1}(X'_s) \\ \sum_i n_i[V_i] &\mapsto \sum_i n_i (X.V_i)_{loc}. \end{aligned}$$

Remark 4.5. Consider the fiber square (8) where i is a $*$-closed immersion of codimension d with conormal sheaf $\mathcal{N}_X Y$ and V is a purely k-dimensional scheme. If $W_\eta = \emptyset$, then

$$(X.V)_{loc} = \{g^*(c(\mathcal{N}_X Y))^* \cap s(W, V)\}_{k-d-1} \in A_{k-d-1}(W_s) = A_{k-d-1}(W),$$

where the $c(\ )^*$ means to multiply the term c_i by $(-1)^i$, $g^*(c(\mathcal{N}_X Y))$ is a notation for the total Chern class of the complex $g^*(\mathcal{E}.)$ for any resolution $\mathcal{E}.$ of $\mathcal{N}_X Y$ by locally free $\mathcal{O}_X$ modules, and $s(W, V)$ is the total Segre class of the closed immersion

https://doi.org/10.1023/A:1001822419774 Published online by Cambridge University Press

$W \to V$ ([7] chapter 4). Indeed, the cycle defined by $\mathrm{Proj}(\mathcal{S}_W V[z])$ is contained in the special fiber. So, we do not need to localize the Chern classes and the result follows exactly as in [7] proposition 6.1-a).

PROPOSITION 4.6. *Consider a Cartesian diagram*

$$\begin{tikzcd} X'' \arrow[r, "i''"] \arrow[d, "l"'] & Y'' \arrow[d, "h"] \\ X' \arrow[r, "i'"] \arrow[d, "g"'] & Y' \arrow[d, "f"] \\ X \arrow[r, "i"] & Y \end{tikzcd} \tag{10}$$

with i a $*$-closed immersion of codimension d.

(a) *If h is proper, and $\alpha \in Z_k(Y'')$, then*

$$i^!_{loc} h_*(\alpha) = l_{s*}(i^!_{loc}\alpha) \in A_{k-d-1}(X'_s).$$

(b) *If h is flat of relative dimension n, and $\alpha \in Z_k(Y')$, then*

$$i^!_{loc} h^*(\alpha) = l^*_s(i^!_{loc}\alpha) \in A_{k+n-d-1}(X''_s).$$

Proof. (a) We may assume that $\alpha = [Y'']$ and $h(Y'') = Y'$. Consider the diagram

$$\begin{tikzcd} \mathrm{Proj}(\mathcal{S}_{X''}Y''[z]) \arrow[r] & \mathrm{Proj}(l^*\mathcal{S}_{X'}Y'[z]) \arrow[r, "t"] \arrow[d, "l_2"] & \mathrm{Proj}(l^*g^*\mathcal{S}_X Y[z]) \arrow[r, "q''"] \arrow[d, "l_1"] & X'' \arrow[d, "l"] \\ & \mathrm{Proj}(\mathcal{S}_{X'}Y'[z]) \arrow[r, "j"] & \mathrm{Proj}(g^*\mathcal{S}_X Y[z]) \arrow[r, "q'"] \arrow[d, "g_1"] & X' \arrow[d, "g"] \\ & & \mathrm{Proj}(\mathcal{S}_X Y[z]) \arrow[r, "q"] & X \end{tikzcd} \tag{11}$$

With the same notation as before, we have

$$i^!_{loc}([Y']) = q'_{s*}((-1)^{d+1} c^{\mathbb{P}}_{d+1,\mathbb{P}_s}(\mathcal{F}.) \cap [\mathrm{Proj}(\mathcal{S}_{X'} Y'[z])]).$$

The first equality follows from:

$$l_{1*}[\mathrm{Proj}(\mathcal{S}_{X''} Y''[z])] = \deg(Y''/Y')[\mathrm{Proj}(\mathcal{S}_{X'} Y'[z])].$$

(b) We may assume $\alpha = [Y']$ and $h(Y'') = Y'$. As h is flat, $\mathcal{S}_{X''} Y'' = l^*\mathcal{S}_{X'} Y'$. Hence, $l_1^*[\mathrm{Proj}(\mathcal{S}_{X'} Y'[z])] = [\mathrm{Proj}(\mathcal{S}_{X''} Y''[z])]$, which implies the needed relation. □

https://doi.org/10.1023/A:1001822419774 Published online by Cambridge University Press

THEOREM 4.7 (Localized Excess Formula). *Consider the Cartesian diagram (10) where i is a $*$-closed immersion of codimension d. Assume that i' is a regular embedding of codimension d' and let $\mathcal{J}$ be the ideal sheaf of X' in Y' and $M = (\mathcal{J}/\mathcal{J}^2)^\vee$ be the normal bundle on X'. Assume that Y'' has pure dimension k. Let $\mathcal{E}.$ be a resolution of $\mathcal{N}_X Y$ by locally free $\mathcal{O}_X$-modules and let $\mathcal{F}.$ be the complex of locally free $\mathcal{O}_{X'}$-modules*

$$0 \to g^*(\mathcal{E}_n) \to g^*(\mathcal{E}_{n-1}) \to \ldots\ldots g^*(\mathcal{E}_1) \to F \to 0,$$

where F is the kernel of the surjection $g^*\mathcal{E}_0 \to \mathcal{J}/\mathcal{J}^2$ (called an excess complex). Then $\mathcal{F}.$ satisfies conditions (P) relatively to the closed immersion $X'_s \to X'$ and we have

$$(X.Y'')_{loc} = \sum_{j=e+1}^{d+1} (-1)^j c^{X'}_{j,X'_s}(\mathcal{F}.) \cap \{c(l^*M) \cap s(X'', Y'')\}_{k+j-d-1} \in A_{k-d-1}(X''_s),$$

where $e = d - d'$, $c(l^*M)$ is the total Chern class of the locally free $\mathcal{O}_{X''}$-module l^*M, and $s(X'', Y'')$ is the Segre class of the closed immersion $X'' \to Y''$.

In particular, if Y' is a purely k-dimensional scheme, then

$$(X.Y')_{loc} = (-1)^{e+1} c^{X'}_{e+1,X'_s}(\mathcal{F}.) \cap [X'] \in A_{k-d-1}(X'_s).$$

Proof. It's easy to see that $\mathcal{F}.$ satisfies the conditions (P) (see also the proof of Lemma 4.1). Consider the diagram (11), put $\mathbb{P} = \mathrm{Proj}(\mathcal{S}_X Y[z])$, and define L and ξ by the exact sequences

$$0 \to L \to q^*(\mathcal{E}_0 \oplus \mathcal{O}_X) \to \mathcal{O}_{\mathbb{P}}(1) \to 0,$$
$$0 \to \xi \to q^*(\mathcal{N}_X Y \oplus \mathcal{O}_X) \to \mathcal{O}_{\mathbb{P}}(1) \to 0.$$

We denote by $\mathcal{G}.$ the complex of locally free $\mathcal{O}_{\mathbb{P}}$-modules:

$$0 \to q^*\mathcal{E}_n \to q^*\mathcal{E}_{n-1} \to \ldots\ldots q^*\mathcal{E}_1 \to L \to 0.$$

Finally we define a locally free module ξ' over $\mathbb{P}' = \mathrm{Proj}(\mathcal{S}_{X'} Y'[z])$ by

$$0 \to \xi' \to j^*q'^*(\mathcal{J}/\mathcal{J}^2 \oplus \mathcal{O}_{X'}) \to \mathcal{O}_{\mathbb{P}'}(1) \to 0.$$

LEMMA 4.8. *We have over* $\mathrm{Proj}(\mathcal{S}_{X'} Y'[z])$ *the exact sequence of complexes of locally free modules*

$$0 \to j^*q'^*\mathcal{F}. \to j^*g_1^*\mathcal{G}. \to \xi' \to 0.$$

https://doi.org/10.1023/A:1001822419774 Published online by Cambridge University Press

Proof. The only thing to check is the exact sequence in degree 0

$$0 \to j^*q'^*F \to j^*g_1^*L \to \xi' \to 0. \tag{12}$$

Consider the following diagram on $\mathbb{P}'$

$$\begin{array}{ccccccc} 0 \to \xi' & \to & j^*q'^*(\mathcal{J}/\mathcal{J}^2 \oplus \mathcal{O}_{X'}) & \to & \mathcal{O}_{\mathbb{P}'}(1) \to 0 \\ \uparrow & & \uparrow & & \| \\ 0 \to j^*g_1^*L & \to & j^*g_1^*q^*(\mathcal{E}_0 \oplus \mathcal{O}_X) & \to & \mathcal{O}_{\mathbb{P}'}(1) \to 0 \end{array}$$

The second vertical morphism is a surjection being a composition of two surjections, and has the locally free module $j^*q'^*F$ as a kernel. The snake lemma implies that the first vertical morphism is surjective with kernel $j^*q'^*F$, this finishes the proof of (12). □

The localized intersection product is given by

$$(X.Y'')_{loc} = q''_{s*}t_{s*}((-1)^{d+1}c_{loc}(\mathcal{G}.) \cap [\mathrm{Proj}(\mathcal{S}_{X''}Y''[z])]),$$

where $c_{loc}(\mathcal{G}.) = c^{\mathbb{P}}_{d+1,\mathbb{P}_s}(\mathcal{G}.)$. Proposition 3.1(b) applied to the exact sequence of Lemma 4.8 implies

$$\begin{aligned} &c_{loc}(\mathcal{G}.) \cap [\mathrm{Proj}(\mathcal{S}_{X''}Y''[z])] \\ &\quad = \sum_{j=e+1}^{d+1} c^{X'}_{j,X'_s}(\mathcal{F}.) \cap c_{d+1-j}(l_2^*\xi') \cap [\mathrm{Proj}(\mathcal{S}_{X''}Y''[z])] \\ &\quad = \sum_{j=e+1}^{d+1} c^{X'}_{j,X'_s}(\mathcal{F}.) \cap \{c(l_2^*\xi') \cap [\mathrm{Proj}(\mathcal{S}_{X''}Y''[z])]\}_{k+j-d-1}. \end{aligned}$$

On the other hand, we have on $\mathbb{P}'$:

$$c(\xi') = c(j^*q'^*M^\vee)/c(\mathcal{O}_{\mathbb{P}'}(1)) = c(j^*q'^*M^\vee)\sum_{n \geqslant 0}(-1)^n c_1(\mathcal{O}_{\mathbb{P}'}(1))^n.$$

Let δ be the projection $\mathrm{Proj}(\mathcal{S}_{X''}Y''[z]) \to X''$. The proper push-forward property of bivariant classes (C_1) gives

$$\begin{aligned} (X.Y'')_{loc} &= (-1)^{d+1}\sum_{j=e+1}^{d+1} c^{X'}_{j,X'_s}(\mathcal{F}.) \cap \delta_*\{c(\delta^*l^*M^\vee) \cap \sum_n (-1)^n c_1(\mathcal{O}(1))^n \cap [\mathrm{Proj}(\mathcal{S}_{X''}Y''[z])]\}_{k+j-d-1} \\ &= (-1)^{d+1}\sum_{j=e+1}^{d+1} c^{X'}_{j,X'_s}(\mathcal{F}.) \cap \{c(l^*M^\vee) \cap \delta_*(\sum_n (-1)^n c_1(\mathcal{O}(1))^n \cap [\mathrm{Proj}(\mathcal{S}_{X''}Y''[z])])\}_{k+j-d-1} \\ &= \sum_{j=e+1}^{d+1} (-1)^j c^{X'}_{j,X'_s}(\mathcal{F}.) \cap \{c(l^*M) \cap \delta_*(\sum_n c_1(\mathcal{O}(1))^n \cap [\mathrm{Proj}(\mathcal{S}_{X''}Y''[z])])\}_{k+j-d-1}. \end{aligned}$$

In order to pass from the first to the second equality we used the projection formula for the classical Chern classes. The third equality is only a sign verification. But

https://doi.org/10.1023/A:1001822419774 Published online by Cambridge University Press

the Segre class of $X'' \to Y''$ is

$$s(X'', Y'') = \delta_*(\sum_{n \geqslant 0} c_1(\mathcal{O}(1))^n \cap [\mathrm{Proj}(\mathcal{S}_{X''} Y''[z])]).$$

This finishes the proof of the Theorem 4.7. □

COROLLARY 4.9. *Let $i: X \to Y$ be a $*$-closed immersion of codimension d. Let V be a purely k-dimensional scheme and $h: V \to Y$ be a morphism. Assume that $W = X \times_Y V \to V$ is an isomorphism. Then*

$$(X.V)_{loc} = (-1)^{d+1} c^X_{d+1, X_s}(\mathcal{N}_X Y) \cap [V] \in A_{k-d-1}(V_s).$$

As a particular case, we have the localized self-intersection formula: *if X has pure dimension k, then*

$$(X.X)_{loc} = (-1)^{d+1} c^X_{d+1, X_s}(\mathcal{N}_X Y) \cap [X] \in A_{k-d-1}(X_s).$$

Remark 4.10. The localized Gysin homomorphism does not pass to rational equivalence. Consider a diagram (10) where $i': X' \to Y'$ is a Cartier divisors, and take $Y'' = V$ to be an irreducible and reduced k-dimensional subscheme of Y'. The localized excess formula gives

$$(X.V)_{loc} = (-1)^d c^{X'}_{d, X'_s}(\mathcal{F}.) \cap (X'.[V]) + (-1)^{d+1} c^{X'}_{d+1, X'_s}(\mathcal{F}.) \cap \{s(V \cap X', V)\}_k,$$

where $(X'.[V])$ is the usual intersection with the Cartier divisor X' on Y'. The first term of the right hand side in this equation passes to rational equivalence, but the second does not (the Segre class $\{s(V \cap X', V)\}_k = [V]$ if $V \subset X'$ and vanishes otherwise).

PROPOSITION 4.11. *Let $i: X \to Y$ be a $*$-closed immersion of codimension d. The localized intersection product $i^!_{loc}$ defined in this section is the unique localized intersection product compatible with proper push-forward and flat pull-back and satisfying the localized excess formula in codimension* 1 *and* 0. *More precisely, consider a fiber square*

$$\begin{array}{ccc} W & \xrightarrow{j} & V \\ \downarrow & & \downarrow \\ X & \xrightarrow{i} & Y \end{array}$$

such that V is a purely k-dimensional scheme and j is either an isomorphism or a regular embedding of codimension 1 *(i.e., W is an effective Cartier divisor on V). A localized intersection product satisfies the localized excess formula in codimension* 1 *and* 0 *if for any fiber square as above, we have:*

$$(X.V)_{loc} = (-1)^{d+\epsilon} c^W_{d+\epsilon, W_s}(\mathcal{F}.) \cap [W],$$

https://doi.org/10.1023/A:1001822419774 Published online by Cambridge University Press

where $\epsilon = 1$ if j is an isomorphism and 0 otherwise, and $\mathcal{F}.$ is the excess complex constructed over W as in Theorem 4.7.

Proof. Let V be an irreducible and reduced k-dimensional scheme and let $W = X \times_Y V$. We will prove that the conditions of the proposition are enough to compute $(X.V)_{loc}$. If W is isomorphic to V, the excess formula in codimension 0 gives $(X.V)_{loc}$. Suppose that $W \neq V$ and denote by $\widetilde{V}$ the blow-up of V along W and by $\widetilde{W}$ the exceptional divisor:

$$\begin{array}{ccc} \widetilde{W} & \longrightarrow & \widetilde{V} \\ \pi \downarrow & & \downarrow \rho \\ W & \rightarrow & V \\ \downarrow & & \downarrow \\ X & \rightarrow & Y \end{array}$$

The compatibility with proper push-forward implies that $i^!_{loc}[V] = \pi_{s*}(i^!_{loc}[\widetilde{V}])$. But $\widetilde{W}$ is a Cartier divisor on $\widetilde{V}$. Then, the excess formula in codimension 1 gives this localized product. $\square$

Let X be an arithmetic scheme over S of relative dimension d. The diagonal closed immersion $\Delta_X \to X \times_S X$ is a $*$-closed immersion of codimension d. Therefore, we can associate with any fiber square

$$\begin{tikzcd} W \arrow[r] \arrow[d] & V \arrow[d] \\ \Delta_X \arrow[r] & X \times_S X \end{tikzcd}$$

where V is a scheme of pure dimension k, the localized intersection product $(X.V)_{loc} \in A_{k-d-1}(W_s)$. For $V = \Delta_X$, the self-intersection formula gives:

$$(\Delta_X.\Delta_X)_{loc} = (-1)^{d+1} c^X_{d+1, X_s}(\Omega^1_{X/S}) \cap [X] \in A_0(X_s).$$

Bloch [3] gives this formula as a Definition of $(\Delta_X.\Delta_X)_{loc}$ but did not define a general localized intersection theory.

5. Localized Intersection Over Arithmetic Surfaces

Let X be an arithmetic surface over S. We prove in this section a residual intersection formula for the localized intersection theory associated with the diagonal closed immersion $\Delta_X \to X \times_S X$. Then, we use this formula to write the Lefschetz number of an automorphism over X as a sum of local contributions supported on the scheme of fixed points. We begin by recalling some known facts about the dualizing sheaf of X over S which will be used later.

5.1. THE DUALIZING SHEAF

1.–*Projective resolution of* $\Omega^1_{X/S}$: Fix an embedding of X in a scheme P smooth over S. As X and P are regular, this embedding is regular. Hence, it induces a resolution of $\Omega^1_{X/S}$ by locally free $\mathcal{O}_X$-modules

$$0 \to \mathcal{E}_1 = \mathcal{N}_X P \to \mathcal{E}_0 = \Omega^1_{P/S}|_X \to \Omega^1_{X/S} \to 0.$$

Indeed, the kernel of $\mathcal{E}_1 \to \mathcal{E}_0$ is locally free (as X is a regular surface) of rank 0.

Let $\omega_X/S = \mathcal{H}om(\det(\mathcal{N}_X P), \det(\Omega^1_{P/S}|_X))$ be the dualizing sheaf of X over S. There exists a canonical map $\rho : \Omega^1_{X/S} \to \omega_{X/S}$ defined locally as follows: given a local section τ of $\Omega^1_{X/S}$, let $\widetilde{\tau}$ be a lifting of τ to $\Omega^1_{P/S}$, then ρ is given by $\tau \mapsto (\alpha \mapsto \alpha \wedge \widetilde{\tau})$.

2.–*The semi-stable case*: Assume that X is a semi-stable surface over S and let \$ be the set of singular points in X_s. Then, the map ρ is injective because $\Omega^1_{X/S}$ is R-torsion-free, and its cokernel is a skyscraper sheaf:

$$0 \to \Omega^1_{X/S} \to \omega_{X/S} \to \bigoplus_{x \in \$} k \to 0. \tag{13}$$

3.–*A differential invariant*: Here, we do not assume that X is semi-stable. Let D be a finite and flat scheme over S and let $g\colon D \to X$ be an S-morphism. Let τ and γ be respectively the kernel and the cokernel of the map induced by ρ:

$$0 \to \tau \to g^*\Omega^1_{X/S} \xrightarrow{\rho_D} g^*\omega_{X/S} \to \gamma \to 0.$$

LEMMA 5.1.

(i) *The module* τ *is the* R*-torsion submodule of* $g^*\Omega^1_{X/S}$.

(ii) *The modules* γ *and* τ *have the same finite* R*-length. Put*

$$\delta_X(D \to X) := \mathrm{Leng}_R(\gamma) = \mathrm{Leng}_R(\tau).$$

If D is a horizontal effective Cartier divisor over X, we denote $\delta_X(D) = \delta_X(D \to X)$.

(iii) *If X is semi-stable and if D is a horizontal effective Cartier divisor over X, then $\delta_X(D)$ is the number of singular points of X_s contained in D without multiplicity (i.e. the cardinality of \$ $\cap D_s$).*

Proof. (i) *Let C be the image of ρ_D. We have an exact sequence $0 \to \tau \to g^*\Omega^1_{X/S} \to C \to 0$. As C is a submodule of $g^*\omega_{X/S}$, it is R-torsion-free. Therefore, τ is the R-torsion submodule of $g^*\Omega^1_{X/S}$.*

(ii) By pull-back to D, $0 \to g^*\mathcal{E}_1 \to g^*\mathcal{E}_0 \to g^*\Omega^1_{X/S} \to 0$ is exact and gives a resolution of $g^*\Omega^1_{X/S}$ by locally free modules. Indeed, the kernel N of $g^*\mathcal{E}_1 \to g^*\mathcal{E}_0$ is R-torsion-free because it injects in $g^*\mathcal{E}_1$ and D is flat over R. Hence N is R-locally free. But N vanishes on the generic fiber of D because $\Omega^1_{X_\eta/K}$ is locally free. So N vanishes. We consider ρ_D as a map of complexes $\rho_D\colon g^*\mathcal{E}. \to g^*\omega_{X/S}$, where the second complex is concentrated in degree 0. It is a quasi-isomorphism on

https://doi.org/10.1023/A:1001822419774 Published online by Cambridge University Press

the generic fiber D_η. The rational map $t\colon g^*\omega_{X/S} \dashrightarrow g^*\omega_{X/S}$ induces by ρ_D is the identity. Then, by example 3.8, we have $\mathrm{Leng}(\gamma) - \mathrm{Leng}(\tau) = \mathrm{ord}(t) = 0$.

(iii) One easily proves that the restriction of the exact sequence (13) to the divisor D is

$$0 \to \bigoplus_{x \in \mathcal{S} \cap D_s} k \to g^*\Omega^1_{X/S} \to g^*\omega_{X/S} \to \bigoplus_{x \in \mathcal{S} \cap D_s} k \to 0. \tag{14}$$

4.–*Relative differentials*: Let $f\colon X \to Y$ be a dominant morphism of arithmetic surfaces over S. By [3] Lemma 7.2, there exists an exact sequence

$$0 \to f^*\Omega^1_{Y/S} \to \Omega^1_{X/S} \to \Omega^1_{X/Y} \to 0.$$

LEMMA 5.2. *Assume that $f_\eta\colon X_\eta \to Y_\eta$ is étale. Then, there exists a canonical rational map $f^*\omega_{Y/S} \dashrightarrow \omega_{X/S}$, and*

$$c^X_{1,X_s}(f^*\omega_{Y/S} \dashrightarrow \omega_{X/S}) = c^X_{1,X_s}(\Omega^1_{X/Y}) \in A^1(X_s \to X).$$

Proof. In [3], Lemma 7.2, Bloch proves that we can find resolutions of length 1 which fit in the exact sequence

$$\begin{tikzcd}
0 \arrow[r] & f^*\Omega^1_{Y/S} \arrow[r] & \Omega^1_{X/S} \arrow[r] & \Omega^1_{X/Y} \arrow[r] & 0 \\
0 \arrow[r] & f^*F_0 \arrow[r] \arrow[u, two heads] & E_0 \arrow[r] \arrow[u, two heads] & H_0 \arrow[r] \arrow[u, two heads] & 0 \\
0 \arrow[r] & f^*F_1 \arrow[r] \arrow[u] & E_1 \arrow[r] \arrow[u] & H_1 \arrow[r] \arrow[u] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

The Lemma follows from Proposition 3.6. □

5.2. A RESIDUAL INTERSECTION FORMULA

Let V be an irreducible scheme of dimension k with a map $f\colon V \to X \times_S X$, and W be the restriction of V to Δ_X:

$$\begin{tikzcd}
W \arrow[r] \arrow[d, "g"'] & V \arrow[d, "f"] \\
\Delta_X \arrow[r] & X \times_S X
\end{tikzcd} \tag{15}$$

PROPOSITION 5.3. *Assume that $W \neq V$. Then, the localized intersection product $(\Delta_X.V)_{loc} \in A_{k-2}(W_s)$ can be defined as follows:*

(1) *If W is a Cartier divisor on V, then diagram (15) induces a surjection $g^*\Omega^1_{X/S} \to \mathcal{O}_V(-W)|_W$ which is an isomorphism over the generic fiber of W. Therefore, we get a rational map $g^*\omega_{X/S} \dashrightarrow \mathcal{O}_V(-W)|_W$, and we have*

$$(\Delta_X.V)_{loc} = c^W_{1,W_s}(g^*\omega_{X/S} \dashrightarrow \mathcal{O}_V(-W)|_W) \cap [W] \in A_{k-2}(W_s).$$

(2) *Since $V \neq W$, we reduce the general case to the first case by taking the blow-up of V along W.*

Proof. By Proposition 4.6, (2) follows from (1). Let $\mathcal{E}.$ be a resolution of $\Omega^1_{X/S}$ as in Section 5.1. Diagram (15) gives a surjective map $g^*\mathcal{E}. \to \mathcal{O}_V(-W)|_W$ where the second complex is concentrated in degree 0. Let $\mathcal{F}.$ be its kernel. By the localized excess formula 4.7,

$$(\Delta_X.V)_{loc} = -c^W_{1,W_s}(\mathcal{F}.) \cap [W] \in A_{k-2}(W_s).$$

Corollary 3.7 implies the Proposition. □

Consider now an irreducible scheme Γ of dimension 2 with a map $\Gamma \to X \times_S X$. Let W be the restriction of Γ to Δ_X, and assume that $W \neq \Gamma$. Consider the diagram

$$\begin{tikzcd} & V \arrow[d, "b"] & \\ H \arrow[r, "a"] & W \arrow[r, "j"] \arrow[d, "g"] & \Gamma \arrow[d] \\ & \Delta_X \arrow[r] & X \times_S X \end{tikzcd} \tag{16}$$

where H is a Cartier divisor on Γ and V is the residual scheme to H in W, which means that V is defined by its ideal sheaf $I_V = I_W \otimes \mathcal{O}_\Gamma(H) \subset \mathcal{O}_\Gamma$, where I_W is the ideal sheaf of W in Γ, and $\mathcal{O}_\Gamma(H)$ is the invertible sheaf associated with H ([7] Definition 9.2.1). Let $\omega = g^*\omega_{X/S}$ and $\mathcal{O}(H) = j^*\mathcal{O}_\Gamma(H)$.

PROPOSITION 5.4. *Assume that V is vertical (i.e. $V_\eta = \emptyset$). Then, there exists a canonical rational section $\omega \dashrightarrow \mathcal{O}(-H)$ over H, and we have:*

$$\begin{aligned}(\Delta_X.\Gamma)_{loc} &= c^H_{1,H_s}(\omega \dashrightarrow \mathcal{O}(-H)) \cap [H] \\ &\quad + \{c(\omega \otimes \mathcal{O}(2H))^* \cap s(V,\Gamma)\}_{\dim=0} \in A_0(W_s),\end{aligned}$$

where $c(\ )^*$ means to multiply c_i by $(-1)^i$ and $s(V,\Gamma)$ is the total Segre class of the closed immersion $V \to \Gamma$.

Proof. The existence of the rational section $\omega \dashrightarrow \mathcal{O}(-H)$ follows from diagram (16) as $V_\eta = \emptyset$. Let $\pi : \widetilde{\Gamma} \to \Gamma$ be the blow-up of Γ along V, and $\widetilde{V}$, $\widetilde{H}$ and $\widetilde{W}$ be the

https://doi.org/10.1023/A:1001822419774 Published online by Cambridge University Press

inverse images of V, H and W. Then, the relation $\widetilde{W} = \widetilde{V} + \widetilde{H}$ holds between Cartier divisors over $\widetilde{\Gamma}$. We have a rational map $\pi^*\omega \dashrightarrow \mathcal{O}(-\widetilde{W})|_{\widetilde{W}}$ over $\widetilde{W}$, and by Proposition 5.3,

$$(\Delta_X.\Gamma)_{loc} = \pi_{s*}c^{\widetilde{W}}_{1,\widetilde{W}_s}(\pi^*\omega \dashrightarrow \mathcal{O}(-\widetilde{W})|_{\widetilde{W}}) \cap [\widetilde{W}].$$

Recall that $\mathcal{O}(-\widetilde{H}) = \pi^*\mathcal{O}(-H)$. Hence, we get:

$$\begin{aligned} &c^{\widetilde{W}}_{1,\widetilde{W}_s}(\pi^*\omega \dashrightarrow \mathcal{O}(-\widetilde{W})|_{\widetilde{W}}) \cap [\widetilde{W}] \\ &\quad = c^{\widetilde{H}}_{1,\widetilde{H}_s}(\pi^*\omega \dashrightarrow \mathcal{O}(-\widetilde{W})|_{\widetilde{W}}) \cap [\widetilde{H}] + c_1(\pi^*\omega \dashrightarrow \mathcal{O}(-\widetilde{W})|_{\widetilde{W}}) \cap [\widetilde{V}] \\ &\quad = c^{\widetilde{H}}_{1,\widetilde{H}_s}(\pi^*\omega \dashrightarrow \pi^*\mathcal{O}(-H)|_H) \cap [\widetilde{H}] \\ &\qquad - c_1(\pi^*[\omega \otimes \mathcal{O}(2H)]) \cap [\widetilde{V}] - c_1(\mathcal{O}(\widetilde{V})) \cap [\widetilde{V}]. \end{aligned}$$

The π-push-forward of this formula gives the Proposition. □

5.3. LEFSCHETZ NUMBERS

Let σ be an S-automorphism of X and denote by $\Gamma \subset X \times_S X$ the graph of σ. The scheme of σ-fixed points $\mathrm{fix}(\sigma)$ is defined by the fiber square

$$\begin{array}{ccc} \mathrm{fix}(\sigma) & \longrightarrow & \Gamma \\ \downarrow & & \downarrow \\ \Delta_X & \longrightarrow & X \times_S X \end{array} \tag{17}$$

The localized intersection $(\Delta_X.\Gamma)_{loc}$ is a 0-cycle class in the closed fiber of X. Therefore, we can take its degree.

DEFINITION 5.5. The Lefschetz number of an S-automorphism σ of X is the degree of the localized intersection $(\Delta_X.\Gamma)_{loc} \in A_0(X_s)$. It is equally denoted $(\Delta_X.\Gamma)_{loc}$.

Assume from now on that σ is *non-trivial*. Let $\mathcal{I}$ be the ideal sheaf of $\mathrm{fix}(\sigma)$ in X. Let $x \in \mathrm{fix}(\sigma)$ be a closed fixed point and A be the local ring of X at x. Then σ induces an automorphism of A, and the ideal $\mathcal{I}$ is locally generated at x by $\sigma(a) - a$ where a runs over A. For any $a, b \in A$, we have:

$$\sigma(ab) - ab = \sigma(a)(\sigma(b) - b) + b(\sigma(a) - a). \tag{18}$$

Hence, by Nakayama's lemma, $\mathcal{I}_x$ is generated by $\sigma(\theta_1) - \theta_1$ and $\sigma(\theta_2) - \theta_2$ for two local parameters θ_1 and θ_2 of A. Let Y be the Cartier divisor over X defined locally by the greatest common divisor of all functions in the ideal sheaf of $\mathrm{fix}(\sigma)$ in X, and denote by $\mathfrak{R}$ the residual scheme to Y in $\mathrm{fix}(\sigma)$.

LEMMA 5.6. *The closed immersion $\mathfrak{R} \to \Gamma_\sigma$ is a regular embedding. Any point x in $\mathfrak{R}$ is a singular point of X_s.*

Proof. The ideal sheaf $\mathcal{I}$ is locally generated by two equations. Therefore, the ideal sheaf $\mathcal{I}_{\mathfrak{R}}$ of $\mathfrak{R}$ in Γ is also locally generated by two equations, namely the quotient of two equations defining $\mathrm{fix}(\sigma)$ by an equation defining the Cartier divisor Y. Since Γ is regular and $\mathrm{codim}(\Gamma, \mathfrak{R}) = 2$, the closed immersion $\mathfrak{R} \to \Gamma$ is regular (see [7] the remark after Corollary 9.2.1). We have a canonical surjection $\Omega^1_{X/S}|_{\mathrm{fix}(\sigma)} \to \mathcal{I}|_{\mathrm{fix}(\sigma)}$, which induces the surjective map

$$\{\Omega^1_{X/S} \otimes \mathcal{O}(Y)\}|_{\mathfrak{R}} \to \mathcal{I}_{\mathfrak{R}}|_{\mathfrak{R}}.$$

It follows that for any point x of $\mathfrak{R}$, $\dim_{k(x)}(\Omega^1_{X/S}(x)) \geqslant \dim_{k(x)}(\mathcal{I}_{\mathfrak{R}}(x)) = 2$. This proves the second statement in the Lemma. □

Remark 5.7. Let x be a point in $\mathfrak{R}$ and $l(x)$ be its multiplicity in $\mathfrak{R}$. Then, $l(x)$ is also the algebraic multiplicity of Γ along $\mathfrak{R}$ at x because $\mathfrak{R}$ is regularly embedded in Γ ([7] example 4.3.5-c)).

PROPOSITION 5.8. *Let $Y = H + V$ be the decomposition of Y into a horizontal Cartier divisor H and a vertical one V. Then, there exists a rational section $\omega|_H \dashrightarrow \mathcal{O}(-Y)|_H$ over H, and the following formula holds*

$$\begin{aligned}(\Delta_X.\Gamma)_{loc} = {} & c^H_{1,H_s}(\omega|_H \dashrightarrow \mathcal{O}(-Y)|_H) \cap [H] \\ & - (\omega + H + V.V) + \sum_{x \in \mathfrak{R}} l(x)[x] \in A_0(X_s).\end{aligned}$$

Proof. By Proposition 5.4,

$$\begin{aligned}(\Delta_X.\Gamma)_{loc} &= c^Y_{1,Y_s}(\omega|_Y \dashrightarrow \mathcal{O}(-Y)|_Y) \cap [Y] + s_0(\mathfrak{R}, \Gamma) \\ &= c^H_{1,H_s}(\omega|_H \dashrightarrow \mathcal{O}(-Y)|_H) \cap [H] - (\omega + H + V.V) + s_0(\mathfrak{R}, \Gamma).\end{aligned}$$

The relation $s_0(\mathfrak{R}, \Gamma) = \sum l(x)[x]$ follows from the remark above. □

6. A Projection Formula

Let $f : X \to Y$ be a morphism of finite degree n between two arithmetic surfaces over S, and σ and τ be non-trivial S-automorphisms, of respectively, X and Y such that $\tau \circ f = f \circ \sigma$. Put $\Gamma = \Gamma_\sigma \subset X \times_S X$ and $\Gamma_\tau \subset Y \times_S Y$ the graphs of σ and τ, and consider the Cartesian diagram

$$\begin{tikzcd} W \arrow[r] \arrow[d] & X \times_Y X \arrow[r] \arrow[d] & \Delta_Y \arrow[d, "j"] \\ \Gamma \arrow[r, "i"] & X \times_S X \arrow[r] & Y \times_S Y \end{tikzcd} \tag{19}$$

where W is the intersection of Γ with $X \times_Y X$. The closed immersions i and j are $*$-closed immersions of codimension 1 and conormal sheaves, respectively, $\sigma^*\Omega^1_{X/S}$ and $\Omega^1_{Y/S}$. On the one hand, $X \times_Y X$ has pure dimension 2. Its localized intersection with Γ is a cycle class of dimension 0 over W_s. On the other hand, the localized intersection of Γ with Δ_Y is also a cycle class of dimension 0 over W_s. We expect that these cycle classes are the same:

$$(\Gamma.[X \times_Y X])_{loc} = (\Delta_Y.\Gamma)_{loc} \in A_0(W_s). \tag{20}$$

A weaker version of this formula consists in the equality of the degrees of the above cycles:

$$(\Gamma_\sigma.[X \times_Y X])_{loc} = n(\Delta_Y.\Gamma_\tau)_{loc}. \tag{21}$$

We used in this equation the same notation for a localized intersection product and its degree.

Let σ and σ' be two automorphisms of X such that $f \circ \sigma = f \circ \sigma' = \tau \circ f$. Then, $\iota = \sigma' \circ \sigma^{-1}$ is an isomorphism of X over Y:

$$\begin{tikzcd} X \arrow[rr, "\sim", "\iota"'] \arrow[dr] & & X \arrow[dl] \\ & Y & \end{tikzcd}$$

The automorphism $(\mathrm{id} \times \iota)\colon X \times_S X \to X \times_S X$ sends Γ_σ onto $\Gamma_{\sigma'}$ and preserves $X \times_Y X$. So, $\mathrm{id} \times \iota$ induces an isomorphism between W and W', and we have

$$(\Gamma_\sigma.[X \times_Y X])_{loc} = (\Gamma_{\sigma'}.[X \times_Y X])_{loc} \in A_0(W_s) = A_0(W'_s).$$

Hence, the projection formula does not depend on the choice of the lifting of τ to X. Unfortunately, the proof we have is based on the existence of a good lifting.

Good lifting of an automorphism over curves. We consider here smooth projective and irreducible curves over $\eta = \mathrm{Spec}(K)$. Let $f : D \to C$ be a finite morphism between such curves, and τ be a non-trivial K-automorphism of C. A lifting σ of τ to D is a K-automorphism of D such that $f \circ \sigma = \tau \circ f$. Denote $\mathrm{fix}(\sigma) \subset D$ and $\mathrm{fix}(\tau) \subset C$ the schemes of fixed points. Then, there exists a canonical closed immersion $\mathrm{fix}(\sigma) \subset \mathrm{fix}(\tau) \times_C D$.

DEFINITION 6.1. A good lifting of τ to D is a lifting σ such that $\mathrm{fix}(\sigma) = \mathrm{fix}(\tau) \times_C D$.

https://doi.org/10.1023/A:1001822419774 Published online by Cambridge University Press

LEMMA 6.2. *Let $f: D \to C$ be an étale covering of curves and τ be a non-trivial K-automorphism of C.*

(i) *If* $\mathrm{fix}(\tau) = \emptyset$, *then any lifting of τ is a good lifting.*
(ii) *If* $\mathrm{fix}(\tau) \neq \emptyset$, *then there exists at most one good lifting of τ.*

Proof. The first point is clear. For the second, consider two good liftings σ and σ'. Then $\mathrm{fix}(\sigma) = \mathrm{fix}(\sigma') = \mathrm{fix}(\tau) \times_C D \neq \emptyset$. Therefore, we can find a point $x \in \mathrm{fix}(\sigma) = \mathrm{fix}(\sigma') \subset D$ such that $\sigma(x) = \sigma'(x) = x$ and the automorphisms induced by σ and σ' over $K(x)$ are equal to the identity. As f is étale, $\sigma = \sigma'$ ([12] Corollary 3.13 page 26). $\square$

We consider again a morphism $f: X \to Y$ between arithmetic surfaces and a non-trivial S-automorphism τ of Y. A lifting σ of τ to X is a good lifting if it induces a good one over the generic fibers.

THEOREM 6.3. *Let $f: X \to Y$ be a morphism of finite degree n between arithmetic surfaces such that $f_\eta: X_\eta \to Y_\eta$ is étale. Let τ be a non-trivial S-automorphism of Y and σ be a good lifting of τ to X. Assume that $W = \Gamma \cap (X \times_Y X)$ is a Cartier divisor on Γ. Then, the strong projection formula holds for f.*

The rest of this section is devoted to a proof of this Theorem.

6.1. PROOF OF THEOREM 6.3

Fix the notation as follows:

$$\begin{tikzcd} W \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & X \times_Y X \arrow[r, "p_2"] \arrow[d, "p_1"] & X \\ & X \arrow[ur, "\sigma"'] & \end{tikzcd} \tag{22}$$

- α is the closed immersion of W in $X \times_Y X$ and $\mathcal{I}$ is its ideal sheaf.
- p_1 and p_2 are the first and the second projection.
- β is the map $p_1 \circ \alpha$. It satisfies $\sigma \circ \beta = p_2 \circ \alpha$ because of the definition of W.

The map β coincides with the closed immersion $W \to \Gamma$ when we identify Γ with X by the composed map $\Gamma \longrightarrow X \times_S X \xrightarrow{p_1} X$, which we will do subsequently. Hence, β is a closed immersion and makes W a Cartier divisor on X. Let $\mathcal{J}$ be the ideal sheaf defining β. Diagram (22) leads to the following exact sequence:

$$\mathcal{J}/\mathcal{J}^2 \longrightarrow \mathcal{I}/\mathcal{I}^2 \longrightarrow \beta^*\sigma^*\Omega^1_{X/Y} \longrightarrow 0.$$

Let $P = \mathrm{Proj}(\bigoplus_{n \geqslant 0} \mathcal{I}^n/\mathcal{I}^{n+1})$ and $q: P \to W$ be the canonical projection. Then, P

has pure dimension 1. Let $s = q_*[P] \in Z_1(W)$ be the first Segre class of the closed immersion α. As W_η is a Cartier divisor on $(X \times_Y X)_\eta$, the cycle $([W] - s)$ is supported on the closed fiber of W. The morphisms over P:

$$q^*\mathcal{J}/\mathcal{J}^2 \to q^*\mathcal{I}/\mathcal{I}^2 \to \mathcal{O}(1) = \mathcal{O}_P(1)$$

induce isomorphisms on the generic fiber of P. So, the complex $q^*\mathcal{J}/\mathcal{J}^2 \to \mathcal{O}(1)$ is exact off P_s and we can consider its Chern classes localized in P_s. We now state a proposition which implies Theorem 6.3.

PROPOSITION 6.4. *With the above notation, we have:*

$$\begin{aligned} &q_{s*}\{c^P_{1,P_s}(q^*\mathcal{J}/\mathcal{J}^2 \to \mathcal{O}(1)) \cap [P]\} \\ &\quad = c^X_{1,X_s}(\sigma^*\Omega^1_{X/Y}) \cap \beta_*[W] + \\ &\qquad + (c_1(\mathcal{J}/\mathcal{J}^2) - c_1(\beta^*\sigma^*\omega_{X/S})) \cap ([W] - s) \in A_0(W_s). \end{aligned}$$

First, we prove that Proposition 6.4 implies Theorem 6.3. We extend diagram (19):

$$\begin{tikzcd} \widetilde{W} \arrow[r] \arrow[d] & \widetilde{X \times_Y X} \arrow[d, "\pi"] & \\ W \arrow[r] \arrow[d] & X \times_Y X \arrow[r] \arrow[d] & \Delta_Y \arrow[d, "j"] \\ \Gamma \arrow[r, "i"] & X \times_S X \arrow[r] & Y \times_S Y \end{tikzcd} \tag{23}$$

where $\widetilde{X \times_Y X}$ is the blow-up of $X \times_Y X$ along W and $\widetilde{W}$ is the exceptional divisor. Notice that π is birational and that $\widetilde{W}$ is canonically isomorphic to P with conormal sheaf in $\widetilde{X \times_Y X}$ the sheaf $\mathcal{O}(1)$. We subsequently identify $\widetilde{W}$ and P, and also denote by $q\colon \widetilde{W} \to W$ the projection. Diagram (23) induces two rational maps:

(i) $q^*\beta^*\sigma^*\omega_{X/S} \dashrightarrow \mathcal{O}(1)$ over $P = \widetilde{W}$,
(ii) $\beta^*\sigma^*f^*\omega_{Y/S} \dashrightarrow \mathcal{J}/\mathcal{J}^2$ over W.

By Proposition 5.3, we have:

$$(\Delta_Y.\Gamma)_{loc} = c^W_{1,W_s}(\beta^*\sigma^*f^*\omega_{Y/S} \dashrightarrow \mathcal{J}/\mathcal{J}^2) \cap [W] \in A_0(W_s). \tag{24}$$

As π is birational, Proposition 5.3 (more precisely, its analogue for the closed immersion $\Gamma \to X \times_S X$) implies:

$$(\Gamma.[X \times_Y X])_{loc} = q_{s*}\{c^P_{1,P_s}(q^*\beta^*\sigma^*\omega_{X/S} \dashrightarrow \mathcal{O}(1)) \cap [P]\} \in A_0(W_s). \tag{25}$$

By Lemma 5.2, f induces a rational map $f^*\omega_{Y/S} \dashrightarrow \omega_{X/S}$, and we have

$$c^X_{1,X_s}(\sigma^*f^*\omega_{Y/S} \dashrightarrow \sigma^*\omega_{X/s}) = c^X_{1,X_s}(\sigma^*\Omega^1_{X/Y}) \in A^1(X_s \to X). \tag{26}$$

https://doi.org/10.1023/A:1001822419774 Published online by Cambridge University Press

By substituting equations (24), (25) and (26) in the formula of Proposition 6.4, we get the projection formula $(\Gamma.[X \times_Y X])_{loc} = (\Delta_Y.\Gamma)_{loc}$. □

The next step is a reformulation of Proposition 6.4. Denote $B = X \times_Y X$ and consider it as an X-scheme by the first projection. Consider a scheme C smooth over X with an X-closed immersion of B in C:

$$\begin{tikzcd} B \arrow[r] \arrow[dr] & C \arrow[d] \\ & X \end{tikzcd}$$

We will prove the existence of such a factorization in the proof of Lemma 6.6. As X is regular, then C is regular. And as B is an l.c.i. scheme, the closed immersion of B in C is regular (EGA IV 19.3.2). Let $E = \Omega^1_{C/X}|_B$, and $U = \mathcal{N}_B C$ be the conormal sheaf to B in C. We have an exact sequence over B:

$$U \to E \to p_2^*\Omega^1_{X/Y} \to 0,$$

where $p_2\colon B \to X$ is the second projection. As X_η is étale over Y_η, the complex $U \to E$ is exact off B_s. It defines a bivariant class $c_1^{loc} = c^B_{1,B_s}(U \to E) \in A^1(B_s \to B)$.

LEMMA 6.5. *The bivariant class* $c_1^{loc} \in A^1(B_s \to B)$ *does not depend on the scheme* C.

Proof. Consider two schemes C_1 and C_2 as above and the diagram

$$\begin{tikzcd} B \arrow[r] \arrow[dr] & C_1 \times_X C_2 \arrow[d] \\ & X \end{tikzcd}$$

Let $U \to E$, $U_1 \to E_1$ and $U_2 \to E_2$ be the complexes associated to $C_1 \times_X C_2$, C_1 and C_2. Then $E = E_1 \oplus E_2$, and we have an exact sequence $0 \to U_1 \to U \to E_2 \to 0$, which follows from the diagram

$$\begin{tikzcd} B \arrow[r] \arrow[dr] & C_1 \times_X C_2 \arrow[d] \\ & C_1 \end{tikzcd}$$

Therefore, $c^B_{1,B_s}(U \to E) = c^B_{1,B_s}(U_1 \to E_1)$, and the Lemma follows. □

LEMMA 6.6. *With the above notation, we have:*

$$c_1^{loc} \cap [W] = c^X_{1,X_s}(\sigma^*\Omega^1_{X/Y}) \cap \beta_*[W] \in A_0(W_s).$$

https://doi.org/10.1023/A:1001822419774 Published online by Cambridge University Press

Proof. There exists a Y-closed immersion of X in a scheme Z which is smooth over Y. Then, Z is regular and the closed immersion of X in Z is l.c.i. (EGA IV 19.3.2). Let $E_0 = \Omega^1_{Z/Y}|_X$ and $E_1 = \mathcal{N}_X Z$ be the conormal sheaf of X in Z. We have an exact sequence

$$0 \to E_1 \to E_0 \to \Omega^1_{X/Y} \to 0.$$

Indeed, the kernel of $E_1 \to E_0$ is torsion-free with a generic rank 0. Therefore, Lemma 6.6 is equivalent to

$$\begin{aligned} c_1^{loc} \cap [W] &= c^X_{1,X_s}(\sigma^* E_1 \to \sigma^* E_0) \cap \beta_*[W] \\ &= c^W_{1,W_s}(\beta^*\sigma^* E_1 \to \beta^*\sigma^* E_0) \cap [W] \in A_0(W_s). \end{aligned} \tag{27}$$

Consider now the diagram

$$\begin{tikzcd} B = X \times_Y X \arrow[r, "i"] \arrow[dr] & C = X \times_Y Z \arrow[d, "p_1"] \\ & X \end{tikzcd}$$

The map p_1 is smooth and i is a closed immersion. So, this diagram can be used to compute the localized Chern class c_1^{loc}. Let $U = \mathcal{N}_B C$ be the conormal sheaf to B in C and $E = \Omega^1_{C/X}|_B$. It is easily seen that $E = p_2^* E_0$, where $p_2\colon B \to X$ is the second projection. Furthermore, we have a surjective map $p_2^* E_1 \to U$ which is an isomorphism as E_1 and U are locally free of the same rank. Hence,

$$c_1^{loc} = c^B_{1,B_s}(U \to E) = c^B_{1,B_s}(p_2^* E_1 \to p_2^* E_0).$$

We then apply this relation to W and use the equation $p_2 \circ \alpha = \sigma \circ \beta$ in diagram (22), to get relation (27). □

By Lemmas 6.5 and 6.6, Proposition 6.4 is equivalent to the following:

$$\begin{aligned} &q_{s*}\{c^P_{1,P_s}(q^*\mathcal{J}/\mathcal{J}^2 \to \mathcal{O}(1)) \cap [P]\} \\ &\quad = c_1^{loc} \cap [W] + (c_1(\mathcal{J}/\mathcal{J}^2) - c_1(\beta^*\sigma^*\omega_{X/S})) \cap ([W] - s) \in A_0(W_s). \end{aligned} \tag{28}$$

Taken in $A_0(W)$, equation (28) will follow from the commutativity of Fulton's intersection theory for well chosen regular closed immersions. For this purpose, we need to leave the category of schemes and work with formal schemes. Equation (28) taken in $A_0(W_s)$ turns out to be a refined version of Fulton's commutativity.

6.2 REGULAR IMMERSIONS OF FORMAL SCHEMES

DEFINITION 6.7. (i) A closed immersion of formal schemes $\mathfrak{Y} \to \mathfrak{Z}$ defined by a coherent ideal sheaf $\mathfrak{I}$ is a regular immersion if for any point $y \in \mathfrak{Y}$, there exists an open neighborhood U of y in $\mathfrak{Z}$ such that $\mathfrak{I}|_U$ is generated by a regular sequence

https://doi.org/10.1023/A:1001822419774 Published online by Cambridge University Press

of elements of $\Gamma(U, \mathcal{O}_{\mathfrak{Z}})$. One can prove that $\mathfrak{Y} \to \mathfrak{Z}$ is regular if and only if for any point y of $\mathfrak{Y}$, the kernel $\mathfrak{I}_y$ of $\mathcal{O}_{\mathfrak{Z},y} \to \mathcal{O}_{\mathfrak{Y},y}$ is generated by a regular sequence of elements of $\mathcal{O}_{\mathfrak{Z},y}$.

(ii) A formal scheme $\mathfrak{Y}$ is local complete intersection if for any point y of $\mathfrak{Y}$, the local ring $\mathcal{O}_{\mathfrak{Y},y}$ is a local complete intersection ring (i.e. the completion of $\mathcal{O}_{\mathfrak{Y},y}$ is isomorphic to a quotient of a complete local regular ring by a regular sequence of elements).

LEMMA 6.8. (1) *Let Z' be a closed subscheme of a scheme Z and let $\mathfrak{Z}$ be the formal completion of Z along Z'. If Z is l.c.i. then $\mathfrak{Z}$ is l.c.i..*

(2) *Let $i : \mathfrak{Y} \to \mathfrak{Z}$ be a closed immersion of formal schemes and assume that $\mathfrak{Z}$ is regular. Then, i is a regular immersion if and only if $\mathfrak{Y}$ is l.c.i.*

Proof. (1) Being l.c.i. is a local property. Therefore, we can assume that $Z = \mathrm{Spec}(A)$ and Z' is given by an ideal I. Let $\hat{A}$ be the I-adic completion of A. The formal scheme $\mathfrak{Z}$ is isomorphic to $\mathrm{Spf}(\hat{A})$. Let $\mathcal{P}$ be a prime ideal of A containing I, denote by x the associated point either in Z' or in $\mathfrak{Z}$ and by $\mathfrak{P} = \mathcal{P}\hat{A}$. The local ring of $\mathfrak{Z}$ at x is given by:

$$\mathcal{O}_{\mathfrak{Z},x} = \varinjlim_{f \notin \mathfrak{P}} \hat{A}\langle f^{-1}\rangle,$$

where $\hat{A}\langle f^{-1}\rangle$ is the I-adic completion of $\hat{A}_f$ (this is the localization of $\hat{A}$ at $\{f^n, n \in \mathbb{Z}\}$). Then, we get canonical morphisms:

$$\hat{A}_{\mathfrak{P}} \to \mathcal{O}_{\mathfrak{Z},x} \to \hat{A}\langle(\hat{A} - \mathfrak{P})^{-1}\rangle \simeq (\hat{A}_{\mathfrak{P}})^{\wedge},$$

where $\hat{A}_{\mathfrak{P}}$ is the localization of $\hat{A}$ at $\mathfrak{P}$ and $\hat{A}\langle(\hat{A} - \mathfrak{P})^{-1}\rangle$ is canonically isomorphic to the I-adic completion of $\hat{A}_{\mathfrak{P}}$. It follows that the I-adic completion of $\mathcal{O}_{\mathfrak{Z},x}$ is canonically isomorphic to the I-adic completion of $\hat{A}_{\mathfrak{P}}$. On the one hand, the I-adic completion of $\hat{A}_{\mathfrak{P}}$ is canonically isomorphic to the I-adic completion of $A_{\mathcal{P}}$. On the other hand the $\mathcal{P}$-adic completion of $A_{\mathcal{P}}$ is canonically isomorphic to the $\mathcal{P}$-adic completion of the I-adic completion of $A_{\mathcal{P}}$. The same remark applies for $\mathcal{O}_{\mathfrak{Z},x}$. So, the completions of the local rings $\mathcal{O}_{\mathfrak{Z},x}$ and $\mathcal{O}_{Z,x}$ at their maximal ideals are isomorphic.

(2) Follows from EGA VI 19.3.2. $\square$

We return to our problem and consider the Cartesian diagram:

$$\begin{tikzcd}
V \arrow[r] \arrow[d] & W \arrow[r] \arrow[d] & \Gamma \arrow[d] \\
\Delta_X \arrow[r] & X \times_Y X \arrow[r] & X \times_S X
\end{tikzcd}$$

where V is the scheme of σ-fixed points over X. Let $\mathcal{A}$ be the formal completion of $X \times_S X$ along Γ, $\mathcal{B}$ be the formal completion of $X \times_Y X$ along W and $\mathcal{C}$ be the formal

https://doi.org/10.1023/A:1001822419774 Published online by Cambridge University Press

completion of Δ_X along V. Then, we have canonical embeddings $\mathcal{C} \subset \mathcal{B} \subset \mathcal{A}$. We consider $\mathcal{C}$, $\mathcal{B}$ and $\mathcal{A}$ as X-formal schemes by means of the first projection $X \times_S X \to X$.

LEMMA 6.9. *The closed immersions $V \subset W$ and $\mathcal{C} \subset \mathcal{B}$ induce isomorphisms $V_\eta \simeq W_\eta$ and $\mathcal{C}_\eta \simeq \mathcal{B}_\eta$ on the generic fibers.*

Proof. Notice first that $W = \mathrm{fix}(\tau) \times_{\Gamma_\tau^Y} \Gamma_\sigma^X$ and $V = \mathrm{fix}(\sigma)$. As σ is a good lifting of τ , $V_\eta = W_\eta$. The map $f_\eta \colon X_\eta \to Y_\eta$ is étale. So, $(\Delta_X)_\eta$ is a connected component of $(X \times_Y X)_\eta$. Taking completions along $V_\eta \simeq W_\eta$, we get an isomorphism $\mathcal{C}_\eta \simeq \mathcal{B}_\eta$.□

LEMMA 6.10 ([3] Lemma 7.5). *There exist a formal scheme $\mathcal{Z}$ over X, locally isomorphic to $\hat{\mathbb{A}}^2 \times X$, and an X-closed immersion of $\mathcal{A}$ in $\mathcal{Z}$.*

Proof. We give the construction only for $\sigma = \mathrm{id}$ since it is similar for any σ. Let $\mathcal{I}$ be the ideal sheaf of the closed immersion $X \to \mathcal{A}$ induced by the diagonal closed immersion. Since X is regular, $\Omega^1_{X/S} = \mathcal{I}/\mathcal{I}^2$ is locally generated by two sections. One can lift these generators to get a local surjection $\mathcal{O}_X[[x, y]] \to \mathcal{O}_{\mathcal{A}}$. Hence, locally such a $\mathcal{Z}$ exists. For the globalization of this construction, we proceed as follows. There exists a covering of X by two affine open subschemes X_1 and X_2 such that $\Omega^1_{X/S}|_{X_i}$ is generated by two sections over X_i. To construct X_1 and X_2, fix in each component of X_s a closed point. The sheaf $\Omega^1_{X/S}$ is generated by two sections over the associated semi-local ring. Choose any open affine neighborhood X_1 of these points over which $\Omega^1_{X/S}$ is generated by two sections. There exist only finitely many closed points of X which are not in X_1. Over the corresponding semi-local ring, the sheaf $\Omega^1_{X/S}$ is generated by two sections. Lift these to an open affine neighborhood X_2.

Let $\mathcal{A}_i = \mathcal{A} \times_X X_i$ for $i = 1, 2$. These are affine formal schemes. Indeed, consider for example $\mathcal{A}_1$. It is canonically isomorphic to the formal completion of $X_1 \times_S X$ along X_1 diagonally embedded. But X_1 is affine and so are all its infinitesimal neighborhoods, because they are finite over X_1. Put $\mathcal{A}_i = \mathrm{Spf}(A_i)$. Let I_i be the associated ideals of A_i. Let α_i, α_i' be two elements of I_i lifting generators of $I_i/I_i^2 = \Omega^1_{X/S}|_{X_i}$. We get a map $\varphi_i \colon \mathcal{O}_{X_i}[[x, y]] \to A_i$. It is a surjective map. Indeed, for any integer n, $\varphi_i^n \colon \mathcal{O}_{X_i}[x, y]/(x, y)^n \to A_i/I_i^n$ is surjective. The last statement follows by induction from the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (x,y)^n/(x,y)^{n+1} & \longrightarrow & \mathcal{O}_{X_i}[x,y]/(x,y)^{n+1} & \longrightarrow & \mathcal{O}_{X_i}[x,y]/(x,y)^n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I_i^n/I_i^{n+1} & \longrightarrow & A_i/I_i^{n+1} & \longrightarrow & A_i/I_i^n & \longrightarrow & 0
\end{array}$$

because the left vertical map is surjective.

I claim that there exists an automorphism φ of $\mathcal{O}_{X_1 \cap X_2}[[x, y]]$ such that $\varphi \equiv \mathrm{id} \bmod (x, y)$ and $\varphi_1 \circ \varphi = \varphi_2$ over $X_1 \cap X_2$. Then, we can glue φ_1 and φ_2 to get the formal scheme $\mathcal{Z}$. We prove the claim in two steps.

https://doi.org/10.1023/A:1001822419774 Published online by Cambridge University Press

Step 1: We construct a homomorphism φ satisfying the needed properties. There exist $a, b, c, d \in \mathcal{O}_{X_1 \cap X_2}$ such that the relations

$$\overline{\alpha}_2 = a\overline{\alpha}_1 + b\overline{\alpha}'_1, \qquad \overline{\alpha}'_2 = c\overline{\alpha}_1 + d\overline{\alpha}'_1 \tag{29}$$

hold in $\mathcal{I}/\mathcal{I}^2$ over $X_1 \cap X_2$. Hence, the differences $\alpha_2 - a\alpha_1 - b\alpha'_1$ and $\alpha'_2 - c\alpha_1 - d\alpha'_1$ are in $\mathcal{I}^2$. As φ_1 is surjective, we find two formal series $f(x, y)$ and $f'(x, y)$ in $(x, y)^2$ mapping by φ_1 to these differences. Put

$$g(x, y) = ax + by + f(x, y),$$
$$g'(x, y) = cx + dy + f'(x, y),$$

and define

$$\varphi\colon \mathcal{O}_{X_1 \cap X_2}[[x, y]] \to \mathcal{O}_{X_1 \cap X_2}[[x, y]]$$
$$x; y \mapsto g(x, y); g'(x, y).$$

The only point is to check that we can take φ to be an isomorphism. It suffices to take φ to be an isomorphism mod $(x, y)^2$ (i.e. to find $a, b, c, d \in \mathcal{O}_{X_1 \cap X_2}$ such that $ad - bc$ is invertible).

Step 2: We can choose a, b, c, d such that $ad - bc$ does not vanish at all the generic points of $(X_1 \cap X_2)_s$. Let $\kappa_1, \ldots \kappa_r$ be the generic points of $(X_1 \cap X_2)_s$ and let $P_1, \ldots P_r$ be different closed points of $X_1 \cap X_2$ such that P_j is in the closure of κ_j. If $ad - bc$ does not vanish at P_j then it does not vanish at κ_j. So, we are reduced to find a, b, c, d satisfying this property. For each $j = 1, \ldots, r$, we can find $a_j, b_j, c_j, d_j \in \mathcal{O}_{X_1 \cap X_2}$ such that:

(1) the relations $\begin{cases} \overline{\alpha}_2 = a_j\overline{\alpha}_1 + b_j\overline{\alpha}'_1 \\ \overline{\alpha}'_2 = c_j\overline{\alpha}_1 + d_j\overline{\alpha}'_1 \end{cases}$ hold at P_j, and

(2) $a_jd_j - b_jc_j$ does not vanish at P_j.

Statement (1) implies the existence of f_j, g_j in the maximal ideal of P_j and $a'_j, b'_j, c'_j, d'_j \in \mathcal{O}_{X_1 \cap X_2}$ such that the relations

$$\overline{\alpha}_2 = a_j\overline{\alpha}_1 + b_j\overline{\alpha}'_1 + f_j(a'_j\overline{\alpha}_1 + b'_j\overline{\alpha}'_1), \qquad \overline{\alpha}'_2 = c_j\overline{\alpha}_1 + d_j\overline{\alpha}'_1 + g_j(c'_j\overline{\alpha}_1 + d'_j\overline{\alpha}'_1)$$

hold over $X_1 \cap X_2$. As the P_j are distinct closed points, there exist elements h_j with $\sum h_j = 1$ such that h_j belongs to the maximal ideal of any P_i except P_j. Take a, b, c, d to be:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \sum_j h_j \begin{pmatrix} (a_j + f_ja'_j) & (b_j + f_jb'_j) \\ (c_j + g_jc'_j) & (d_j + g_jd'_j) \end{pmatrix}.$$

Relations (29) are trivially satisfied and $ad - bc$ does not vanish at any point P_j and therefore at any point κ_j. As a consequence, $ad - bc$ fails to be invertible at only finitely many closed points of $X_1 \cap X_2$. Let $\mathfrak{m}_i$ be maximal ideals of $\mathcal{O}_{X_2}$ corresponding to these closed points. Let $\mathfrak{n}_j$ be the maximal ideals of $\mathcal{O}_{X_2}$ corresponding to the finite set of closed points of X_2 which are not contained in X_1. Take any element f of $\mathcal{O}_{X_2}$ which is in any $\mathfrak{m}_i$, but which is not in any of the $\mathfrak{n}_j$. Replace X_2 by its localization at f. Clearly X_1 and X_2 continue to cover X. This finishes the construction of φ and gives the formal scheme $\mathcal{Z}$. $\square$

The formal scheme $\mathcal{Z}$ constructed in Lemma 6.10 fits in the Cartesian diagram:

$$\begin{tikzcd} V \arrow[r] \arrow[d] & W \arrow[r] \arrow[d] & \Gamma \arrow[d] \\ \mathcal{C} \arrow[r] \arrow[drr] & \mathcal{B} \arrow[r] \arrow[dr] & \mathcal{Z} \arrow[d] \\ & & X \end{tikzcd} \tag{30}$$

The scheme $X \times_Y X$ is l.c.i., hence so is the formal scheme $\mathcal{B}$ by Lemma 6.8. By the same Lemma, the closed immersions $\mathcal{C} \to \mathcal{Z}$, $\mathcal{B} \to \mathcal{Z}$ and $\Gamma \to \mathcal{Z}$ are l.c.i. of codimension 2. Put $H = \mathcal{N}_\Gamma \mathcal{Z}$ the conormal sheaf to Γ in $\mathcal{Z}$, $\mathcal{U} = \mathcal{N}_{\mathcal{B}} \mathcal{Z}$ the conormal sheaf to $\mathcal{B}$ in $\mathcal{Z}$, $\mathcal{F} = \mathcal{N}_{\mathcal{C}} \mathcal{Z}$ the conormal sheaf to $\mathcal{C}$ in $\mathcal{Z}$ and $\mathcal{E} = \Omega^1_{\mathcal{Z}/X}$ the sheaf of relative differentials of $\mathcal{Z}$ over X. Then, we have canonical isomorphisms $H \simeq \mathcal{E}|_\Gamma$ and $\mathcal{F} \simeq \mathcal{E}|_{\mathcal{C}}$, and an exact sequence over $\mathcal{B}$:

$$\mathcal{U} \longrightarrow \mathcal{E}|_{\mathcal{B}} \longrightarrow \hat{\Omega} \longrightarrow 0, \tag{31}$$

where $\hat{\Omega}$ is the formal completion of $p_2^* \Omega^1_{X/Y}$ along W. Put $U = \mathcal{U}|_W$ and $E = \mathcal{E}|_W = H|_W$.

The scheme P defined in the previous subsection is canonically isomorphic to $\mathrm{Proj}(\mathcal{S}_W \mathcal{B})$. We have denoted $q\colon P \to W$ the projection. We consider also $Q = \mathrm{Proj}(\mathcal{S}_V \mathcal{C})$. Then, there exists a canonical closed immersion $Q \subset P$ which induces an isomorphism on the generic fibers (by Lemma 6.9). Thus, the cycle $[P] - [Q]$ is supported over the closed fiber of P. We define the invertible sheaf K_1 over P by the exact sequence

$$0 \longrightarrow K_1 \longrightarrow q^* E \xrightarrow{\delta} \mathcal{O}(1) \longrightarrow 0, \tag{32}$$

where δ is the composed map $q^* E \to q^* \mathcal{I}/\mathcal{I}^2 \to \mathcal{O}(1)$ and the map $E \to \mathcal{I}/\mathcal{I}^2$ is induced by diagram (30). We denote K_2 the excess bundle relative to the diagram

$$\begin{tikzcd} W \arrow[r] \arrow[d] & \Gamma \arrow[d] \\ \mathcal{B} \arrow[r] & \mathcal{Z} \end{tikzcd}$$

In other words, K_2 is the invertible sheaf over W defined by the exact sequence:

$$0 \longrightarrow K_2 \longrightarrow U \longrightarrow \mathcal{J}/\mathcal{J}^2 \longrightarrow 0. \tag{33}$$

https://doi.org/10.1023/A:1001822419774 Published online by Cambridge University Press

These sheaves are related by the commutative diagram over P:

$$\begin{tikzcd} 0 \arrow[r] & q^*K_2 \arrow[r] \arrow[d] & q^*U \arrow[r] \arrow[d] & q^*(\mathcal{J}/\mathcal{J}^2) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & K_1 \arrow[r] & q^*E \arrow[r] & \mathcal{O}(1) \arrow[r] & 0 \end{tikzcd} \tag{34}$$

where the second vertical map is induced by the exact sequence (31), and the third one was defined in Subsection 6.1 (we leave to the reader to check the commutativity of this diagram). All the vertical maps are isomorphisms off P_s. The following Proposition states the refined commutativity $(\Gamma.[\mathcal{B}] - [\mathcal{C}]) = ([\mathcal{B}] - [\mathcal{C}].\Gamma)$ of the intersection products in $\mathcal{Z}$.

PROPOSITION 6.11. *We have the relation:*

$$\begin{aligned} &q_{s*}\{c_1(K_1) \cap ([P] - [Q])\} \\ &\quad = c_1(K_2) \cap \{[W] - s_1(V, \Gamma)\} - q_{s*}\{c^P_{1,P_s}(q^*K_2 \to K_1) \cap [Q]\} \in A_0(W_s), \end{aligned} \tag{35}$$

where $s_1(V, \Gamma)$ is the first Segre class of the closed immersion of V in Γ (notice that $\{[W] - s_1(V, \Gamma)\}$ is a cycle over W_s, by Lemma 6.9).

We postpone the proof of (35) to the next subsection, and we continue the proof of Equation (28).

LEMMA 6.12. *Let D be an irreducible reduced component of W appearing with a non-vanishing multiplicity in $([W] - s)$. Then, $\mathcal{I}/\mathcal{I}^2|_D$ is locally free of rank 2 over D.*

Proof. Notice first that $\mathcal{I}/\mathcal{I}^2$ is locally generated over W by two sections, namely the pull-back of two local sections generating $\sigma^*\Omega^1_{X/S}$ over Γ. Let κ be the generic point of D. It is enough to see that the stalk $(\mathcal{I}/\mathcal{I}^2)_\kappa$ is generated by two sections. If $(\mathcal{I}/\mathcal{I}^2)_\kappa$ is generated by one section, then we have an isomorphism $\mathcal{O}_{W,\kappa}[T] \to (\oplus_{n \geqslant 0}\mathcal{I}^n/\mathcal{I}^{n+1})_\kappa$. It follows that $P_\kappa = \mathrm{Proj}[(\oplus_{n \geqslant 0}\mathcal{I}^n/\mathcal{I}^{n+1})_\kappa] \simeq \mathrm{Spec}(\mathcal{O}_{W,\kappa})$. We deduce that D does not appear in the cycle $([W] - s)$. The Lemma is proved. □

PROPOSITION 6.13. *We have the relation:*

$$c_1(E) \cap ([W] - s) = c_1(\beta^*\sigma^*\omega_{X/S}) \cap ([W] - s) \in A_0(W_s).$$

Proof. Let D be an irreducible component of W as in Lemma 6.12 (notice that D is vertical). The surjective map $E|_D \to \mathcal{I}/\mathcal{I}^2|_D$, induced by diagram (30), is an isomorphism because both sheaves are locally free of rank 2. Also, the surjective map $\beta^*\sigma^*\Omega^1_{X/S}|_D \to \mathcal{I}/\mathcal{I}^2|_D$, induced by diagram (23), is an isomorphism because $\Omega^1_{X/S}$ is locally generated over X by two sections. Therefore, $E|_D \simeq \beta^*\sigma^*\Omega^1_{X/S}|_D$.

https://doi.org/10.1023/A:1001822419774 Published online by Cambridge University Press

Hence,

$$\begin{aligned} c_1(E)\cap[D] &= c_1(E|_D)\cap[D]\\ &= c_1(\beta^*\sigma^*\Omega^1_{X/S}|_D)\cap[D]\\ &= c_1(\sigma^*\Omega^1_{X/S})\cap\beta_*[D] \quad \text{(by [3] lemma 7.4)}\\ &= c_1(\sigma^*\omega_{X/S})\cap\beta_*[D]\\ &= c_1(\beta^*\sigma^*\omega_{X/S})\cap[D]. \end{aligned}$$

Proposition 6.13 is proved. $\square$

Now, we prove relation (28). From diagram (34), we get

$$\begin{aligned} &q_{s*}\{c^P_{1,P_s}(q^*\mathcal{J}/\mathcal{J}^2\to\mathcal{O}(1))\cap[P]\}\\ &\quad = q_{s*}\{c^P_{1,P_s}(q^*U\to q^*E)\cap[P]\} - q_{s*}\{c^P_{1,P_s}(q^*K_2\to K_1)\cap[P]\}\\ &\quad = c^W_{1,W_s}(U\to E)\cap s - q_{s*}\{c_1(q^*K_2\to K_1)\cap([P]-[Q])\}\\ &\qquad - q_{s*}\{c^P_{1,P_s}(q^*K_2\to K_1)\cap[Q]\}\\ &\quad = c^W_{1,W_s}(U\to E)\cap s + c_1(K_2)\cap(s - s_1(V,\Gamma))\\ &\qquad - q_{s*}\{c_1(K_1)\cap([P]-[Q])\} - q_{s*}\{c^P_{1,P_s}(q^*K_2\to K_1)\cap[Q]\}\in A_0(W_s). \end{aligned}$$

In the last equation, we used that $q_*[Q]=s_1(V,\Gamma)$. Indeed, $q_*[Q]=s_1(V,\mathcal{C})=s_1(V,\Delta_X)$ where s_1 denotes the first Segre class of a closed immersion. As V is the scheme of σ-fixed point, $s_1(V,\Gamma)=s_1(V,\Delta_X)$. We deduce, using Proposition 6.11, that:

$$\begin{aligned} &q_{s*}\{c^P_{1,P_s}(q^*\mathcal{J}/\mathcal{J}^2\to\mathcal{O}(1))\cap[P]\} = c^W_{1,W_s}(U\to E)\cap[W]+\\ &\quad + c_1(U\to E)\cap(s-[W]) + c_1(K_2)\cap(s-[W])\in A_0(W_s). \end{aligned}$$

I claim that $c^W_{1,W_s}(U\to E)\cap[W]=c_1^{loc}\cap[W]\in A_0(W_s)$. Indeed, the diagram of formal schemes

$$\begin{tikzcd} \mathcal{B} \arrow[r] \arrow[dr] & \mathcal{Z} \arrow[d] \\ & X \end{tikzcd}$$

can be used to compute the class c_1^{loc} (see Lemma 6.5). Therefore,

$$\begin{aligned} &q_{s*}\{c^P_{1,P_s}(q^*\mathcal{J}/\mathcal{J}^2\to\mathcal{O}(1))\cap[P]\}\\ &\quad = c_1^{loc}\cap[W] + (c_1(K_2)+c_1(E)-c_1(U))\cap(s-[W])\\ &\quad = c_1^{loc}\cap[W] + (c_1(E)-c_1(\mathcal{J}/\mathcal{J}^2))\cap(s-[W]) \quad \text{(by the sequence (33))}\\ &\quad = c_1^{loc}\cap[W] + (c_1(\beta^*\sigma^*\omega_{X/S})-c_1(\mathcal{J}/\mathcal{J}^2))\cap(s-[W]) \quad \text{(by Proposition 6.13).} \end{aligned}$$

Equation (28) is proved. $\square$

https://doi.org/10.1023/A:1001822419774 Published online by Cambridge University Press

6.3. REFINED COMMUTATIVITY OF FULTON'S INTERSECTION THEORY

Using the commutativity of Fulton's intersection theory, we get relation (35) only in $A_0(W)$. Proposition 6.11 is a refined version of this commutativity. Consider the Cartesian diagram

$$\begin{tikzcd}
 & & V'' \arrow[rr] \arrow[dl] & & W'' \arrow[rr] \arrow[dl] & & \Gamma'' \arrow[dl] \\
 & V' \arrow[rr] \arrow[dl] \arrow[dd, "e"] & & W' \arrow[rr] \arrow[dl] \arrow[dd, "h"] & & \Gamma' \arrow[dl] \arrow[dd, "\pi"] & \\
\mathcal{C}' \arrow[rr] \arrow[dd, "f"] & & \mathcal{B}' \arrow[rr] \arrow[dd, "g"] & & \mathcal{Z}' \arrow[dd] & & \\
 & V \arrow[rr] \arrow[dl] & & W \arrow[rr] \arrow[dl] & & \Gamma \arrow[dl, "i"] & \\
\mathcal{C} \arrow[rr] & & \mathcal{B} \arrow[rr] & & \mathcal{Z} & &
\end{tikzcd}$$

where

(i) $\mathcal{Z}'$ is obtained from $\mathcal{Z}$ by a sequence of two blow-ups, the first along $\mathcal{B}$, and the second along the inverse image of $\mathcal{C}$. Hence, $\mathcal{C}'$ and $\mathcal{B}'$, the inverse images of $\mathcal{C}$ and $\mathcal{B}$, are Cartier divisors over $\mathcal{Z}'$. Put $\mathcal{L} = \mathcal{O}_{\mathcal{Z}'}(-\mathcal{B}')|_{\mathcal{B}'}$ and $\mathcal{G} = \mathcal{O}_{\mathcal{Z}'}(-\mathcal{C}')|_{\mathcal{C}'}$.

(ii) Γ', W' and V' are the inverse images of Γ, W and V respectively in $\mathcal{Z}'$, $\mathcal{B}'$ and $\mathcal{C}'$.

(iii) Γ'' is the blow-up of Γ along V and W'' and V'' are the inverse images of W and V in Γ''. They are Cartier divisors over Γ''. By the universal property of blowing-up, there exists a canonical closed immersion of Γ'' in $\mathcal{Z}'$ such that $\mathcal{B}'$ and $\mathcal{C}'$ restrict respectively to W'' and V''. This closed immersion factors into $\Gamma'' \to \Gamma' \to \mathcal{Z}'$. Notice that V'' is canonically isomorphic to Q.

The residual formal scheme to $\mathcal{C}'$ in $\mathcal{B}'$, denoted $\mathcal{T}$, is the Cartier divisor over $\mathcal{Z}'$ associated to the ideal sheaf $\mathcal{O}_{\mathcal{Z}'}(-\mathcal{B}' + \mathcal{C}') \subset \mathcal{O}_{\mathcal{Z}'}$. As $\mathcal{C}'_\eta = \mathcal{B}'_\eta$, $\mathcal{T}$ is supported over $\mathcal{B}'_s$. We label the closed immersions which appear in the diagram as follows

(1) $i : \Gamma \to \mathcal{Z}$, $b : \mathcal{B} \to \mathcal{Z}$ and $c : \mathcal{C} \to \mathcal{Z}$,

(2) $b' : \mathcal{B}' \to \mathcal{Z}'$, $c' : \mathcal{C}' \to \mathcal{Z}'$ and $t : \mathcal{T} \to \mathcal{Z}'$.

Remark 6.14. We use the notion of Gysin map associated to a regular closed immersion as defined by Fulton in [7] chapter 6.2. Moreover, we have to consider

https://doi.org/10.1023/A:1001822419774 Published online by Cambridge University Press

the following situation

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & \mathcal{Y}' \arrow[d] \\ \mathcal{X} \arrow[r, "r"] & \mathcal{Y} \end{tikzcd}$$

where r is a regular closed immersion of formal schemes and $\mathcal{Y}'$ is a formal scheme. If X' is a scheme, then we can consider $r^!([\mathcal{Y}'])$ as a cycle class over X'. It is defined exactly as in [7] chapter 6. Indeed, the normal cones needed to define this class are schemes because X' is a scheme. The same remark applies for other bivariant classes like Chern classes or localized Chern classes. For example, if $\mathcal{X}$ is a formal scheme, $\mathcal{E}$ is a locally free sheaf of $\mathcal{X}$, and $X \to \mathcal{X}$ is a morphism from a scheme X to $\mathcal{X}$, then we can consider $c_i(\mathcal{E})$ as an operator on cycles of X.

As usual, we introduce the excess bundles $\mathcal{K}_1$ over $\mathcal{C}'$ and $\mathcal{K}_2$ over $\mathcal{B}'$ defined by the exact sequences:

$$0 \to \mathcal{K}_2 \to g^*\mathcal{U} \to \mathcal{L} \to 0,$$
$$0 \to \mathcal{K}_1 \to f^*(\mathcal{E}|_{\mathcal{C}}) \to \mathcal{G} \to 0.$$

We get the commutative diagram over $\mathcal{C}'$:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{K}_2|_{\mathcal{C}'} \arrow[r] \arrow[d] & f^*(\mathcal{U}|_{\mathcal{C}}) \arrow[r] \arrow[d] & \mathcal{L}|_{\mathcal{C}'} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{K}_1 \arrow[r] & f^*(\mathcal{E}|_{\mathcal{C}}) \arrow[r] & \mathcal{G} \arrow[r] & 0 \end{tikzcd} \tag{36}$$

We denote in the following c_i^{loc} for the localized Chern classes $c_{i,\mathcal{C}'_s}^{\mathcal{C}'}$.

LEMMA 6.15. *The vertical maps in diagram* (36) *are exact off* $\mathcal{C}'_s$ *and the following relation holds between localized Chern classes over* $\mathcal{C}'$*:*

$$c_1(\mathcal{K}_2)c_1^{loc}(\mathcal{L}|_{\mathcal{C}'} \to \mathcal{G}) + c_1^{loc}(\mathcal{K}_2|_{\mathcal{C}'} \to \mathcal{K}_1)c_1(\mathcal{G})$$
$$= c_2^{loc}(f^*(\mathcal{U}|_{\mathcal{C}}) \to f^*(\mathcal{E}|_{\mathcal{C}})) + c_1^{loc}(f^*(\mathcal{U}|_{\mathcal{C}}) \to f^*(\mathcal{E}|_{\mathcal{C}}))c_1(f^*(\mathcal{U}|_{\mathcal{C}}))$$

Proof. Given two invertible sheaves U and V over $\mathcal{C}'$ and a morphism $U \to V$ which is an isomorphism on the generic fiber of $\mathcal{C}'$, one can prove that:

$$c_2^{loc}(U \to V) = -c_1(U)c_1^{loc}(U \to V).$$

https://doi.org/10.1023/A:1001822419774 Published online by Cambridge University Press

We apply this remark to the complexes $\mathcal{K}_2|_{\mathcal{C}'} \to \mathcal{K}_1$ and $\mathcal{L}|_{\mathcal{C}'} \to \mathcal{G}$. Then, using the exact sequences (36) and the relation $c_2(f^*(\mathcal{U}|_{\mathcal{C}})) = c_1(\mathcal{K}_2|_{\mathcal{C}'})c_1(\mathcal{L}|_{\mathcal{C}'})$, we get:

$$
\begin{aligned}
&c_2^{loc}(f^*(\mathcal{U}|_{\mathcal{C}}) \to f^*(\mathcal{E}|_{\mathcal{C}})) + c_1^{loc}(f^*(\mathcal{U}|_{\mathcal{C}}) \to f^*(\mathcal{E}|_{\mathcal{C}}))c_1(f^*(\mathcal{U}|_{\mathcal{C}}))\\
&\quad = c_2^{loc}(\mathcal{K}_2|_{\mathcal{C}'} \to \mathcal{K}_1) + c_2^{loc}(\mathcal{L}|_{\mathcal{C}'} \to \mathcal{G}) + c_1^{loc}(\mathcal{K}_2|_{\mathcal{C}'} \to \mathcal{K}_1)c_1^{loc}(\mathcal{L}|_{\mathcal{C}'} \to \mathcal{G}) +\\
&\qquad + [c_1(\mathcal{K}_2|_{\mathcal{C}'}) + c_1(\mathcal{L}|_{\mathcal{C}'})][c_1^{loc}(\mathcal{K}_2|_{\mathcal{C}'} \to \mathcal{K}_1) + c_1^{loc}(\mathcal{L}|_{\mathcal{C}'} \to \mathcal{G})]\\
&\quad = [c_1(\mathcal{L}|_{\mathcal{C}'}) + c_1^{loc}(\mathcal{L}|_{\mathcal{C}'} \to \mathcal{G})]c_1^{loc}(\mathcal{K}_2|_{\mathcal{C}'} \to \mathcal{K}_1) + c_1(\mathcal{K}_2|_{\mathcal{C}'})c_1^{loc}(\mathcal{L}|_{\mathcal{C}'} \to \mathcal{G})\\
&\quad = c_1(\mathcal{G})c_1^{loc}(\mathcal{K}_2|_{\mathcal{C}'} \to \mathcal{K}_1) + c_1(\mathcal{K}_2|_{\mathcal{C}'})c_1^{loc}(\mathcal{L}|_{\mathcal{C}'} \to \mathcal{G}).
\end{aligned}
$$

The last equality follows from the fact that a product of two localized bivariant classes is the product of one of them by the other class taken without localization. □

LEMMA 6.16. *Let* $\alpha = i^!([\mathcal{Z}']) \in A_2(\Gamma')$. *Then, there exist two cycle classes* $\gamma \in A_2(V')$ *and* $\delta \in A_2(W_s')$ *such that* $\alpha - [\Gamma''] = \gamma + \delta \in A_2(\Gamma')$.

Proof. I claim that $\alpha - [\Gamma'']$ is in the image of $A_2(W') \to A_2(\Gamma')$. Let $\Omega = \Gamma - W$ and $\Omega' = \Gamma' - W'$. The map π induces an isomorphism $\Omega' \simeq \Omega$. Consider the commutative diagram

$$
\begin{array}{ccccccc}
A_2(W_s') & \longrightarrow & A_2(W') & \longrightarrow & A_1(W_\eta') & \longrightarrow & 0\\
\downarrow & & \downarrow & & \downarrow & & \\
A_2(\Gamma_s') & \longrightarrow & A_2(\Gamma') & \longrightarrow & A_1(\Gamma_\eta') & \longrightarrow & 0\\
\downarrow & & & & \downarrow & & \\
A_2(\Omega_s') & & & & A_1(\Omega_\eta') & & \\
\downarrow & & & & \downarrow & & \\
0 & & & & 0 & &
\end{array}
$$

where the vertical and horizontal sequences are localization sequences. Since $\Omega_s' \simeq \Omega_s$ is an open subscheme of Γ_s, its dimension is at most one, so $A_2(\Omega_s') = 0$. Therefore, any class in $A_2(\Gamma')$ which image vanishes in $A_1(\Omega_\eta')$, is the image of a class in $A_2(W')$. The image of $\alpha - [\Gamma'']$ in $A_1(\Omega_\eta')$ coincides with the image of $\pi_*(\alpha - [\Gamma''])$ in $A_1(\Omega_\eta)$ via the isomorphism $\Omega' \simeq \Omega$. But $\pi_*(\alpha - [\Gamma'']) = 0 \in A_2(\Gamma)$ by compatiblity of intersection product with push-forward. The claim follows. The irreducible components of W' are either contained in V' or in the closed fiber W_s'. Then we have a surjection $A_2(V') \oplus A_2(W_s') \longrightarrow A_2(W')$. The Lemma follows. □

The commutativity of Fulton's intersection theory ([7] Theorem 6.4) implies

$$i^!([\mathcal{T}]) = t^!(\alpha) \in A_1(W_s'), \tag{37}$$

$$i^!([\mathcal{C}']) = c'^!(\alpha) \in A_1(V'). \tag{38}$$

https://doi.org/10.1023/A:1001822419774 Published online by Cambridge University Press

Indeed, the first relation is localized in the closed fiber of W' because $\mathcal{T}$ is vertical. From these relations, we get:

$$\begin{aligned} & c_1(\mathcal{K}_2)i^!([\mathcal{T}]) - c_1^{loc}(\mathcal{K}_{2|\mathcal{C}'} \to \mathcal{K}_1)i^!([\mathcal{C}']) \\ & \quad = (c_1(\mathcal{K}_2)t^! - c_1^{loc}(\mathcal{K}_{2|\mathcal{C}'} \to \mathcal{K}_1)c'^!)(\alpha) \in A_0(W'_s). \end{aligned} \tag{39}$$

LEMMA 6.17. *We have the following relation between cycle classes over W_s:*

$$\begin{aligned} & h_{s*}\{(c_1(\mathcal{K}_2)t^! - c_1^{loc}(\mathcal{K}_2|_{\mathcal{C}'} \to \mathcal{K}_1)c'^!)(\alpha)\} \\ & \quad = h_{s*}\{(c_1(\mathcal{K}_2)t^! - c_1^{loc}(\mathcal{K}_2|_{\mathcal{C}'} \to \mathcal{K}_1)c'^!)([\Gamma''])\} \in A_0(W_s). \end{aligned}$$

Proof. By Lemma 6.16, it is enough to prove:

$$h_{s*}\{(c_1(\mathcal{K}_2)t^! - c_1^{loc}(\mathcal{K}_2|_{\mathcal{C}'} \to \mathcal{K}_1)c'^!)(\gamma)\} = 0 \in A_0(V_s), \tag{40}$$

$$h_{s*}\{(c_1(\mathcal{K}_2)t^! - c_1^{loc}(\mathcal{K}_2|_{\mathcal{C}'} \to \mathcal{K}_1)c'^!)(\delta)\} = 0 \in A_0(W_s). \tag{41}$$

Restricted to cycles over V', the bivariant class $t^!$ coincides with the localized Chern class $c_1^{loc}(\mathcal{L}|_{\mathcal{C}'} \to \mathcal{G})$, and the bivariant class $c'^!$ coincides with the Chern class $-c_1(\mathcal{G})$. Therefore,

$$\begin{aligned} & (c_1(\mathcal{K}_2)t^! - c_1^{loc}(\mathcal{K}_2|_{\mathcal{C}'} \to \mathcal{K}_1)c'^!)(\gamma) \\ & \quad = (c_1(\mathcal{K}_2)c_1^{loc}(\mathcal{L}|_{\mathcal{C}'} \to \mathcal{G}) + c_1^{loc}(\mathcal{K}_2|_{\mathcal{C}'} \to \mathcal{K}_1)c_1(\mathcal{G}))(\gamma) \in A_0(V'_s). \end{aligned}$$

Then, by Lemma 6.15,

$$\begin{aligned} & h_{s*}\{(c_1(\mathcal{K}_2)t^! - c_1^{loc}(\mathcal{K}_2|_{\mathcal{C}'} \to \mathcal{K}_1)c'^!)(\gamma)\} \\ & \quad = (c_2^{loc}(\mathcal{U}|_{\mathcal{C}} \to \mathcal{E}|_{\mathcal{C}}) + c_1^{loc}(\mathcal{U}|_{\mathcal{C}} \to \mathcal{E}|_{\mathcal{C}})c_1(\mathcal{U}|_{\mathcal{C}})) \cap (e_*\gamma) \\ & \quad = 0 \in A_0(V_s), \end{aligned}$$

because $e_*(\gamma) = 0 \in A_2(V)$. The proof of relation (40) is finished. This method cannot apply to the cycle class δ, as it is not supported over V'. But δ is already localized in the closed fiber of W'. Then,

$$\begin{aligned} & (c_1(\mathcal{K}_2)t^! - c_1^{loc}(\mathcal{K}_2|_{\mathcal{C}'} \to \mathcal{K}_1)c'^!)(\delta) \\ & \quad = (c_1(\mathcal{K}_2)(b'^! - c'^!) - (c_1(\mathcal{K}_1) - c_1(\mathcal{K}_2))c'^!)(\delta) \\ & \quad = (c_1(\mathcal{K}_2)b'^! - c_1(\mathcal{K}_1)c'^!)(\delta) \in A_0(W'_s). \end{aligned}$$

By Fulton's excess formula ([7] Theorem 6.3), we have:

$$(c_1(\mathcal{K}_2)t^! - c_1^{loc}(\mathcal{K}_2|_{\mathcal{C}'} \to \mathcal{K}_1)c'^!)(\delta) = -(b^! - c^!)(\delta) \in A_0(W'_s).$$

Taking the push-forward by h_{s*}, we get

$$h_{s*}\{(c_1(\mathcal{K}_2)t^! - c_1^{loc}(\mathcal{K}_2|_{\mathcal{C}'} \to \mathcal{K}_1)c'^!)(\delta)\} = -(b^! - c^!)(h_{s*}(\delta)) \in A_0(W_s).$$

Relation (41) follows because $h_{s*}(\delta) = 0 \in A_2(W_s)$. $\square$

https://doi.org/10.1023/A:1001822419774 Published online by Cambridge University Press

LEMMA 6.18. *We have the following relation between cycle classes over W_s:*

$$h_{s*}\{c_1(\mathcal{K}_2)i^!([\mathcal{T}]) - c_1^{loc}(\mathcal{K}_2|_{\mathcal{C}'} \to \mathcal{K}_1)i^!([\mathcal{C}'])\} = q_{s*}\{c_1(K_1) \cap ([P] - [Q])\} \in A_0(W_s).$$

Proof. The Lemma follows from the two equations:

$$h_{s*}\{c_1(\mathcal{K}_2)i^!([\mathcal{T}]) - c_1^{loc}(\mathcal{K}_2|_{\mathcal{C}'} \to \mathcal{K}_1)i^!([\mathcal{C}'])\} = -i^!([\mathcal{B}] - [\mathcal{C}]) \in A_0(W_s), \tag{42}$$

$$i^!([\mathcal{B}] - [\mathcal{C}]) = -q_{s*}\{c_1(K_1) \cap ([P] - [Q])\} \in A_0(W_s). \tag{43}$$

Equation (43) is a consequence of Fulton's excess formula ([7] Theorem 6.3). The first equation is harder. Let A' and B' be the projective completions of the normal cones respectively to V' in $\mathcal{C}'$ and to W' in $\mathcal{B}'$. They have pure dimension 3. There exists a canonical closed immersion $A' \subset B'$ which induces an isomorphism over the generic fibers. Let $p'\colon B' \to W'$ be the canonical projection. Let ξ' be the invertible sheaf over B' defined by the exact sequence

$$0 \to \xi' \to p'^*(h^*E \oplus \mathcal{O}_{W'}) \to \mathcal{O}_{B'}(1) \to 0.$$

By [7] proposition 6.1,

$$i^![\mathcal{T}] = p'_{s*}\{c_2(\xi') \cap ([B'] - [A'])\} \in A_1(W'_s),$$
$$i^![\mathcal{C}'] = p'_*\{c_2(\xi') \cap [A']\} \in A_1(V').$$

Let A and B be the projective completions of the normal cones respectively to V in $\mathcal{C}$ and to W in $\mathcal{B}$. They have pure dimension 2. There exists a canonical closed immersion $A \subset B$ which induces an isomorphism over the generic fibers. Let $p : B \to W$ be the canonical projection. Let ξ be the invertible sheaf over B defined by the exact sequence

$$0 \to \xi \to p^*(E \oplus \mathcal{O}_W) \to \mathcal{O}_B(1) \to 0.$$

Again by [7] proposition 6.1,

$$i^!([\mathcal{B}] - [\mathcal{C}]) = p_{s*}\{c_2(\xi) \cap ([B] - [A])\} \in A_0(W_s).$$

There exists a canonical closed immersion $B' \subset B \times_W W'$. Therefore, we get a map $\pi\colon B' \to B$ which extends the map $h\colon W' \to W$. It is easily seen that $\pi^*\xi = \xi'$. Therefore, equation (42) is reduced to the following equation:

$$\pi_{s*}\{c_1(\mathcal{K}_2) \cap ([B'] - [A']) - c_1^{loc}(\mathcal{K}_2|_{\mathcal{C}'} \to \mathcal{K}_1) \cap [A']\} = -([B] - [A]) \in A_2(B_s).$$

The scheme B has pure dimension 2. Therefore, $A_2(B) = Z_2(B)$ and $A_2(B_s) = Z_2(B_s)$. As $Z_2(B_s)$ injects in $Z_2(B)$, $A_2(B_s)$ injects in $A_2(B)$. Hence, the equality of two cycles in $A_2(B_s)$ is equivalent to their equality in $A_2(B)$. Therefore, it is enough to prove the relation above in $A_2(B)$. Hence, we are reduced to prove the following:

(i) $\pi_*(c_1(\mathcal{K}_2) \cap [B']) = -[B] \in A_2(B)$,
(ii) $\pi_*(c_1(\mathcal{K}_1) \cap [A']) = -[A] \in A_2(A)$.

https://doi.org/10.1023/A:1001822419774 Published online by Cambridge University Press

We will prove (i), (ii) is similar. Consider the Cartesian diagram:

$$\begin{tikzcd}
W' \arrow[r] \arrow[d] & \mathcal{B}' \arrow[r] \arrow[d] & \mathcal{Z}' \arrow[d] \\
W_1 \arrow[r] \arrow[d] & \mathcal{B}_1 \arrow[r] \arrow[d] & \mathcal{Z}_1 \arrow[d] \\
W \arrow[r] & \mathcal{B} \arrow[r] & \mathcal{Z}
\end{tikzcd}$$

where $\mathcal{Z}_1$ is the blow-up of $\mathcal{Z}$ along $\mathcal{B}$ and $\mathcal{B}_1$ and W_1 are the inverse images of $\mathcal{B}$ and W in $\mathcal{Z}_1$. Let B_1 be the projective completion of the normal cone to W_1 in $\mathcal{B}_1$. We can factor π into $B' \xrightarrow{\rho} B_1 \xrightarrow{\pi_1} B$. Let E be the excess bundle relative to the diagram

$$\begin{tikzcd}
\mathcal{B}_1 \arrow[r] \arrow[d] & \mathcal{Z}_1 \arrow[d] \\
\mathcal{B} \arrow[r] & \mathcal{Z}
\end{tikzcd}$$

Clearly $\mathcal{K}_2$ is the pull back of E. On the other hand, $\rho_*([B']) = [B_1]$ because $\mathcal{B}'$ and $\mathcal{B}_1$ are birational. So, by the projection formula, (i) is reduced to the following:

$$\pi_{1*}(c_1(E) \cap [B_1]) = -[B] \in A_2(B). \tag{44}$$

The map $\mathcal{B}_1 \to \mathcal{B}$ is flat. Then the diagram

$$\begin{tikzcd}
B_1 = \mathrm{Proj}(\mathcal{S}_{W_1}\mathcal{B}_1[z]) \arrow[r] \arrow[d, "\pi_1"'] & W_1 \arrow[d, "h_1"'] \\
B = \mathrm{Proj}(\mathcal{S}_W\mathcal{B}[z]) \arrow[r] & W
\end{tikzcd}$$

is fiber square. But $W_1 = \mathrm{Proj}_W(\mathrm{Sym}(U))$. Hence, $B_1 = \mathrm{Proj}_B(\mathrm{Sym}(U))$. With this identification, the restriction to W_1 of the ideal sheaf of $\mathcal{B}_1$ in $\mathcal{Z}_1$ is the sheaf $\mathcal{O}(1)$. Then, $E|_{W_1}$ is isomorphic to the kernel of the canonical surjection $h_1^*(U) \to \mathcal{O}(1)$. Relation (44) follows. □

We give now the proof of Equation (35). As V'' and W'' are Cartier divisors over Γ'', we have

$$t^!([\Gamma'']) = [W''] - [V''] \in A_1(W_s''),$$
$$c'^!([\Gamma'']) = [V''] \in A_1(V'').$$

Therefore, Lemma 6.17 becomes

$$\begin{aligned}&h_{s*}\{(c_1(\mathcal{K}_2)t^! - c_1^{loc}(\mathcal{K}_2|_{C'} \to \mathcal{K}_1)c'^!)(\alpha)\}\\ &\quad = h_{s*}\{c_1(\mathcal{K}_2) \cap ([W''] - [V'']) - c_1^{loc}(\mathcal{K}_2|_{C'} \to \mathcal{K}_1) \cap [V'']\} \in A_0(W_s).\end{aligned}$$

Observe that $\mathcal{K}_1|_{V''} = K_1$ under the identification $V'' \simeq Q$, and $\mathcal{K}_2|_{W''}$ is isomorphic to the pull-back of the line bundle K_2 over W. Hence,

$$\begin{aligned}&h_{s*}\{(c_1(\mathcal{K}_2)t^! - c_1^{loc}(\mathcal{K}_2|_{C'} \to \mathcal{K}_1)c'^!)(\alpha)\}\\ &\quad = c_1(K_2) \cap ([W] - s_1(V, \Gamma)) - q_{s*}\{c_{1,Q_s}^{Q}(q^*K_2 \to K_1) \cap [Q]\} \in A_0(W_s).\end{aligned}$$

Taking the push-forward of Equation (39) by h_s, we get (35) as a consequence of Lemma 6.18 and the above relation. □

7. The Weak Projection Formula for Birational Morphisms

In this Section we focus on the weak projection formula for birational morphisms. The first part is devoted to the proof of a key formula giving the behavior of the Lefschetz numbers under blowing-up. The second part deals with the vertical contribution to the projection formula. In the last part, we prove the weak projection formula for birational morphisms and for morphisms obtained from an arithmetic surface by extending the base ring and resolving singularities, called simply base changes. The latter are central in the proof of the Lefschetz fixed point formula, precisely in reducing to semi–stable arithmetic surfaces. In this section, we make heavy use of Theorem 6.3 and the following key formula.

7.1. THE KEY FORMULA

THEOREM 7.1. *Let $\pi\colon X' \to X$ be a birational morphism between two arithmetic surfaces and σ be a non-trivial automorphism of X which can be lifted to an automorphism of X' denoted σ'. Let $\Gamma \subset X \times_S X$ and $\Gamma' \subset X' \times_S X'$ be the graphs of σ and σ'. Then,*

$$(\Delta_{X'}.\Gamma')_{loc} = (\Delta_X.\Gamma)_{loc} + \mathrm{tr}(\sigma')|H^*_{et}(X'_s, \mathbb{Q}_l) - \mathrm{tr}(\sigma)|H^*_{et}(X_s, \mathbb{Q}_l).$$

Proof. Any birational morphism between arithmetic surfaces is obtained by a sequence of blow-ups at closed points [11]. Moreover, if σ can be lifted from X to X', then π is obtained by a sequence of birational maps of two types, namely a blow-up at a fixed point and blow-ups along the (reduced) orbit of a non-fixed point. It is enough to prove the above formula for each type.

The second type: Let x be a closed non-fixed point of X and $O(x)$ be its orbit. Let $\pi\colon X' \to X$ be the blow-up of X along $O(x)$. Put $U = X - O(x)$, then π induces an isomorphism between U and its inverse image $U' = \pi^{-1}(U)$. The schemes of fixed points $\mathrm{fix}(\sigma)$ and $\mathrm{fix}(\sigma')$ are, respectively, closed subschemes of U and U'. Using

Proposition 5.8, we see that all the terms appearing there are the same for the intersections $(\Delta_X.\Gamma)_{loc}$ and $(\Delta_{X'}.\Gamma')_{loc}$. Therefore, $(\Delta_X.\Gamma)_{loc} = (\Delta_{X'}.\Gamma')_{loc}$.

LEMMA 7.2. *Let $\pi\colon X' \to X$ be the blow-up of an arithmetic surface X along a finite set F of closed reduced points. Then,*

$$H^i_{et}(X'_s, \mathbb{Q}_l) \simeq H^i_{et}(X_s, \mathbb{Q}_l) \quad \text{for } i \neq 2,$$
$$0 \to H^2_{et}(X_s, \mathbb{Q}_l) \to H^2_{et}(X'_s, \mathbb{Q}_l) \to \bigoplus_{x\in F} \mathbb{Q}_l \to 0.$$

Proof. By the proper base change theorem, $H^*_{et}(X_s, \mathbb{Q}_l) = H^*_{et}(X, \mathbb{Q}_l)$. Hence, we are reduced to compare the étale cohomology groups of X and X'. We use the Leray spectral sequence for π:

$$H^p(X, R^q\pi_*\mathbb{Q}_l) \Rightarrow H^{p+q}(X', \mathbb{Q}_l).$$

For any $x \in F$, let $i_x\colon \mathrm{Spec}(k) \to X$ be the canonical closed immersion. As the exceptional fibers of π are isomorphic to $\mathbb{P}^1_k$, one gets by the proper base change theorem that $R^q\pi_*\mathbb{Q}_l = \mathbb{Q}_l, 0, \oplus_{x\in F} i_{x*}\mathbb{Q}_l, 0$, respectively, for $i = 0, 1, 2, \geqslant 3$. Hence, the E_∞ terms are

$$E^{p,q}_\infty = \begin{cases} H^p(X, \mathbb{Q}_l) & \text{for } q = 0, \\ \bigoplus_{x\in F} \mathbb{Q}_l & \text{for } (p, q) = (0, 2) \\ 0 & \text{otherwise.} \end{cases},$$

The Lemma follows. $\square$

If in the above Lemma we take $F = O(x)$, we get $\mathrm{tr}(\sigma')|H^*_{et}(X'_s, \mathbb{Q}_l) = \mathrm{tr}(\sigma)|H^*_{et}(X_s, \mathbb{Q}_l)$. The theorem is now proved for morphisms of the second type.

The first type: We consider a blow-up $\pi\colon X' \to X$ of X at a closed fixed point x. As in Section 5.3, let Y (resp. Y') be the Cartier divisor of X (resp. X') defined locally by the greatest common divisor of all functions in the ideal sheaf of $\mathrm{fix}(\sigma)$ (resp. $\mathrm{fix}(\sigma')$), and $\mathfrak{R}$ (resp. $\mathfrak{R}'$) be its residual scheme in $\mathrm{fix}(\sigma)$ (resp. $\mathrm{fix}(\sigma')$). Denote $E = (\pi^{-1}(x))_{red}$ the exceptional fiber of π, which is isomorphic to $\mathbb{P}^1_k$ with self-intersection -1.

LEMMA 7.3. *Under the above conditions, we have $Y' = \pi^*Y + \alpha E$ where α is an integer such that*

$$\alpha(\alpha + 1) + \deg \mathfrak{R}' - \deg \mathfrak{R} = 1,$$

and $\deg \mathfrak{R}$ *and* $\deg \mathfrak{R}'$ *are the sums of the multiplicities of all closed points in* $\mathfrak{R}$ *and* $\mathfrak{R}'$ *respectively.*

Remark 7.4. The scheme of fixed points of an automorphism σ over a scheme X was defined globally by the fiber square (17). We can also define it locally as follows. Let $\mathcal{I}$ be its ideal sheaf in X. If $X = \mathrm{Spec}(A)$ then $\mathcal{I}$ is generated by $\sigma(a) - a$ where a runs over A. In general, let $U = \mathrm{Spec}(A)$ and $V = \mathrm{Spec}(B)$ be two affine open

https://doi.org/10.1023/A:1001822419774 Published online by Cambridge University Press

subschemes of X such that $U \subset V \cap \sigma(V)$. Denote $\phi, \psi : B \to A$ the maps induced respectively by the scheme morphisms $U \to V$ and $U \to \sigma(V) \xrightarrow{\sim} V$. Then, $\mathcal{I}$ is generated over U by $\psi(b) - \phi(b)$ where b runs over B.

Proof of Lemma 7.3. Let A be the local ring of X at x, $\mathfrak{m}$ be its maximal ideal, and t and u be two local parameters of A. The automorphism σ over A is given by

$$\sigma(t) = at + bu, \qquad \sigma(u) = ct + du,$$

where $a, b, c, d \in A$. Let $h = \sigma(t) - t$ and $g = \sigma(u) - u$ be the generators of the ideal of fixed points at x. Let F be the greatest common divisor of h and g. It is a local equation defining the Cartier divisor Y at x. The residual scheme $\mathfrak{R}$ is defined at x by the ideal $(h/F, g/F)$. Its degree is the local intersection at x of the Cartier divisors defined by h/F and g/F. Let n be the order of h/F and m be the order of g/F at x. We assume that $0 \leqslant n \leqslant m$.

All the computations we will do are local. So, we can replace X by $\mathrm{Spec}(A)$. The blow-up of X at x is $X' = \mathrm{Proj}(A[T, U]/(tU - uT))$, and the automorphism σ' lifting σ to X' is given by

$$\sigma'(T) = aT + bU, \qquad \sigma'(U) = cT + dU.$$

The scheme of σ'-fixed points over X' is covered by the open affine subschemes $\Omega_1 = D_+(U) \cap D_+(\sigma'(U))$ and $\Omega_2 = D_+(T) \cap D_+(\sigma'(T))$. As we will remark in the following, the computation depends on the order of $(uh - tg)/F$ at x. This order is always greater or equal than $n + 1$.

Assume first that $\mathrm{ord}_x\{(uh - tg)/F\} = n + 1$. The scheme Ω_1 is isomorphic to the spectrum of the algebra:

$$\{A[T]/(t - uT)\}\left[\frac{1}{cT + d}\right].$$

The exceptional divisor E is defined there by the equation $u = 0$. We have $\sigma(u)/u = cT + d$, and is therefore invertible over Ω_1. Using remark 7.4, we get that $\mathrm{fix}(\sigma')$ is defined over Ω_1 by the equations $g = \sigma(u) - u$ and

$$H = \frac{aT + b}{cT + d} - T = \frac{\sigma(t)}{\sigma(u)} - \frac{t}{u} = \frac{uh - tg}{u\sigma(u)}.$$

We leave to the reader the exercise of writing ϕ and ψ in this case. The greatest common divisor of g and H is Fu^{n-1}. Hence, $\alpha = n - 1$ and the residual scheme $\mathfrak{R}'$ is defined over Ω_1 by the ideal $(g/Fu^{n-1}, (uh - tg)/Fu^n\sigma(u))$.

In the same way, Ω_2 is isomorphic to the spectrum of the algebra:

$$\{A[U]/(u - Ut)\}\left[\frac{1}{a + bU}\right].$$

The exceptional divisor E is defined there by the equation $t = 0$. We have $\sigma(t)/t = a + bU$, and is therefore invertible over Ω_2. The scheme of fixed points is defined over Ω_2 by $h = \sigma(t) - t$ and

$$G = \frac{c + dU}{a + bU} - U = \frac{\sigma(u)}{\sigma(t)} - \frac{u}{t} = \frac{tg - uh}{t\sigma(t)}.$$

The greatest common divisor of h and G is Ft^{n-1} and $\mathfrak{R}'$ is defined over Ω_2 by the ideal $(h/Ft^{n-1}, (tg - uh)/Ft^n\sigma(t))$. I claim that:

$$\deg \mathfrak{R}' = \left(\pi^*\left(\frac{h}{F}\right) - nE.\pi^*\left(\frac{g}{F}\right) - nE\right) + \left(\pi^*\left(\frac{uh - tg}{F}\right) - (n+1)E.E\right).$$

The first intersection number $(\pi^*(h/F) - nE.\pi^*(g/F) - nE)$ is defined as the sum, over all closed points of E, of the local intersection of the effective Cartier divisors $\pi^*(h/F) - nE$ and $\pi^*(g/F) - nE$. These meet properly. The second intersection number can be defined in the same way. We can also use [6] exposé X to define it because one of the factors is E. To prove the claim, choose a closed fixed point y of E. Assume that $y \in \Omega_1$. Then, using Lemma B.1,

$$\begin{aligned}\deg_y \mathfrak{R}' &= \left(\frac{g}{Fu^{n-1}} \cdot \frac{uh - tg}{Fu^{n+1}}\right)_y = \left(\frac{ug}{Fu^n} \cdot \frac{uh - tg}{Fu^{n+1}}\right)_y \\ &= \left(\frac{g}{Fu^n} \cdot \frac{uh - tg}{Fu^{n+1}}\right)_y + \left(u. \frac{uh - tg}{Fu^{n+1}}\right)_y = \left(\frac{g}{Fu^n} \cdot \frac{h}{Fu^n}\right)_y + \left(u. \frac{uh - tg}{Fu^{n+1}}\right)_y.\end{aligned}$$

This proves the claim. By the classical projection formula, the claim implies

$$\deg \mathfrak{R}' = \left(\frac{h}{F} \cdot \frac{g}{F}\right)_x - n^2 + (n+1) = \deg \mathfrak{R} - n(n-1) + 1,$$

which is the needed relation.

Assume now that $\mathrm{ord}_x(uh - tg)/F > n + 1$. In this case, $n = m \geqslant 1$ and there exists $\alpha \in \mathfrak{m}^{n-1}$ such that $h/F - t\alpha \in \mathfrak{m}^{n+1}$ and $g/F - u\alpha \in \mathfrak{m}^{n+1}$. First, over Ω_1 the greatest common divisor of g and H is Fu^n. Hence, $\alpha = n$ and $\mathfrak{R}'$ is defined there by the ideal $(g/Fu^n, (uh - tg)/Fu^{n+1}\sigma(u))$. Second, over Ω_2 the greatest common divisor of h and G is Ft^n. So, the residual scheme $\mathfrak{R}'$ is defined there by the ideal $(h/Ft^n, (tg - uh)/Ft^{n+1}\sigma(t))$. I claim that:

$$\deg \mathfrak{R}' = \left(\pi^*\left(\frac{g}{F}\right) - nE.\pi^*\left(\frac{h}{F}\right) - nE\right) - (E.\pi^*\alpha - (n-1)E).$$

Choose a closed fixed point y of E. Assume that $y \in \Omega_1$. Then, using Lemma B.1,

$$\begin{aligned}\deg_y \mathfrak{R}' &= \left(\frac{g}{Fu^n} \cdot \frac{uh - tg}{Fu^{n+1}\sigma(u)}\right)_y = \left(\frac{g}{Fu^n} \cdot \frac{uh - tg}{Fu^{n+1}}\right)_y - \left(u. \frac{g}{Fu^n}\right)_y \\ &= \left(\frac{g}{Fu^n} \cdot \frac{h}{Fu^n}\right)_y - \left(u. \frac{g}{Fu^n}\right)_y = \left(\frac{g}{Fu^n} \cdot \frac{h}{Fu^n}\right)_y - \left(u. \frac{\alpha}{u^{n-1}}\right)_y\end{aligned}$$

https://doi.org/10.1023/A:1001822419774 Published online by Cambridge University Press

Indeed,

$$\frac{g}{Fu^n} - \frac{\alpha}{u^{n-1}} \in \frac{1}{u^n}\mathfrak{m}^{n+1}$$

and is therefore a multiple of u over Ω_1. This proves the claim. The lemma follows from the claim using the projection formula. $\square$

LEMMA 7.5. *With the same notation as above, we have*

$$(\Delta_{X'}.\Gamma')_{loc} - (\Delta_X.\Gamma)_{loc} = \alpha(\alpha+1) + \deg\mathfrak{R}' - \deg\mathfrak{R}.$$

Proof. Let ω and ω' be the dualizing sheaves of X and X'. These sheaves are related by $\omega' = \pi^*\omega \otimes \mathcal{O}_{X'}(E)$. The Cartier divisors Y over X and Y' over X' are decomposed into vertical and horizontal parts $Y = H + V$ and $Y' = H' + V'$. Then, $\pi_* H' = H$ and $\pi_* V' = V$, and by Lemma 7.3, $Y' = \pi^* Y + \alpha E$. Using Proposition 5.8, we see that $(\Delta_{X'}.\Gamma')_{loc} - (\Delta_X.\Gamma)_{loc}$ is a sum of three differences:

(i) The horizontal term:

$$\begin{aligned} &c^{H'}_{1,H'_s}(\omega'|_{H'} \dashrightarrow \mathcal{O}(-Y')|_{H'}) \cap [H'] - c^{H}_{1,H_s}(\omega|_{H} \dashrightarrow \mathcal{O}(-Y)|_{H}) \cap [H] \\ &\quad = -(\alpha+1)(E.H'). \end{aligned}$$

(ii) The vertical term:

$$\begin{aligned} &-(\omega' + Y'.V') + (\omega + Y.V) \\ &\quad = -(\alpha+1)(E.V') = -(\alpha+1)(p^*Y + \alpha E - H'.E) \\ &\quad = \alpha(\alpha+1) + (\alpha+1)(E.H'). \end{aligned}$$

(iii) The 0-dimensional term: $\deg\mathfrak{R}' - \deg\mathfrak{R}$.

Lemma 7.5 is now proved. $\square$

Finally by Lemma 7.2, $\mathrm{tr}(\sigma')|H^*_{et}(X'_s, \mathbb{Q}_l) - \mathrm{tr}(\sigma)|H^*_{et}(X_s, \mathbb{Q}_l) = 1$. Then, Theorem 7.1 is a consequence of Lemmas 7.3 et 7.5. $\square$

Remark 7.6 (Lefschetz numbers over normal surfaces). Let X be a normal surface over S (i.e. a normal integral scheme of dimension 2 proper and flat over S) and let σ be a non-trivial S-automorphism of X. Fix a resolution X' of X to which σ lifts. The existence of such an X' is a consequence of the theory of minimal resolutions in dimension 2 [11]. The lifting of σ, when it exists, is unique (denoted σ'). Define the Lefschetz number of σ over X by the formula:

$$L(X,\sigma) = (\Delta_{X'}.\Gamma_{\sigma'})_{loc} - \mathrm{tr}(\sigma')|H^*_{et}(X'_s, \mathbb{Q}_l) + \mathrm{tr}(\sigma)|H^*_{et}(X_s, \mathbb{Q}_l).$$

By Theorem 7.1, this definition does not depend on the desingularisation we choose.

7.2. SOME COMBINATORICS ON GRAPHS

We consider a birational S-morphism $f: X \to Y$ from a normal surface X to a regular surface Y. Let $\mathbb{V}$ be the cycle of $X \times_Y X$ given by its irreducible components different from the diagonal. The latter are isomorphic to products of two irreducible components of X_s which collapse to the same point in Y. We write $[X \times_Y X] = [\Delta_X] + \mathbb{V}$ and decompose

$$\mathbb{V} = \sum_{(i,j)} e_{i,j}[E_i \times_k E_j],$$

where the E_i are the irreducible components of X_s which collapse to a point in Y, and the sum above is taken over all couples (i, j) such that E_i and E_j collapse to the same point of Y. Assume, moreover, that we are given an S-automorphism σ of Y which can be lifted to an automorphism of X (also denoted by σ).

THEOREM 7.7. *Let $f: X \to Y$ be as above, then*

$$\sum_{(i,j)} e_{i,j}(\sigma(E_i).E_j) = \mathrm{tr}(\sigma)|H^*_{et}(Y_s, \mathbb{Q}_l) - \mathrm{tr}(\sigma)|H^*_{et}(X_s, \mathbb{Q}_l). \tag{45}$$

The numbers on the left-hand side are the intersection numbers over the normal surface X defined by Mumford [14] and summarized in Appendix A.

This theorem was proved by Bloch [3] for $\sigma = \mathrm{id}$. We begin by giving his proof in more details. Then, we deduce the result for any automorphism.

Reduction step: Let $f: X \to Y$ be a dominant map between arithmetic surfaces over S (i.e. X and Y are assumed to be regular!). Then, f is local complete intersection. This means that f factors into

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "f"'] & P \arrow[d, "g"] \\ & Y \end{tikzcd}$$

where g is smooth and i is a regular closed immersion. Put $f^* = i^! g^* \in A^0(X \to Y)$. Fulton proved that f^* does not depend on the factorization of f ([7] proposition 6.6). Consider the following diagram

$$\begin{tikzcd} X \times_S X \arrow[r, "i \times i"] \arrow[dr, "f \times f"'] & P \times_S P \arrow[d, "g \times g"] \\ & Y \times_S Y \end{tikzcd}$$

As $g \times g$ is smooth and $i \times i$ is a regular closed immersion (it is the composition of $X \times_S X \to X \times_S P$ and $X \times_S P \to P \times_S P$ which are both regular closed immersions), then $f \times f$ is l.c.i. and defines a bivariant class $(f \times f)^* \in A^0(X \times_S X \to Y \times_S Y)$. Consider the following Cartesian diagram

$$\begin{tikzcd} X \times_Y X \arrow[r] \arrow[d] & X \times_S X \arrow[d, "f \times f"] \\ \Delta_Y \arrow[r] & Y \times_S Y \end{tikzcd}$$

I claim that

$$(f \times f)^*[\Delta_Y] = [X \times_Y X] \in A_2(X \times_Y X). \tag{46}$$

Proof of the Claim. We consider the previous factorization of $f \times f$. Then,

$$\begin{aligned} (f \times f)^*[\Delta_Y] &= (i \times i)^!(g \times g)^*([\Delta_Y]) \\ &= (i \times i)^!([P \times_Y P]) \quad \text{(because } g \times g \text{ is flat).} \end{aligned}$$

Consider the fiber square

$$\begin{tikzcd} X \times_Y X \arrow[r] \arrow[d] & P \times_Y P \arrow[d] \\ X \times_S X \arrow[r, "i \times i"'] & P \times_S P \end{tikzcd}$$

Let d be the dimension of P. As $P \times_Y P$ is regular and $X \times_Y X$ is an l.c.i. scheme, then the closed immersion $X \times_Y X \to P \times_Y P$ is regular (EGA IV 19.3.2) of codimension $2d - 4$, which is the same codimension as $i \times i$. Therefore,

$$(i \times i)^!([P \times_Y P]) = [X \times_Y X] \in A_2(X \times_Y X).$$

The claim is proved. $\square$

Consider now two effective *vertical* divisors E and F over Y. I claim that

(i) $f^{-1}E$ is a Cartier divisor over X of associated sheaf $f^*\mathcal{O}_Y(E)$, and $f^*([E]) = [f^{-1}E] \in A_1(f^{-1}E)$.

(ii) $(f \times f)^*[E \times_S F] = [f^{-1}E \times_S f^{-1}F] \in A_2(f^{-1}E \times_S f^{-1}F)$.

Proof. (i) The closed subscheme $f^{-1}E$ is locally defined by one equation in X (the pull-back of a local equation defining E in Y). It is not a zero divisor because the map is dominant. We have, $f^*([E]) = i^!([g^{-1}E])$ because g is flat. Consider

https://doi.org/10.1023/A:1001822419774 Published online by Cambridge University Press

the diagram

$$\begin{tikzcd}
f^{-1}E \arrow[r] \arrow[d] & g^{-1}E \arrow[r] \arrow[d, "j"] & E \arrow[d] \\
X \arrow[r, "i"'] & P \arrow[r, "g"'] & Y
\end{tikzcd}$$

As g is flat, j is a regular closed immersion of codimension 1. The commutativity of intersection products ([7] theorem 6.4) implies that $i^!([g^{-1}E]) = j^![X]$. But $j^![X] = [f^{-1}E]$ because $f^{-1}E$ is a Cartier divisor over X.

(ii) The bivariant class f^* maps $A_1(Y_s)$ to $A_1(X_s)$, and $(f \times f)^*$ maps $A_2(Y_s \times_k Y_s)$ to $A_2(X_s \times_k X_s)$. Moreover, the following diagram commutes:

$$\begin{tikzcd}
A_1(Y_s) \otimes A_1(Y_s) \arrow[r] \arrow[d, "f^* \otimes f^*"'] & A_2(Y_s \times_k Y_s) \arrow[d, "(f\times f)^*"] \\
A_1(X_s) \otimes A_1(X_s) \arrow[r] & A_2(X_s \times_k X_s)
\end{tikzcd}$$

Notice that $[E]$ and $[F]$ are in the image of $A_1(Y_s) \to A_1(Y)$, and $[E \times F]$ is in the image of $A_2(Y_s \times_k Y_s) \to A_2(Y \times_S Y)$. It is enough to prove relation (ii) in $A_2(X_s \times_k X_s)$. Indeed, by dimension argument, $A_2(f^{-1}E \times_S f^{-1}F) = A_2((f^{-1}E)_{\mathrm{red}} \times_k (f^{-1}F)_{\mathrm{red}})$ injects in $A_2(X_s \times_k X_s)$. In $A_2(X_s \times_k X_s)$, the relation is a consequence of (i) and the commutativity of the previous diagram. □

Assume that we are given two birational maps $f : X_2 \to X_1$ and $g : X_1 \to X_0$ between arithmetic surfaces over S. Define the cycles $\mathbb{V}(X_2/X_1)$ over $X_2 \times_{X_1} X_2$, $\mathbb{V}(X_2/X_0)$ over $X_2 \times_{X_0} X_2$ and $\mathbb{V}(X_1/X_0)$ over $X_1 \times_{X_0} X_1$ as before : they are made of the irreducible components of these schemes different from the diagonals. Write

$$\mathbb{V}(X_1/X_0) = \sum_{(i,j)} e_{i,j}[E_i \times E_j],$$

where the E_i are the irreducible components of X_1 which collapse to a point in X_0, and the sum is taken over all couples (i, j) such that E_i and E_j collapse to the same point in X_0. I claim that

$$\mathbb{V}(X_2/X_0) = \mathbb{V}(X_2/X_1) + \sum_{(i,j)} e_{i,j}[f^{-1}E_i \times f^{-1}E_j] \in A_2(X_2 \times_{X_0} X_2). \tag{47}$$

Moreover, this relation holds between cycles because $X_2 \times_{X_0} X_2$ has dimension 2.

https://doi.org/10.1023/A:1001822419774 Published online by Cambridge University Press

Proof of the claim. Put $h = g \circ f$ and consider the Cartesian diagram

$$\begin{tikzcd} X_2 \times_{X_1} X_2 \arrow[r] \arrow[d] & X_2 \times_{X_0} X_2 \arrow[r] \arrow[d] & X_2 \times_S X_2 \arrow[d, "f \times f"] \\ \Delta_{X_1} \arrow[r] & X_1 \times_{X_0} X_1 \arrow[r] & X_1 \times_S X_1 \end{tikzcd}$$

By Equation (46),

$$\begin{aligned}[][X_2 \times_{X_0} X_2] &= (h \times h)^*[\Delta_{X_0}] = (f \times f)^*(g \times g)^*[\Delta_{X_0}] \quad \text{([7] theorem 6.5)} \\ &= (f \times f)^*[X_1 \times_{X_0} X_1] \\ &= (f \times f)^*([\Delta_{X_1}] + \sum_{(i,j)} e_{i,j}[E_i \times E_j]) \\ &= [X_2 \times_{X_1} X_2] + \sum_{(i,j)} e_{i,j}[f^{-1}E_i \times f^{-1}E_j] \in A_2(X_2 \times_{X_0} X_2).\end{aligned}$$

The claim follows by subtracting $[\Delta_{X_2}]$ from this relation. □

Remark 7.8. Equation (47) implies very easily Theorem 7.7 for a regular X. Indeed, it implies that, like the right-hand side, the left-hand side of Equation (45) is additive for the composition of birational maps between arithmetic surfaces. So, it is enough to prove (45) for the two types of birational maps introduced in the proof of Theorem 7.1. For these maps, (45) is obvious.

We come to the general situation of Theorem 7.7 where X is just a normal surface. Fix a resolution Z of X to which σ extends. The composed map $g : Z \to Y$ is birational and therefore is obtained by a sequence of blow-ups at closed points [11]. The map $Z \to X$ is obtained by contracting some of the exceptional curves in Z. Denote Φ the dual graph of the exceptional fibers of g. It is the graph labeled by one vertex for each irreducible component in an exceptional fiber of Z/Y and one edge between intersecting curves. Each vertex is labeled with the self-intersection number of the corresponding curve and each edge with the intersection number of its vertices. We denote I the set of vertices which are contracted in $Z \to X$ and σ the automorphism induced by σ on this graph. Notice that $\sigma(I) = I$.

LEMMA 7.9. *The only datum which determines the left and right-hand sides of equation (45) is* (Φ, I, σ). *The way that they depend on* (Φ, I, σ) *is the same in the pure and in the mixed characteristic situation.*

Proof. The right-hand side of (45) is minus the number of σ-fixed vertices in $\Phi - I$. To prove this statement, write

$$\begin{aligned}&\mathrm{tr}(\sigma)|H^*_{et}(Y_s, \mathbb{Q}_l) - \mathrm{tr}(\sigma)|H^*_{et}(X_s, \mathbb{Q}_l) \\ &\quad = \mathrm{tr}(\sigma)|H^*_{et}(Y_s, \mathbb{Q}_l) - \mathrm{tr}(\sigma)|H^*_{et}(Z_s, \mathbb{Q}_l) + \mathrm{tr}(\sigma)|H^*_{et}(Z_s, \mathbb{Q}_l) - \mathrm{tr}(\sigma)|H^*_{et}(X_s, \mathbb{Q}_l).\end{aligned}$$

As $Z \to Y$ is a sequence of blow-ups, Lemma 7.2 implies that the first difference is minus the number of σ-fixed vertices in Φ. The second difference is the number of σ-fixed vertices in I. Indeed, one can factor $Z \to X$ into a sequence of birational maps each of them is a contraction of a disjoint union of curves isomorphic to $\mathbb{P}^1$ and making an orbit under σ of one of them. Then, the proof of Lemma 7.2 applies in this case. The details are given in Lemma 7.11.

We prove now the lemma for the left hand side of (45). Denote E_i' the strict transform in Z of the irreducible component E_i. First, the E_i' are exactly the vertices of Φ which are not in I. Second, the intersection number $(\sigma(E_i).E_j)$ over X is completely determined by (Φ, I, σ) as explained in Appendix A. Finally, consider two such components E_i and E_j over X and E_i' and E_j' their strict transform in Z. We will prove that the $e_{i,j}$ depends only on Φ. Let $e_{i,j}'$ be the multiplicity of $E_i' \times E_j'$ in $Z \times_Y Z$. The map $Z \to X$ is an isomorphism in a neighborhood of the generic points of E_i and E_j, then $e_{i,j} = e_{i,j}'$. Therefore, it is enough to consider the case of $I = \emptyset$. In this case, the proof is by induction on the number n of blow-ups giving the map $Z \to Y$.

If $n = 1$, then there is only one exceptional fiber E of self-intersection -1 and the multiplicity of $E \times E$ in $Z \times_Y Z$ is 1. Assume the result for any sequence of $(n-1)$ blow-ups and factor the map $Z \to Y$ into $\rho : Z \to Z'$ and $\pi : Z' \to Y$, where the number of blow-ups in ρ is $(n-1)$, and π is a blow-up at a closed point with exceptional fiber E. Let Φ' be the graph associated with $Z \to Z'$. It is the subgraph of Φ obtained by removing the vertex corresponding to the strict transform of E in Z. By the reduction step (equation (47)), the following relation holds between cycles over $Z \times_Y Z$:

$$\mathbb{V}(Z/Y) = \mathbb{V}(Z/Z') + [\rho^* E \times \rho^* E].$$

By the induction hypothesis, the cycle $\mathbb{V}(Z/Z')$ is completely determined by $\Phi' \subset \Phi$. The pull-back $\rho^* E$ is also the pull-back as a Cartier divisor of E and therefore, is determined by the graph Φ. □

DEFINITION 7.10. An admissible triple is a triple (Φ, I, σ) made of

- an admissible graph Φ: the dual graph of a birational map between regular surfaces labeled as explained before (where a regular surface means an irreducible regular scheme of dimension 2 proper and flat over a base which can be either a field or the spectrum of a discrete valuation ring),
- a set of vertices I on Φ,
- an automorphism σ of the labeled graph Φ (i.e. an automorphism which preserves the intersection numbers), such that $\sigma(I) = I$.

The arithmetic situation we begin with provides us with an admissible triple. Moreover, any admissible triple of the form (Φ, I, id) can be realized in this way.

LEMMA 7.11. (i) *Let* Φ *be an admissible graph and* Y *be a regular surface (in the sense of the previous definition). Then, there exist a regular surface* Z *and a birational map* $Z \to Y$ *with dual graph* Φ.

(ii) *Assume that* Y *is either defined over an algebraic closure of a finite field, or over a complete discrete valuation ring. Let* I *be a set of vertices of* Φ. *Then, one can contract in* Z *the curves corresponding to* I, *to get a normal surface* X *and a birational morphism* $X \to Y$. *Moreover,* $\chi(X) - \chi(Y) = \#[\Phi - I]$.

Proof. (i) Obvious.

(ii) For each $i \in I$, let F_i be the associated exceptional curve of Z/Y. The intersection form over the divisors supported over the exceptional curves of Z/Y is definite negative. Therefore, its restriction to the divisors supported over the F_i is also definite negative. This is a necessary and sufficient condition for contractability under the hypothesis of the Lemma ([1] theorem 2.9 for the geometric case, and [13] corollary 4.4 for the arithmetic case). This means that there exists a normal surface X and a birational projective morphism $\varphi : Z \to X$ such that

(1) $\varphi(\cup_{i\in I} F_i)$ is a finite set of points, and
(2) $\varphi : Z - (\cup_{i\in I} F_i) \longrightarrow X - \varphi(\cup_{i\in I} F_i)$ is an isomorphism.

A contraction when it exists is unique up to isomorphism ([11] section 27). It is universal in the following sense: *Let* A *be a normal surface with a birational projective morphism* $\psi : Z \to A$ *such that* $\psi(\cup_{i\in I} F_i)$ *is a finite set of points. Then, there exists a map* $\phi : X \to A$ *such that* $\psi = \phi \circ \varphi$. For this, consider the following diagram

$$\begin{tikzcd} Z \arrow[r] \arrow[dr, "(\psi,\varphi)"'] & X' \arrow[d] \\ & A \times X \end{tikzcd}$$

where X' is the normalization of the image of Z (we use here that Z is regular in order the get the factorization of (ψ, φ)). Obviously, X' satisfies properties (1) and (2). Then, X' is isomorphic to X. The map $\phi : X \to A$ is obtained by composing with the first projection.

Choose a bijection $I \simeq \{1, \dots n\}$. For each $0 \leqslant j \leqslant n$, let X_j be the normal surface obtained by contracting in Z the curves $(F_i \mid 1 \leqslant i \leqslant j)$, and $\varphi_j : Z \to X_j$ be the contraction (hence, $X_n = X$). By the universal property of X_j, there exists a proper morphism $\phi_j : X_j \to X_{j+1}$ such that $\varphi_{j+1} = \phi_j \circ \varphi_j$. The map ϕ_j is an isomorphism outside a single point of X_{j+1}. Its fiber over this point is $C_j = \varphi_j(F_{j+1})$. I claim that φ_j induces an isomorphism $F_{j+1} \simeq \mathbb{P}^1 \to C_j$. Let x be a point of C_j. Its inverse image $\varphi_j^{-1}(x) \subset Z$ is either a point or a connected curve C_x supported over the $(F_i \mid 1 \leqslant i \leqslant j)$. Assume that it is a connected curve. It is enough to see that $F_{j+1} \cap C_x$ is a unique point. If it is more than one point, one can construct a

https://doi.org/10.1023/A:1001822419774 Published online by Cambridge University Press

non-trivial loop with curves among the $(F_i \mid 1 \leqslant i \leqslant j+1)$. This is not possible because Φ is an admissible graph.

It follows from the Leray spectral sequence of ϕ_j that (see Lemma 7.2),

$$H^i_{et}(X_j, \mathbb{Q}_l) \simeq H^i_{et}(X_{j+1}, \mathbb{Q}_l) \quad \text{for } i \neq 2,$$
$$0 \to H^2_{et}(X_{j+1}, \mathbb{Q}_l) \to H^2_{et}(X_j, \mathbb{Q}_l) \to \mathbb{Q}_l \to 0,$$

for a prime number $l \neq p$. The last statement in the Lemma is proved. $\square$

Fix an admissible triple (Φ, I, σ), and let E and F be two vertices of $\Phi - I$. Define the rational numbers $(\alpha_i)_{i\in I}$ and $(\beta_i)_{i\in I}$ by the equations

$$\left(E + \sum_{i\in I} \alpha_i E_i . E_j\right) = 0 \quad \forall j \in I; \qquad \left(F + \sum_{i\in I} \beta_i E_i . E_j\right) = 0 \quad \forall j \in I.$$

These numbers are well defined because $\det(E_i.E_j)_{(i,j)\in I^2} \neq 0$. Put

$$(E.F)_I = \left(E + \sum_{i\in I} \alpha_i E_i . F + \sum_{i\in I} \beta_i E_i\right).$$

Let $Z \to Y$ be a realization of Φ, and X be the surface obtained by contracting the curves in I as in Lemma 7.11. Let E' and F' be the images of E and F in X. Then, $(E.F)_I$ is the intersection number of E' and F' over X defined by Mumford.

We associate to Φ , as in the proof of Lemma 7.9, non-negative integers $(e_{i,j})_{(i,j)\in\Phi^2}$ such that $e_{i,j} = 0$ if and only if i and j are not in the same connected component of Φ. Namely, we define them by induction on the number of vertices of Φ, using equation (47). Finally, we put

$$l(\Phi, I, \sigma) = \sum_{(i,j)\in[\Phi - I]^2} e_{i,j}(\sigma(E_i).E_j)_I, \tag{48}$$

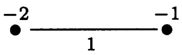
$$r(\Phi, I, \sigma) = -\#\{x \in \Phi - I \ ; \ \sigma(x) = x\}. \tag{49}$$

By Lemma 7.9, Theorem 7.7 is reduced to the combinatorial equation:

$$l(\Phi, I, \sigma) = r(\Phi, I, \sigma), \tag{50}$$

for any admissible triple. The latter will be proved in two steps. Following Bloch [3], the case $\sigma = \mathrm{id}$ is proved by taking a geometric realization of the triple (Φ, I, id) as in Lemma 7.11, and applying the Poincaré duality (Lemma 7.13). For a non-trivial automorphism, it is not clear if one can realize geometrically any admissible triple (Φ, I, σ). Lacking such an elegant proof, we will give a combinatorial one : thanks to some elementary operations on admissible graphs, equation (50) for any σ is reduced to the same equation for $\sigma = \mathrm{id}$.

EXAMPLE 7.12. Consider the graph



,

and take $I = \{-2\}$ and $\sigma = \mathrm{id}$. The geometric (or if you want the arithmetic) realization of this graph is

$$\begin{tikzcd} & X_2 \arrow[dl] \arrow[dr] & \\ X_1 \arrow[dr] & & Y' \arrow[dl] \\ & Y & \end{tikzcd}$$

where Y is a regular surface, X_1 is the blow-up of Y at a closed point with exceptional fiber E_1, and X_2 is the blow-up of X_1 at a closed point of E_1. Denote E_2 the exceptional fiber of X_2/X_1 and E_1' the strict transform of E_1 in X_2. Then, $(E_2.E_2) = -1$, $(E_1'.E_1') = -2$ and $(E_2.E_1') = 1$. Finally, Y' is obtained by contracting E_1' in X_2. Therefore, it is not regular. Let E_2' be the image of E_2 in Y'. Then, $(E_2'.E_2') = -1/2$. By Equation (47), we have:

$$[X_2 \times_Y X_2] = [\Delta_{X_2}] + [E_1' \times E_1'] + [E_1' \times E_2] + [E_2 \times E_1'] + 2[E_2 \times E_2].$$

So, $[Y' \times_Y Y'] = [\Delta_{Y'}] + 2[E_2' \times E_2']$. Thus, $l(\Phi, I, \mathrm{id}) = -1 = r(\Phi, I, \mathrm{id})$. This example shows also that Equation (45) does not hold if the target surface is not regular!

LEMMA 7.13. *Let Φ be an admissible graph and I be a set of vertices of Φ. Then,* $l(\Phi, I, \mathrm{id}) = r(\Phi, I, \mathrm{id})$.

Proof. Let k be an algebraic closure of the finite field $\mathbb{F}_p$. Take $Y = \mathbb{P}_k^2$ and construct a regular surface Z and a birational map $Z \to Y$ with dual graph Φ. By Lemma 7.11, one can contract the curves in Z corresponding the vertices of I to get a normal surface X. Put $g : Z \to X$ and $f : X \to Y$. The idea now is to invert the previous computation over $X \times_k X$. Namely, we first prove the projection formula and then deduce the above equation. The computation cannot be done in the Chow group $A_*(X \times_k X)$ because we lack an intersection product. Instead, we work with the ring of étale cohomology $H^*(X \times_k X, \mathbb{Q}_l)$ for a prime $l \neq p$. For any proper variety V over k of dimension d, we have a canonical isomorphism $H^{2d}(V, \mathbb{Q}_l) \simeq \mathbb{Q}_l$ given by the trace isomorphism $H_c^{2d}(V_0, \mathbb{Q}_l) \simeq \mathbb{Q}_l$, where V_0 is the smooth locus in V. It induces a cup product which will be denoted $\cup$.

Claim: X satisfies Poincaré duality, i.e.,

$$H^i(X, \mathbb{Q}_l) \times H^{4-i}(X, \mathbb{Q}_l) \xrightarrow{\cup} \mathbb{Q}_l$$

is a perfect pairing.

Proof. We factor $g : Z \to X$ as in the proof of Lemma 7.11-(ii) into $Z = X_0 \to X_1 \to \ldots \to X_n = X$. Each map $\phi_j : X_j \to X_{j+1}$ is a contraction of a curve $C_j \simeq \mathbb{P}^1_k$. We have seen that

$$\begin{aligned}
&H^i(X_j, \mathbb{Q}_l) \simeq H^i(X_{j+1}, \mathbb{Q}_l) \quad \text{for } i \neq 2,\\
&0 \to H^2(X_{j+1}, \mathbb{Q}_l) \to H^2(X_j, \mathbb{Q}_l) \to \mathbb{Q}_l \to 0.
\end{aligned}$$

We will prove that if Poincaré duality holds for X_j, then it holds for X_{j+1}. As ϕ_j^* preserves cup products, the statement is clear for $i \neq 2$.

The closed immersion $C_j \to X_j$ induces the morphism

$$H^2(X_j, \mathbb{Q}_l) \to H^2(C_j, \mathbb{Q}_l) \simeq \mathbb{Q}_l.$$

By Poincaré duality over X_j, this map defines a class denoted $[C_j] \in H^2(X_j, \mathbb{Q}_l)$. Obviously, for any $\alpha \in H^2(X_{j+1}, \mathbb{Q}_l)$, $\phi_j^*\alpha \cup [C_j] = 0$. Therefore, the map

$$\begin{aligned}
H^2(X_{j+1}, \mathbb{Q}_l) \oplus \mathbb{Q}_l &\to H^2(X_j, \mathbb{Q}_l)\\
(\alpha, x) &\mapsto \phi_j^*\alpha + x[C_j]
\end{aligned}$$

is an isomorphism and gives an orthogonal decomposition of $H^2(X_j, \mathbb{Q}_l)$. Then, we get the Poincaré duality for X_{j+1}. By induction, Poincaré duality holds for X because it holds for the regular surface Z. □

Let $(E_h)_h$ be the irreducible curves of X collapsing to a point in Y. For any h, define the class $[E_h] \in H^2(X, \mathbb{Q}_l)$ by the map

$$H^2(X, \mathbb{Q}_l) \to H^2(E_h, \mathbb{Q}_l) \simeq \mathbb{Q}_l$$

under Poincaré duality. By the Künneth formula, we have:

$$H^4(X \times X, \mathbb{Q}_l) \simeq \bigoplus_{0 \leqslant i \leqslant 4} H^i(X, \mathbb{Q}_l) \otimes H^{4-i}(X, \mathbb{Q}_l).$$

Now, we define the needed classes. First, for any (h, l), put $[E_h \times E_l] = [E_h] \otimes [E_l] \in H^4(X \times X, \mathbb{Q}_l)$. Second, for any $0 \leqslant i \leqslant 4$, fix a basis $(a_{i,j})_j$ of $H^i(X, \mathbb{Q}_l)$. Let $a^*_{i,j}$ be the dual basis of $H^{4-i}(X, \mathbb{Q}_l)$, and put

$$[\Delta_X] = \sum_{i,j} a_{i,j} \otimes a^*_{i,j} \in H^4(X \times X, \mathbb{Q}_l).$$

Third, define $[X \times_Y X]$ to be the inverse image of $[\Delta_Y] \in H^4(Y \times Y, \mathbb{Q}_l)$. Let $e_{h,l}$ be

https://doi.org/10.1023/A:1001822419774 Published online by Cambridge University Press

the integers associated to Φ as before. Then

$$[X \times_Y X] = [\Delta_X] + \sum_{(h,l)} e_{h,l}[E_h \times E_l] \in H^4(X \times X, \mathbb{Q}_l). \tag{51}$$

Proof. Let $h = f \circ g : Z \to Y$ and denote F_i the exceptional curves in Z. Define in the same way the classes $[\Delta_Z], [Z \times_Y Z]$ and $[F_i \times F_j] \in H^4(Z \times Z, \mathbb{Q}_l)$. As Z is regular, we have a class map $A_2(Z \times Z) \to H^4(Z \times Z, \mathbb{Q}_l)$. Notice that the classes previously defined are the images by the cycle map of the usual cycles over $Z \times Z$. Therefore, the following relation

$$[Z \times_Y Z] = [\Delta_Z] + \sum_{(i,j)} e_{i,j}[F_i \times F_j] \in H^4(Z \times Z, \mathbb{Q}_l) \tag{52}$$

holds (as the image of the same relation in $A_2(Z \times Z)$ by the cycle map).
Define

$$(g \times g)_* = \bigoplus_{0 \leqslant i \leqslant 4} g_* \otimes g_* : H^4(Z \times Z, \mathbb{Q}_l) \to H^4(X \times X, \mathbb{Q}_l),$$

where $g_* : H^i(Z, \mathbb{Q}_l) \to H^i(X, \mathbb{Q}_l)$ is the adjoint of g^* under Poincaré duality. I claim that:

(i) $(g \times g)_*[Z \times_Y Z] = [X \times_Y X]$,
(ii) $(g \times g)_*[\Delta_Z] = [\Delta_X]$,
(iii) $(g \times g)_*[F_i \times F_j] = [g(F_i) \times g(F_j)]$ if $g(F_i)$ and $g(F_j)$ are curves, and 0 otherwise.

The equalities (i) and (ii) are purely formal. We have injections $H^2(Y, \mathbb{Q}_l) \to H^2(X, \mathbb{Q}_l) \to H^2(Z, \mathbb{Q}_l)$. One takes bases of these spaces by taking a basis of the subspace and a basis of its orthogonal relatively to Poincaré duality, and notices that the push-forward maps the orthogonal part to zero. Equality (iii) is a consequence of the functoriality of pull-back and Poincaré duality. Finally, relation (51) is the image by $(g \times g)_*$ of Equation (52). $\square$

The following cup products are computed over $X \times X$:

(1) $[\Delta_X] \cup [\Delta_X] = \chi(X)$,
(2) $[\Delta_X] \cup [X \times_Y X] = \chi(Y)$,
(3) $[\Delta_X] \cup [E_h \times E_l]$ is Mumford's intersection number $(E_h.E_l)$ over the normal surface X.

As before, (1) and (2) are purely formal. But (3) needs some details : from the definition of the diagonal, $[\Delta_X] \cup [E_h \times E_l] = [E_h] \cup [E_l]$ where the cup product is now over X. We have $[E_h] \cup [E_l] = g^*[E_h] \cup g^*[E_l]$. Let E'_h be the strict transform of E_h in Z. We have seen that for any exceptional curve F of Z/X, $g^*[E_h] \cup [F] = 0$. Moreover, $g^*[E_h] - [E'_h]$ is the image by the class map of an exceptional divisor. Then, the pull-back $g^*[E_h]$ is the class of the divisor g^*E_h defined

https://doi.org/10.1023/A:1001822419774 Published online by Cambridge University Press

in Appendix A. As Z is regular, $[E_h] \cup [E_l]$ is the intersection number $(g^* E_h . g^* E_l)$ which is the claim.

Now, we deduce the lemma: take the cup product of Equation (51) with $[\Delta_X]$. By Lemma 7.11, $\chi(Y) - \chi(X) = r(\Phi, I, \mathrm{id})$, and by (3),

$$[\Delta_X] \cup (\sum_{(h,l)} e_{h,l}[E_h \times E_l]) = l(\Phi, I, \mathrm{id}). \qquad \square$$

We introduce some basic operations on admissible graphs:

(i) If Φ is the admissible graph associated to a birational map $Z \to Y$, then the connected components of Φ correspond to the exceptional fibers of Z/Y.

(ii) The subgraph made of some connected components of an admissible graph is an admissible graph. Indeed, by (i) one can contract the exceptional curves in the other connected components to get a regular surface which realizes the new graph over the base surface.

(iii) Let Φ be an admissible graph. Take any realization $Z \to Y$ of Φ, and let Z' be the regular surface obtained from Z by contracting all the special curves of self-intersection -1. Denote $\Phi[1]$ the dual graph of Z'/Y. The graph $\Phi[1]$ can be directly computed from Φ without any geometric realization. Indeed, as a set $\Phi[1]$ is obtained by removing the vertices of self-intersection -1. The intersection numbers are computed by the rules introduced in appendix A. Finally, we put an edge between two different vertices if their intersection number is non–zero. Moreover, an automorphism of Φ induces an automorphism of $\Phi[1]$.

(iv) The number of connected components of $\Phi[1]$ is less or equal than the number of connected components of Φ (this is a consequence of (i)).

(v) Let Φ be an admissible connected graph. By (iv), $\Phi[1]$ is either connected or empty. Define the admissible connected graphs $\Phi[1], \Phi[2], \Phi[3], \ldots$ by induction as in (iii). There exists an integer n such that $\Phi[n] = \emptyset$ and $\Phi[n-1] \neq \emptyset$. Then, $\Phi[n-1]$ contains only vertices of self-intersection -1. But over an admissible graph, two vertices of self-intersection -1 cannot be connected. Therefore $\Phi[n-1]$ is a single vertex. Define the base vertex E_1 of Φ to be the strict transform of $\Phi[n-1]$ in Φ. A base vertex of a connected graph is fixed by any automorphism. To give a geometric picture, fix any realization $Z \to Y$ of Φ and let y be the unique point of Y with an exceptional fiber. Then, we can factor the map Z/Y into $Z \to Y' \to Y$, where Y' is a blow-up of Y at y. Let E be the exceptional fiber in Y'. The strict transform of E in Z is the base vertex E_1.

We come to the proof of Equation (50) for an admissible triple (Φ, I, σ). The action of σ on the connected components of Φ induces the decomposition $\Phi = \bigsqcup_{i=1}^{n} \Phi_i$, where the Φ_i are the orbits of the connected components of Φ. Denote $I_i = \Phi_i \cap I$. By (ii),

https://doi.org/10.1023/A:1001822419774 Published online by Cambridge University Press

the triples (Φ_i, I_i, σ) are admissible. Moreover,

$$l(\Phi, I, \sigma) = \sum_{i=1}^{n} l(\Phi_i, I_i, \sigma), \tag{53}$$

$$r(\Phi, I, \sigma) = \sum_{i=1}^{n} r(\Phi_i, I_i, \sigma). \tag{54}$$

Indeed, (54) follows directly from Definition (49). Equation (53) follows from Definition (48) if we notice that for any (i, j) in the same connected component of Φ, $e_{i,j}$ depends only on the connected component of Φ which contains them.
By (53) and (54), we are reduced to consider two cases:

(1) The graph Φ has no fixed connected component by σ. First, $r(\Phi, I, \sigma) = 0$ because there are no fixed vertices. Second, $l(\Phi, I, \sigma)$ is defined as the sum over couples (E_i, E_j) in the same connected component of Φ of $e_{i,j}(\sigma(E_i).E_j)$. By hypotheses on Φ, if E_i and E_j are in the same connected component, then $\sigma(E_i)$ and E_j are not in the same one. Hence, their intersection number is 0. The equality $l(\Phi, I, \sigma) = 0 = r(\Phi, I, \sigma)$ follows.
(2) The graph Φ is connected. Let E_1 be its base vertex and $\Phi' = \Phi - \{E_1\}$. The graph Φ' is admissible. Indeed, if $Z \to Y$ is a realization of Φ and $Z \to Y_1 \to Y$ is its factorization as in (iv), then $Z \to Y_1$ realizes the graph Φ'. Finally, σ fixes E_1 and therefore induces an automorphism of Φ'. There are two cases, either $E_1 \in I$ or $E_1 \notin I$.

2–(i) If $E_1 \in I$, put $I' = I - \{E_1\}$. Remark that $r(\Phi, I, \sigma) = r(\Phi', I', \sigma)$. By formula (48),

$$l(\Phi, I, \sigma) = \sum_{(i,j)\in[\Phi - I]^2} e_{i,j}(\sigma(E_i).E_j)_I,$$
$$l(\Phi', I', \sigma) = \sum_{(i,j)\in[\Phi' - I']^2} e'_{i,j}(\sigma(E_i).E_j)_{I'}.$$

We consider Φ' as a subgraph of Φ, then $[\Phi - I] = [\Phi' - I']$. First, we compare the integers $e_{i,j}$ and $e'_{i,j}$. Let E be the exceptional fiber of Y_1/Y (its strict transform in Z is E_1). Define the integers $(\alpha_i)_{i\in\Phi-\{1\}}$ by the relations

$$\left(E_1 + \sum_{i\in\Phi-\{1\}} \alpha_i E_i.E_j\right) = 0 \quad \forall j \neq 1.$$

Then, by Equation (47),

$$\mathbb{V}(Z/Y) = \mathbb{V}(Z/Y_1) + [(E_1 + \sum_{i\neq 1} \alpha_i E_i) \times (E_1 + \sum_{i\neq 1} \alpha_i E_i)].$$

It follows that for $i, j \neq 1$, $e_{i,j} = e'_{i,j} + \alpha_i\alpha_j$. Second, I claim that for any

https://doi.org/10.1023/A:1001822419774 Published online by Cambridge University Press

$(i,j) \in [\Phi - I]^2$,

$$(\sigma(E_i).E_j)_I = (\sigma(E_i).E_j)_{I'} - \frac{(\sigma(E_i).E_1)_{I'}(E_j.E_1)_{I'}}{(E_1.E_1)_{I'}}.$$

This is an easy computation if we remark that $\sigma(E_1) = E_1$. Therefore,

$$\begin{aligned} l(\Phi, I, \sigma) - l(\Phi', I', \sigma) &= -\sum_{(i,j)\in[\Phi-I]^2} e_{i,j}\frac{(\sigma(E_i).E_1)_{I'}(E_j.E_1)_{I'}}{(E_1.E_1)_{I'}} + \\ &\quad + \sum_{(i,j)\in[\Phi-I]^2} \alpha_i\alpha_j(\sigma(E_i).E_j)_{I'} \\ &= -\sum_{(i,j)\in[\Phi-I]^2} e_{i,j}\frac{(\sigma(E_i).E_1)_{I'}(E_j.E_1)_{I'}}{(E_1.E_1)_{I'}} + \\ &\quad + \left(\sigma(\sum_{i\in[\Phi-I]} \alpha_i E_i). \sum_{j\in[\Phi-I]} \alpha_j E_j\right)_{I'}. \end{aligned}$$

Using again $\sigma(E_1) = E_1$, we get that $\alpha_{\sigma(i)} = \alpha_i$ and $(\sigma(E_i).E_1)_{I'} = (E_i.E_1)_{I'}$. Hence,

$$\begin{aligned} l(\Phi, I, \sigma) - l(\Phi', I', \sigma) &= l(\Phi, I, \mathrm{id}) - l(\Phi', I', \mathrm{id}) \\ &= r(\Phi, I, \mathrm{id}) - r(\Phi', I', \mathrm{id}) \quad \text{(by Lemma 7.13)} \\ &= 0 \\ &= r(\Phi, I, \sigma) - r(\Phi', I', \sigma). \end{aligned}$$

2–(ii) If $E_1 \notin I$, put $I = I'$. First, $r(\Phi, I, \sigma) = r(\Phi', I', \sigma) - 1$ because E_1 is fixed by σ. Second, $[\Phi - I] = [\Phi' - I'] \sqcup \{E_1\}$, and for any $i, j \notin I \sqcup \{E_1\}$, $(E_i.E_j)_I = (E_i.E_j)_{I'}$. Therefore,

$$\begin{aligned} &l(\Phi, I, \sigma) - l(\Phi', I', \sigma) \\ &\quad = \sum_{i\in[\Phi'-I']} e_{1,i}(\sigma(E_1).E_i)_I + e_{i,1}(\sigma(E_i).E_1)_I + e_{1,1}(\sigma(E_1).E_1)_I + \\ &\qquad + \sum_{(i,j)\in[\Phi'-I']^2} \alpha_i\alpha_j(\sigma(E_i).E_j)_I \\ &\quad = \sum_{i\in[\Phi'-I']} e_{1,i}(\sigma(E_1).E_i)_I + e_{i,1}(\sigma(E_i).E_1)_I + e_{1,1}(\sigma(E_1).E_1)_I + \\ &\qquad + \left(\sigma(\sum_{i\in[\Phi'-I']} \alpha_i E_i). \sum_{j\in[\Phi'-I']} \alpha_j E_j\right)_I. \end{aligned}$$

https://doi.org/10.1023/A:1001822419774 Published online by Cambridge University Press

Again because $\sigma(E_1) = E_1$, we get

$$\begin{aligned} l(\Phi, I, \sigma) - l(\Phi', I', \sigma) &= l(\Phi, I, \mathrm{id}) - l(\Phi', I', \mathrm{id}) \\ &= r(\Phi, I, \mathrm{id}) - r(\Phi', I', \mathrm{id}) \quad \text{(by Lemma 7.13)} \\ &= 1 \\ &= r(\Phi, I, \sigma) - r(\Phi', I', \sigma). \end{aligned}$$

In both cases, we conclude that equation (50) holds for (Φ, I, σ) if and only if it holds for (Φ', I', σ). But Φ' has less vertices than Φ. Then, by induction $l(\Phi, I, \sigma) = r(\Phi, I, \sigma)$. □

7.3. THE WEAK PROJECTION FORMULAS

We prove the weak projection formulas announced in the introduction of this section.

LEMMA 7.14. *Let X be an arithmetic surface over S and σ be an S-automorphism of X. Denote $\Gamma \subset X \times_S X$ its graph and fix E and F two irreducible components of X_s. Then $(\Gamma.[E \times F])_{loc} = (\sigma(E).F)$.*

Proof. Consider the automorphism $\theta := (\mathrm{id} \times_S \sigma) : X \times_S X \to X \times_S X$. Then, $\theta(\Delta_X) = \Gamma$ and $\theta(E \times \sigma^{-1}(F)) = E \times F$. We deduce that

$$(\Gamma.[E \times F])_{loc} = (\Delta_X.[E \times \sigma^{-1}(F)])_{loc}.$$

Therefore, the lemma is equivalent to the relation $(\Delta_X.[E \times F])_{loc} = (E.F)$. As $E \times F$ is vertical, this relation is a consequence of remark 4.5. □

PROPOSITION 7.15. *The weak projection formula holds for any birational morphism of arithmetic surfaces.*

Proof. Let $f : X \to Y$ be a birational morphism between arithmetic surfaces over S, and let σ be a non-trivial S-automorphism of Y which can be lifted to an automorphism of X. Put $\Gamma = \Gamma_\sigma^X \subset X \times_S X$ and $\Gamma_\sigma^Y \subset Y \times_S Y$ the graphs of σ acting respectively, on X and Y. Define, as in the last section, the cycle $\mathbb{V} = \sum_{(i,j)} e_{i,j}[E_i \times E_j]$. The weak projection formula to be shown is $(\Gamma.[X \times_Y X])_{loc} = (\Delta_Y.\Gamma_\sigma^Y)_{loc}$. By Theorem 7.7 and Lemma 7.14, it is equivalent to:

$$\begin{aligned} (\Delta_Y.\Gamma_\sigma^Y)_{loc} &= (\Gamma.\Delta_X)_{loc} + (\Gamma.\mathbb{V})_{loc} \\ &= (\Delta_X.\Gamma)_{loc} + \sum_{(i,j)} e_{i,j}(\Gamma.[E_i \times E_j])_{loc} \\ &= (\Delta_X.\Gamma)_{loc} + \sum_{(i,j)} e_{i,j}(\sigma(E_i).E_j) \\ &= (\Delta_X.\Gamma)_{loc} + \mathrm{tr}(\sigma)|H_{et}^*(Y_s, \mathbb{Q}_l) - \mathrm{tr}(\sigma)|H_{et}^*(X_s, \mathbb{Q}_l). \end{aligned}$$

This is the key formula of Theorem 7.1. □

https://doi.org/10.1023/A:1001822419774 Published online by Cambridge University Press

We need to extend this result to the following situation. Let L be a finite Galois extension of K of degree n and Galois group G, B be the integral closure of R in L, and $T = \mathrm{Spec}(B)$. Let Y be an arithmetic surface over S and σ be a non–trivial S-automorphism of Y.

DEFINITION 7.16. A T-base change of (Y, σ) is an S-morphism $f : X \to Y$, where X is an arithmetic surface over T, such that:

(i) its generic fiber over T is isomorphic to $Y_K \times_K L$,
(ii) the canonical action of G over $Y_K \times_K L$ extends to an action on X (if such an action exists then it is unique),
(iii) there exists a T-automorphism of X lifting σ over Y.

Remark 7.17. By (i), there exists only one possibility to lift σ over the generic fibers to an L-automorphism. It is the L-automorphism $(\sigma \times \mathrm{id})$ of $Y_K \times_K L$. Therefore, the lifting of σ to a T-automorphism of X, if it exists, is unique (denoted by σ). It is clearly a good lifting of σ over Y (Definition 6.1).

Fix a T–base change $f : X \to Y$ and let $\mathbb{V}$ be the cycle of $X \times_Y X$ given by its irreducible components which are not of finite degree over Y. Write

$$\mathbb{V} = \sum_{(i,j)} e_{i,j}[E_i \times E_j],$$

where the E_i are the irreducible components of X_s which collapse to a point in Y, and the sum is taken over all pairs of such components which collapse to the same point.

LEMMA 7.18. *With the above notation, we have*

$$\sum_{(i,j)} e_{i,j}(\sigma(E_i).E_j) = n\mathrm{tr}(\sigma)|H^*_{et}(Y_s, \mathbb{Q}_l) - \sum_{\tau\in G} \mathrm{tr}(\sigma\tau)|H^*_{et}(X_s, \mathbb{Q}_l).$$

Proof. Let Y' be the quotient of X by G. It is a normal surface over S, birational to Y. Moreover, σ descends to Y'. Indeed, for any $\tau \in G$, $\sigma\tau = \tau\sigma$ over X. The reason is that $\tau^{-1}\sigma\tau$ is a T-automorphism of X which lifts σ over Y, by Remark 7.17, it is equal to σ. We factor $f : X \to Y$ into the quotient map $\pi : X \to Y'$ followed by $Y' \to Y$.

Let $\mathbb{V}'$ be the cycle of $Y' \times_Y Y'$ given by all its irreducible components except the diagonal. Write

$$\mathbb{V}' = \sum_{(h,l)} a_{h,l}[F_h \times F_l],$$

where the F_h are the irreducible components of Y'_s which collapse to a point in Y, and the sum is taken over such components which collapse to the same point in Y. I claim that

$$\mathbb{V} = \sum_{(i,j)} e_{i,j}[E_i \times E_j] = \sum_{(h,l)} a_{h,l}[\pi^* F_h \times \pi^* F_l] \in A_2(X \times_Y X), \tag{55}$$

where $\pi^* F_h$ is the cycle of X defined in Appendix A. Notice that the above relation holds on the cycle level because $X \times_Y X$ has dimension 2.

Proof. Choose V and W two arithmetic surfaces which fit in the following diagram

$$\begin{tikzcd}
W \arrow[r, "\kappa"] \arrow[d, "g"'] & X \arrow[d, "\pi"] \arrow[dr, "f"] & \\
V \arrow[r, "\rho"'] & Y' \arrow[r] & Y
\end{tikzcd}$$

such that ρ and κ are birational, the action of G over X extends to an action over W, and this action satisfies $g\tau = g$ for any $\tau \in G$. Let F_h' be the strict transform of F_h in V. In the proof of Lemma 7.9, we have seen that

$$\mathbb{V}(V/Y) = \sum_{(h,l)} a_{h,l}[F_h' \times F_l'] + D_1 \in Z_2(V \times_Y V).$$

The cycle D_1 is a sum of cycles $[G_1 \times G_2]$ where G_1 and G_2 are irreducible curves over V such that at least one of them collapse to a point in Y'. Using the definition of Appendix A, the above relation implies:

$$\mathbb{V}(V/Y) = \sum_{(h,l)} a_{h,l}[\rho^* F_h \times \rho^* F_l] + D_2 \in Z_2(V \times_Y V), \tag{56}$$

where the cycle D_2 has the same property as D_1. The morphisms g and $g \times g$ are l.c.i.. They induce refined Gysin maps g^* and $(g \times g)^*$, and we have:

$$\begin{aligned}
[W \times_Y W] &= [W \times_V W] + (g \times g)^* \mathbb{V}(V/Y) \\
&= [W \times_V W] + \sum_{(h,l)} a_{h,l}[g^* \rho^* F_h \times g^* \rho^* F_l] \\
&\quad + (g \times g)^* D_2 \in A_2(W \times_Y W).
\end{aligned}$$

The first equation follows from the reduction step in the previous Subsection. The second equation follows from (56) and the same reduction step. Let Γ_τ^W be the graph of an automorphism $\tau \in G$. By subtracting $\sum_{\tau \in G}[\Gamma_\tau^W]$ from the previous equation, we find

$$\mathbb{V}(W/Y) = \mathbb{V}(W/V) + \sum_{(h,l)} a_{h,l}[g^* \rho^* F_h \times g^* \rho^* F_l] + (g \times g)^* D_2 \in A_2(W \times_Y W),$$

where the cycles $\mathbb{V}(-/-)$ are defined in the obvious way. Taking the push-forward by

$(\kappa \times \kappa)$, we get

$$\begin{aligned}\mathbb{V}(X/Y) &= (\kappa \times \kappa)_* \mathbb{V}(W/Y) \\ &= \sum_{(h,l)} a_{h,l}[\kappa_* g^* \rho^* F_h \times \kappa_* g^* \rho^* F_l] + \\ &\quad + (\kappa \times \kappa)_* (\mathbb{V}(W/V) + (g \times g)^* D_2) \in A_2(X \times_Y X).\end{aligned}$$

The cycles $\mathbb{V}(W/V)$ and $(g \times g)^* D_2$ are sums of cycles $[C_1 \times C_2]$, where C_1 and C_2 are curves over W such that at least one of them (for instance C_1) collapses to a point in Y'. As π is finite, C_1 collapses to a point in X. We deduce that

$$(\kappa \times \kappa)_* \mathbb{V}(W/V) = (\kappa \times \kappa)_* (g \times g)^* D_2 = 0 \in A_2(X \times_Y X).$$

The claim is now proved because $\pi^* = \kappa_* g^* \rho^*$. □

We deduce from (55) that

$$\sum_{(i,j)} e_{i,j}(\sigma(E_i).E_j) = \sum_{(h,l)} a_{h,l}(\pi^* \sigma(F_h).\pi^*(F_l)).$$

Using relation (8) of Appendix A, we find that

$$\sum_{(i,j)} e_{i,j}(\sigma(E_i).E_j) = n \sum_{(h,l)} a_{h,l}(\sigma(F_h).F_l).$$

The canonical morphism $X_{s,\mathrm{red}}/G \to Y'_{s,\mathrm{red}}$ is purely inseparable. So, $H^*_{et}(Y'_s, \mathbb{Q}_l) = H^*_{et}(X_s, \mathbb{Q}_l)^G$. We deduce that

$$n\mathrm{tr}(\sigma)|H^*_{et}(Y'_s, \mathbb{Q}_l) = \sum_{\tau \in G} \mathrm{tr}(\sigma\tau)|H^*_{et}(X_s, \mathbb{Q}_l).$$

By the last two relations, Lemma 7.18 is reduced to the following equation:

$$\sum_{(h,l)} a_{h,l}(\sigma(F_h).F_l) = \mathrm{tr}(\sigma)|H^*_{et}(Y_s, \mathbb{Q}_l) - \mathrm{tr}(\sigma)|H^*_{et}(Y'_s, \mathbb{Q}_l).$$

This is the statement of Theorem 7.7. Thus, Lemma 7.18 is proved. □

LEMMA 7.19. *Let $f : X \to Y$ be a T-base change. The weak projection formula for f is equivalent to the following relation:*

$$n(\Delta_Y.\Gamma^Y_\sigma)_{loc} = \sum_{\tau \in G} (\Delta_X.\Gamma^X_{\sigma\tau})_{loc} - \sum_{\tau \in G} \mathrm{tr}(\sigma\tau)|H^*_{et}(X_s, \mathbb{Q}_l) + n\mathrm{tr}(\sigma)|H^*_{et}(Y_s, \mathbb{Q}_l), \tag{57}$$

where $\Gamma^Y_\sigma \subset Y \times_S Y$ and $\Gamma^X_{\sigma\tau} \subset X \times_S X$ denote the graphs of, respectively, σ over Y and $\sigma\tau$ over X.

Proof. The weak projection formula for f is $n(\Delta_Y.\Gamma^Y_\sigma)_{loc} = (\Gamma^X_\sigma.[X \times_Y X])_{loc}$. Put $\Gamma_\sigma = \Gamma^X_\sigma$ and $\Gamma_\tau = \Gamma^X_\tau$. From the definitions, the relation $[X \times_Y X] = \sum_{\tau \in G}[\Gamma_\tau]$

https://doi.org/10.1023/A:1001822419774 Published online by Cambridge University Press

$+\mathbb{V}$ holds between cycles over $X \times_S X$. Then

$$(\Gamma_\sigma.[X \times_Y X])_{loc} = \sum_{\tau \in G} (\Gamma_\sigma.\Gamma_\tau)_{loc} + (\Gamma_\sigma.\mathbb{V})_{loc} = \sum_{\tau \in G} (\Delta_X.\Gamma_{\sigma\tau})_{loc} + (\Gamma_\sigma.\mathbb{V})_{loc}.$$

To prove the last equation, consider the automorphism

$$\begin{array}{rcl} \theta : X \times_S X & \to & X \times_S X \\ (x, y) & \mapsto & (y, \sigma(x)) . \end{array}$$

It maps Δ_X to Γ_σ and $\Gamma_{\sigma\tau^{-1}}$ to Γ_τ. Indeed, $\tau\sigma\tau^{-1} = \sigma$ by Remark 7.17. Therefore, $(\Gamma_\sigma.\Gamma_\tau)_{loc} = (\Delta_X.\Gamma_{\sigma\tau^{-1}})_{loc}$, which implies the needed equation. Finally, by Lemmas 7.14 and 7.18,

$$(\Gamma_\sigma.\mathbb{V})_{loc} = \sum_{(i,j)} e_{i,j}(\sigma(E_i).E_j) = n\mathrm{tr}(\sigma)|H^*_{et}(Y_s, \mathbb{Q}_l) - \sum_{\tau \in G} \mathrm{tr}(\sigma\tau)|H^*_{et}(X_s, \mathbb{Q}_l).$$

Lemma 7.19 is proved. □

PROPOSITION 7.20. *The weak projection formula* (21) *, or equivalently equation* (57)*, holds for any base change.*

Proof. Let $f : X \to Y$ be a T-base change. Assume first that we can find $g : X' \to X$ a birational map of arithmetic surfaces such that:

(i) the automorphism σ and the action of G over X can be lifted to X'. Hence, $f \circ g : X' \to Y$ is a T-base change.
(ii) the projection formula holds for $f \circ g : X' \to Y$.

By (ii) and Lemma 7.19,

$$n(\Delta_Y.\Gamma^Y_\sigma)_{loc} = \sum_{\tau \in G} (\Delta_{X'}.\Gamma^{X'}_{\sigma\tau})_{loc} - \sum_{\tau \in G} \mathrm{tr}(\sigma\tau)|H^*_{et}(X'_s, \mathbb{Q}_l) + n\mathrm{tr}(\sigma)|H^*_{et}(Y_s, \mathbb{Q}_l).$$

By Theorem 7.1,

$$(\Delta_{X'}.\Gamma^{X'}_{\sigma\tau})_{loc} - \mathrm{tr}(\sigma\tau)|H^*_{et}(X'_s, \mathbb{Q}_l) = (\Delta_X.\Gamma^X_{\sigma\tau})_{loc} - \mathrm{tr}(\sigma\tau)|H^*_{et}(X_s, \mathbb{Q}_l).$$

Hence, we get

$$n(\Delta_Y.\Gamma^Y_\sigma)_{loc} = \sum_{\tau \in G} (\Delta_X.\Gamma^X_{\sigma\tau})_{loc} - \sum_{\tau \in G} \mathrm{tr}(\sigma\tau)|H^*_{et}(X_s, \mathbb{Q}_l) + n\mathrm{tr}(\sigma)|H^*_{et}(Y_s, \mathbb{Q}_l),$$

which is relation (57) for the morphism f.

In order to prove the existence of X' as above, we use Theorem 6.3. By remark 7.17, the hypotheses of this theorem, except the hypothesis on the dimension of $W = \Gamma \cap [X \times_Y X]$, are satisfied. I claim that the latter can be satisfied after a sequence of blow-ups. Indeed, W is isomorphic to the inverse image of $\mathrm{fix}(\sigma) \subset Y$ in X. If it is not a Cartier divisor, we blow-up X at closed points until this condition will be satisfied. We can do these blow-ups in such a way that σ

https://doi.org/10.1023/A:1001822419774 Published online by Cambridge University Press

and the action of G lift to the final step X'. Then, the scheme W' associated with X' has dimension 1. $\square$

8. Lefschetz Formula: The Semi–Stable Case

Let X be a semi-stable arithmetic surface over S and σ be a non-trivial S-automorphism of X. Denote by \$ the set of singular points in X_s. Let $\Gamma \subset X \times_S X$ be the graph of σ, $\mathrm{fix}(\sigma)$ be the scheme of σ-fixed points, and $\mathcal{I}$ be its ideal sheaf in Γ. Let Y be the Cartier divisor on X defined locally by the greatest common divisor of all functions in the ideal $\mathcal{I}$, and let F be the residual scheme to Y in $\mathrm{fix}(\sigma)$. We decompose Y as $D + Z$, where D is a horizontal Cartier divisor over S and Z is a vertical one.

By Lemma 5.6, any closed point x in F is singular in X_s. The following lemma describes the points of F among the singular points in X_s.

LEMMA 8.1. *Let x be a closed fixed point of X that is singular in X_s.*

(i) *If σ stabilizes the branches of X_s through x, then $x \in F$ and its multiplicity in F is* 1.
(ii) *If σ switches the branches through x, then $x \notin F$ and x is contained in the horizontal Cartier divisor D.*

Proof. The completion of the local ring of X at x is isomorphic to $A = R[[t, \epsilon]]/(t\epsilon - \pi)$, where π denotes a uniformizing element of R. Let $\mathfrak{m}$ be its maximal ideal. The branches of X_s through x are defined by t and ϵ. The automorphism σ is given over A by

$$\sigma(t) = ut + v\epsilon, \qquad \sigma(\epsilon) = u't + v'\epsilon.$$

Using $\sigma(t\epsilon) = \sigma(\pi) = \pi$, one finds $uu't^2 + vv'\epsilon^2 + (uv' + vu')\pi = \pi$. Hence $uu' \in \epsilon A$ and $vv' \in tA$.

- If u is a unit in A, then $u' \in \epsilon A$. As $(uv' - vu') \in A^*$, then $v' \in A^*$. Therefore, $v \in tA$. We conclude that σ is given by

  $$\sigma(t) = \alpha t, \qquad \sigma(\epsilon) = \beta\epsilon,$$

  where α and β are in A^* with $\alpha\beta = 1$ and $\alpha \neq 1$. In this case, σ stabilizes the branches.
- If $u \notin A^*$, then $v \in A^*$ because $(uv' - vu') \in A^*$. Therefore, $v' \in tA^*$. Again this implies that $u' \in A^*$, hence $u \in \epsilon A$. We conclude that σ is given by

  $$\sigma(t) = \alpha\epsilon, \qquad \sigma(\epsilon) = \beta t,$$

  where α and β are in A^* with $\alpha\beta = 1$. In this case σ switches the branches.

(i) In the first case, $I_x = ((\alpha - 1)t, (\beta - 1)\epsilon) = (\alpha - 1)\mathfrak{m}$. So, the Cartier divisor Y is locally defined at x by $\alpha - 1$, and F by $\mathfrak{m}$. Therefore, $x \in F$ and $l(x) = 1$.

https://doi.org/10.1023/A:1001822419774 Published online by Cambridge University Press

(ii) In the second case, $I_x = (\alpha\epsilon - t)$. So, the Cartier divisor Y is defined by $\alpha\epsilon - t$, and $x \notin F$. The equation $\alpha\epsilon - t$ defines a horizontal divisor. $\square$

For a singular point x in X_s, we put

$$l_x = \begin{cases} 1 & \text{if } \sigma \text{ fixes } x \text{ and stabilizes the branches through it,} \\ -1 & \text{if } \sigma \text{ fixes x and switches the branches through it,} \\ 0 & \text{otherwise.} \end{cases} \tag{58}$$

Lemma 8.1 implies the following relation:

$$\sum_{x\in\$} l_x = \sum_{x\in F} 1 - \sum_{x\in D_s\cap(\$-F)} 1. \tag{59}$$

Indeed, by Lemma 8.1, the set of fixed points of $\$$ is a disjoint union of F and $D_s \cap (\$ - F)$. The first subset corresponds to points with $l_x = 1$, and the second to points with $l_x = -1$.

PROPOSITION 8.2. *With these notation,* $(\Delta_X.\Gamma)_{loc} = -(\omega + D + Z.Z) + \sum_{x\in\$} l_x$.

Proof. This Proposition is a consequence of the residual intersection formula and equation (59). We need first to prove a slightly different version of the residual formula given in Proposition 5.8. Let $\widetilde{\Gamma}$ be the blow-up of Γ along F. As any point of F has multiplicity 1, $\widetilde{\Gamma}$ can also be obtained from Γ by blowing-up successively the points of F. In particular, $\widetilde{\Gamma}$ is regular. For any $x \in F$, let E_x be the inverse image of x. Let $W = \mathrm{fix}(\sigma)$ be the scheme of fixed points, and $\widetilde{W}$ be its inverse image in $\widetilde{\Gamma}$.

$$\begin{tikzcd} \widetilde{W} \arrow[r] \arrow[d] \arrow[dd, bend right, "\tilde{g}"'] & \widetilde{\Gamma} \arrow[d, "\rho"] \\ W \arrow[r] \arrow[d, "g"] & \Gamma \arrow[d] \\ \Delta_X \arrow[r] & X \times_S X \end{tikzcd} \tag{60}$$

Denote by $\widetilde{D}$, $\widetilde{Z}$ and $\widetilde{F}$ the inverse images in $\widetilde{\Gamma}$ of D , Z and F. Then, the relations $\widetilde{W} = \widetilde{D} + \widetilde{Z} + \widetilde{F}$ and $\widetilde{F} = \sum_{x\in F} E_x$ hold between Cartier divisors over $\widetilde{\Gamma}$. As ρ is birational, $\deg(\Delta_X.\Gamma)_{loc} = \deg(\Delta_X.\widetilde{\Gamma})_{loc}$. Let $\mathcal{E}.$ be a resolution of $\Omega^1_{X/S}$ as in Section 5.1-1). Diagram (60) induces a surjection of complexes $\tilde{g}^*(\mathcal{E}.) \to \mathcal{O}_X(-\widetilde{W})|_{\widetilde{W}}$ where the second complex is concentrated in degree 0. Let $\mathcal{F}.$ be the kernel of this surjection. The localized excess formula (Theorem 4.7) implies:

$$(\Delta_X.\widetilde{\Gamma}_\sigma)_{loc} = -c^{\widetilde{W}}_{1,\widetilde{W}_s}(\mathcal{F}.) \cap [\widetilde{W}] = -c^{\widetilde{W}}_{1,\widetilde{W}_s}(\mathcal{F}.) \cap [\widetilde{D}] - c^{\widetilde{W}}_{1,\widetilde{W}_s}(\mathcal{F}.) \cap [\widetilde{Z} + \widetilde{F}].$$

https://doi.org/10.1023/A:1001822419774 Published online by Cambridge University Press

As $\widetilde{Z}+\widetilde{F}$ is vertical,

$$\begin{aligned}
c^{\widetilde{W}}_{1,\widetilde{W}_s}(\mathcal{F}.)\cap[\widetilde{Z}+\widetilde{F}] &= c_1(\mathcal{F}.)\cap[\widetilde{Z}+\widetilde{F}]\\
&= c_1(\tilde{g}^*\mathcal{E}.)\cap[\widetilde{Z}+\widetilde{F}]+c_1(\mathcal{O}(\widetilde{W}))\cap[\widetilde{Z}+\widetilde{F}],\\
\deg c^{\widetilde{W}}_{1,\widetilde{W}_s}(\mathcal{F}.)\cap[\widetilde{Z}+\widetilde{F}] &= (\omega+Z+D.Z)-\sum_{x\in F}1.
\end{aligned}$$

We compute now the contribution of $\widetilde{D}$. Let $\overline{D}$ be the strict transform of D in $\widetilde{\Gamma}$ (i.e. the blow-up of D along $F\cap D$). Then, $\widetilde{D}=\overline{D}+\sum_{x\in F}\mu_x(D)[E_x]$ as Cartier divisors, where $\mu_x(D)$ is the multiplicity of D at x. Therefore,

$$\begin{aligned}
c^{\widetilde{W}}_{1,\widetilde{W}_s}(\mathcal{F}.)\cap[\widetilde{D}] &= c^{\overline{D}}_{1,\overline{D}_s}(\mathcal{F}.)\cap[\overline{D}]+c_1(\mathcal{F}.)\cap(\sum_{x\in F}\mu_x(D)[E_x])\\
&= c^{\overline{D}}_{1,\overline{D}_s}(\mathcal{F}.)\cap[\overline{D}]-\sum_{x\in F}\mu_x(D).
\end{aligned}$$

The divisor $\overline{D}$ is finite and flat over R. Let $i:\overline{D}\to\widetilde{W}$ be its inclusion in $\widetilde{W}$, and consider the exact sequence:

$$0\longrightarrow i^*\mathcal{F}.\longrightarrow i^*\tilde{g}^*(\mathcal{E}.)\xrightarrow{\gamma}\mathcal{O}(-\widetilde{W})|_{\overline{D}}\longrightarrow 0.$$

The surjective map γ is a quasi-isomorphism on the generic fiber of $\overline{D}$. Therefore, it defines a rational map $t:i^*\tilde{g}^*\omega_{X/S}\dashrightarrow\mathcal{O}(-\widetilde{W})|_{\overline{D}}$. Corollary 3.7. implies:

$$\deg c^{\overline{D}}_{1,\overline{D}_s}(\mathcal{F}.)\cap[\overline{D}]=-\mathrm{ord}(t).$$

LEMMA 8.3. $\mathrm{ord}(t)=-\sum_{x\in F}\mu_x(D)-\sum_{x\in D_s\cap(\$-F)}1.$

Proof. Let $h:\overline{D}\to D$ be the blow-up of D along $D\cap F$, and consider the diagram:

$$\begin{tikzcd}
i^*\tilde{g}^*\Omega^1_{X/S} \arrow[r, "f"] \arrow[dr, "\alpha"'] & i^*\tilde{g}^*(\omega_{X/S}) \arrow[r] \arrow[d, dashed, "t"] & h^*(\bigoplus_{x\in\$\cap D_s}k) \arrow[r] & 0\\
& \mathcal{O}(-\widetilde{W})|_{\overline{D}} & &
\end{tikzcd} \tag{61}$$

where the top sequence is obtained by taking the h-pull-back of the exact sequence (14) and α is the surjective map induced by diagram (60). The kernel of f is R-torsion. Hence, it maps to zero by α because $\overline{D}$ is flat over R. Therefore, α factors through the image C of f in $i^*\tilde{g}^*\omega_{X/S}$. Denote by $\beta:C\to\mathcal{O}(-\widetilde{W})|_{\overline{D}}$ the induced map. First, β is surjective because α is surjective. Second, β is an isomorphism over the generic fiber of $\overline{D}$. Then, its kernel is R-torsion. But C is R-torsion-free as a sub–bundle of

https://doi.org/10.1023/A:1001822419774 Published online by Cambridge University Press

$i^*\tilde{g}^*\omega_{X/S}$. Then, β is injective, and hence bijective. We have now the diagram

$$\begin{tikzcd} 0 \arrow[r] & C \arrow[r] \arrow[dr, "\beta"'] & i^*\tilde{g}^*\omega_{X/S} \arrow[r] \arrow[d, dotted, "t"] & h^*\left(\oplus_{x\in(\$\cap D_s)}k\right) \arrow[r] & 0 \\ & & \mathcal{O}(-\widetilde{W})|_{\overline{D}} & & \end{tikzcd}$$

from which we deduce that

$$\mathrm{ord}(t) = -\dim_k h^*(\bigoplus_{x\in\$\cap D_s} k) = -\sum_{x\in F}\mu_x(D) - \sum_{x\in D_s\cap(\$-F)} 1. \qquad \square$$

We proved the relation:

$$\deg(\Delta_X.\widetilde{\Gamma})_{loc} = -(\omega + D + Z.Z) + \sum_{x\in F} 1 - \sum_{x\in D_s\cap(\$-F)} 1.$$

Proposition 8.2 follows from the above equation and Equation (59). $\square$

PROPOSITION 8.4. *With the above notation,* $(\omega + D + Z.Z) = 0$.

Proof. Let C be an irreducible reduced component of X_s which appears in Z with multiplicity $i \geqslant 1$. Denote by $\mathcal{J}$ and $\mathcal{I}$ the ideal sheaves of, respectively, C and fix(σ) in Γ ($\mathcal{I} \subset \mathcal{J}^i$). For any integer $n \geqslant 0$, $(\sigma - 1)$ induces a map $\mathcal{O}_{nC} \to \mathcal{O}_{nC}$.

Claim 1. $(\sigma - 1) : \mathcal{O}_{nC} \to \mathcal{O}_{nC}$ vanishes if and only if $n \leqslant i$.

Indeed $(\sigma - 1)(\mathcal{O}_X) = \mathcal{I}$ is contained in $\mathcal{J}^i$ but not in $\mathcal{J}^{i+1}$.

Consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{J}^i/\mathcal{J}^{i+1} \arrow[r] \arrow[d, "\sigma-1"'] & \mathcal{O}_{(i+1)C} \arrow[r] \arrow[d, "\sigma-1"'] & \mathcal{O}_{iC} \arrow[r] \arrow[d, "\sigma-1"'] & 0 \\ 0 \arrow[r] & \mathcal{J}^i/\mathcal{J}^{i+1} \arrow[r] & \mathcal{O}_{(i+1)C} \arrow[r] & \mathcal{O}_{iC} \arrow[r] & 0 \end{tikzcd}$$

Claim 1 implies that the composed map $\mathcal{O}_{(i+1)C} \to \mathcal{O}_{iC}$ vanishes. Then, we get a map

$$(\sigma - 1) : \mathcal{O}_{(i+1)C} \to \mathcal{J}^i/\mathcal{J}^{i+1} = \mathcal{O}_C(-iC).$$

Claim 2. $(\sigma - 1)$ vanishes on $\mathcal{J}/\mathcal{J}^{i+1}$.

This is a local question. Let y be a point of C and let A be the completion of the local ring of X at y. Let t be an equation defining the Cartier divisor C. We should prove that $\sigma(t) - t \in t^{i+1}A_t$, where A_t is the localization of A at tA. We can write $\pi = vt$ where v is a t-unit in A. Hence, by (18):

$$0 = \sigma(vt) - vt = \sigma(v)(\sigma(t) - t) + t(\sigma(v) - v),$$

and the fact that $\sigma(v) - v \in \mathcal{I} \subset \mathcal{J}^i$, we find that $\sigma(t) - t \in t^{i+1}A_t$. The claim is proved.

By Claim 2, we get a map $D_{\sigma,C} : \mathcal{O}_C \to \mathcal{O}_C(-iC)$.

Claim 3. $D_{\sigma,C}$ is a k-derivation.

It follows from the formula $\sigma(ab) - ab = \sigma(a)(\sigma(b) - b) + b(\sigma(a) - a)$ if we remark that $\sigma(a) = a$ modulo $\mathcal{I} \subset \mathcal{J}$.

We denote also by $D_{\sigma,C}$ the $\mathcal{O}_C$-linear map $\Omega^1_{C/k} \to \mathcal{O}_C(-iC)$ induced by $D_{\sigma,C}$.

LEMMA 8.5. *The cokernel κ of $D_{\sigma,C} : \Omega^1_{C/k} \to \mathcal{O}_C(-iC)$ is supported over a finite set of points in C. Its total length is*

$$\mathrm{Leng}_{\mathcal{O}_C}(\kappa) = (C.D) + (C.Z - iC) - (C.C) + \delta,$$

where δ is the number of nodes of C, and $(C.*)$ is the intersection number with C.

Proof. Let y be a closed point of C and let A be the completion of $\mathcal{O}_{X,y}$. We denote κ_y the stalk of κ at y.

If y is smooth in X_s, then $A = R[[t]]$, and C is defined in A by π a uniformizing element of R. The ideal sheaf of $\mathrm{fix}(\sigma)$ is generated by $\sigma(t) - t$. Put $\sigma(t) - t = \pi^i f$, where f is a local equation defining the horizontal Cartier divisor D, and i the multiplicity of C in Z. Therefore

$$\mathrm{Leng}_{\mathcal{O}_C}(\kappa_y) = \mathrm{Leng}_A[A/((\sigma(t) - t)/\pi^i, \pi)] = \mathrm{Leng}_A[A/(f, \pi)] = (C.D)_y,$$

where $(C.D)_y$ is the local intersection of C and D at y.

If y is singular in X_s, then $A = R[[t, \epsilon]]/(t\epsilon - \pi)$. Let $\mathfrak{m}$ be its maximal ideal. We have two cases:

(1) y is a smooth point of C. Then C is defined by the equation $t = 0$. The equation $\epsilon = 0$ defines C', the other component of X_s through x. As C is fixed by σ, C' is stabilized by σ, and we have $\sigma(t) = \alpha t$, $\sigma(\epsilon) = \beta\epsilon$ and $\alpha\beta = 1$ (see the proof of Lemma 8.1). The ideal defining Y is generated by $\alpha - 1$. Write $\alpha - 1 = t^i \epsilon^j f$, where i is the multiplicity of C, and j is the multiplicity of C' in Y, and f is an equation defining D. On the other hand, ϵ gives a local parameter of C at y. Therefore,

$$\begin{aligned}\mathrm{Leng}_{\mathcal{O}_C}(\kappa_y) &= \mathrm{Leng}_A[A/((\sigma(\epsilon) - \epsilon)/t^i, t)] \\ &= \mathrm{Leng}_A[A/((\beta - 1)\epsilon/t^i, t)] \\ &= \mathrm{Leng}_A[A/(\epsilon^{j+1} f, t)] = (j + 1)(C.C')_y + (C.D)_y.\end{aligned}$$

(2) y is a singular point in C (a node). Then C is defined in A by the equation $t\epsilon = 0$. As σ fixes the branches through x , we have the same description of the action of σ on A, and the Cartier divisor Y is defined by $\alpha - 1$. Write $\alpha - 1 = t^i \epsilon^i f$ where f defines D.

https://doi.org/10.1023/A:1001822419774 Published online by Cambridge University Press

Therefore,

$$\begin{aligned}\mathrm{Leng}_{\mathcal{O}_C}(\kappa_y) &= \mathrm{Leng}_A[A/((\sigma(t)-t)/t^i\epsilon^i, (\sigma(\epsilon)-\epsilon)/t^i\epsilon^i, t\epsilon)] \\ &= \mathrm{Leng}_A[A/(tf, \epsilon f, \epsilon t)] \\ &= \mathrm{Leng}_A[A/(f\mathfrak{m}, \epsilon t)] \\ &= \mathrm{Leng}_A[A/(f, t\epsilon)] + \mathrm{Leng}[A/(\mathfrak{m}, t\epsilon)] \quad \text{(by Lemma B.1)} \\ &= (C.D)_y + 1.\end{aligned}$$

The Lemma follows by adding all these contributions. □

LEMMA 8.6. $\mathrm{Leng}_{\mathcal{O}_C}(\kappa) = -i(C.C) - (2g_C - 2) + \delta$.

Proof. Let $\omega_{C/k}$ be the dualizing sheaf of C over k, and Sing be the set of singular points of C. We have an exact sequence

$$0 \longrightarrow \mathcal{N} \longrightarrow \Omega^1_{C/k} \xrightarrow{\rho} \omega_{C/k} \longrightarrow \bigoplus_{x\in \mathrm{Sing}} k \longrightarrow 0,$$

where $\mathcal{N}$ is torsion. Since $\mathcal{O}_C(-iC)$ is torsion-free, then $D_{\sigma,C}$ maps $\mathcal{N}$ to zero. Hence, it factors through the image $\mathcal{H}$ of ρ. Moreover, the kernel of the induced map $\mathcal{H} \to \mathcal{O}_C(-iC)$ is torsion, and $\mathcal{H}$ is torsion free (as a subsheaf of $\omega_{C/k}$). Therefore, we have an exact sequence $0 \to \mathcal{H} \to \mathcal{O}_C(-iC) \to \kappa \to 0$. The lemma follows by observing that $\deg_C \omega_{C/k} = 2g_C - 2$. □

Lemmas 8.5 and 8.6 imply that $(2g_C - 2) - (C.C) = -(C.Z + D)$. So $(\omega + D + Z.C) = 0$ by the adjunction formula. This finishes the proof of Proposition 8.4. □

COROLLARY 8.7. *Let X be a semi-stable arithmetic surface over S, and σ be a non-trivial S-automorphism of X. Then,*

$$(\Delta_X.\Gamma_\sigma)_{loc} = \mathrm{tr}(\sigma)|H^*_{et}(X_s, \mathbb{Q}_l) - \mathrm{tr}(\sigma)|H^*_{et}(X_{\overline{\eta}}, \mathbb{Q}_l).$$

Proof. Propositions 8.2 and 8.4 imply that $(\Delta_X.\Gamma_\sigma)_{loc} = \sum_{x\in\$} l_x$. Therefore, Corollary 8.7 is equivalent to the following relation:

$$\mathrm{tr}(\sigma)|H^*_{et}(X_s, \mathbb{Q}_l) - \mathrm{tr}(\sigma)|H^*_{et}(X_{\overline{\eta}}, \mathbb{Q}_l) = \sum_{x\in\$} l_x. \tag{62}$$

Let M be the $\mathbb{Q}_l$-vector space with generators $\theta_{1,x}$ and $\theta_{2,x}$ for $x \in \$$, and relations $\theta_{1,x} + \theta_{2,x} = 0$. The automorphism σ acts over M and $\mathrm{tr}(\sigma)|M = \sum_{x\in\$} l_x$. Let $\overline{S}$ be the spectrum of the integral closure of R in $\overline{K}$, $\overline{X} = X \times_S \overline{S}$, and $a : X_{\overline{\eta}} \to \overline{X}$.

https://doi.org/10.1023/A:1001822419774 Published online by Cambridge University Press

By [6], we have

$$H^m(\overline{X}, R^n a_* \mathbb{Q}_l) = \begin{cases} M & m = 0, n = 1, \\ H^m(X_s, \mathbb{Q}_l) & n = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{63}$$

Then, the Leray spectral sequence for a implies Equation (62). □

9. Base Change

By the semi-stable reduction theorem and the projection formula proved in Proposition 7.20, the proof of the Lefschetz fixed point formula in Theorem 1.1 is reduced to computing Lefschetz numbers on a semi-stable arithmetic surface. This is the aim of this section.

Let L be a finite Galois extension of K of degree n and Galois group $G(L/K)$. Let B be the integral closure of R in L. Put $T = \mathrm{Spec}(B)$, and denote t its closed point, g its generic point, and $\overline{g}$ a geometric generic point. We choose $\overline{g}$ such that the induced geometric point of S is $\overline{\eta}$. Let $d_{T/S}$ be the discriminant of T/S.

Let X be a semi-stable arithmetic surface over T, and σ be a non-trivial automorphism of X lifting an automorphism $\tau \in G(L/K)$ of T. In particular, σ is an S-automorphism of X. Then, one can form the fiber square

$$\begin{tikzcd} \mathrm{fix}(\sigma) \arrow[r] \arrow[d] & \Gamma_\sigma \arrow[d] \\ \Delta_X \arrow[r] & X \times_S X \end{tikzcd}$$

and compute the intersection product $(\Delta_X.\Gamma_\sigma)_{loc}$ relatively to this diagram.

THEOREM 9.1. *If* $\tau = 1$, *then*

$$(\Delta_X.\Gamma_\sigma)_{loc} = \mathrm{tr}(\sigma)|H^*_{et}(X_t, \mathbb{Q}_l) - (\mathrm{sw}_{L/K}(1) + n)\mathrm{tr}(\sigma)|H^*_{et}(X_{\overline{g}}, \mathbb{Q}_l). \tag{64}$$

If $\tau \neq 1$, *then*

$$(\Delta_X.\Gamma_\sigma)_{loc} = \mathrm{tr}(\sigma)|H^*_{et}(X_t, \mathbb{Q}_l) - \mathrm{sw}_{L/K}(\tau)\ \mathrm{tr}(\sigma)|H^*_{et}(X_{\overline{g}}, \mathbb{Q}_l). \tag{65}$$

Remark 9.2. The automorphism σ acts canonically over X_t and X_g, but it acts canonically over $X_{\overline{g}} = X \times_T \mathit{Spec}(\overline{K})$ only if $\tau = 1$. However, $\mathrm{tr}(\sigma)|H^*_{\mathrm{et}}(X_{\overline{g}}, \mathbb{Q}_l)$ is well defined for any τ as it will be explained at the end of the proof of formula (65).

Proof of formula (64). As $\tau = 1$, σ is a T–automorphism of X. Hence, one can compute the localized intersection product relatively to the fiber square:

$$\begin{tikzcd} \mathrm{fix}(\sigma) \arrow[r] \arrow[d] & \Gamma_\sigma \arrow[d] \\ \Delta_X \arrow[r] & X \times_T X \end{tikzcd}$$

https://doi.org/10.1023/A:1001822419774 Published online by Cambridge University Press

Denote this product (and its degree) by $(\Delta_X.\Gamma_\sigma)^T_{loc}$ in order to distinguish it from the previous one referred by $(\Delta_X.\Gamma_\sigma)^S_{loc}$. The first step compares these numbers.

LEMMA 9.3. *Let V be a scheme of pure dimension 2 with a map $h: V \to X \times_T X$. Then*

$$\deg(\Delta_X.[V])^S_{loc} - \deg(\Delta_X.[V])^T_{loc} = d_{T/S}\ (X_{\bar{g}}.V_{\bar{g}}),$$

where $(X_{\bar{g}}.V_{\bar{g}})$ is the geometric intersection number of $V_{\bar{g}}$ with $X_{\bar{g}}$ diagonally embedded in $X_{\bar{g}} \times X_{\bar{g}}$.

Proof. Let $f: X \to T$ be the structural map. We have an exact sequence ([3] corollary 1.2)

$$0 \to f^*\Omega^1_{T/S} \to \Omega^1_{X/S} \to \Omega^1_{X/T} \to 0. \tag{66}$$

Fix resolutions $\mathcal{F}.$, $\mathcal{E}.$ and $\mathcal{G}.$ of, respectively, $f^*(\Omega^1_{T/S})$, $\Omega^1_{X/S}$ and $\Omega^1_{X/T}$ by locally free $\mathcal{O}_X$-modules extending the exact sequence (66). Let W be the scheme given by the fiber square

$$\begin{tikzcd} W \arrow[r] \arrow[d, "g"'] & V \arrow[d, "h"] \\ \Delta_X \arrow[r] & X \times_T X \end{tikzcd}$$

Then, the diagram

$$\begin{tikzcd} W \arrow[r] \arrow[d, "g"'] & V \arrow[d, "h"] \\ \Delta_X \arrow[r] & X \times_S X \end{tikzcd}$$

is also Cartesian. Let $\mathbb{P} = \mathbb{P}(\mathcal{S}_W V[z])$ and $q: \mathbb{P} \to W$ be the canonical projection. Define the complexes $\mathcal{E}'.$ and $\mathcal{G}'.$ of locally free $\mathcal{O}_{\mathbb{P}}$-modules by

$$0 \to \mathcal{E}'_0 \to q^*g^*(\mathcal{E}_0 \oplus \mathcal{O}_X) \to \mathcal{O}_{\mathbb{P}}(1) \to 0,$$
$$0 \to \mathcal{G}'_0 \to q^*g^*(\mathcal{G}_0 \oplus \mathcal{O}_X) \to \mathcal{O}_{\mathbb{P}}(1) \to 0,$$

and $\mathcal{E}'_i = q^*g^*(\mathcal{E}_i)$ and $\mathcal{G}'_i = q^*g^*(\mathcal{G}_i)$ for $i > 0$. Then, by definition

$$(\Delta_X.V)^S_{loc} = q_{s*}(c^{\mathbb{P}}_{2,\mathbb{P}_t}(\mathcal{E}'.) \cap [\mathbb{P}]) \in A_0(W_s) = A_0(W_t),$$
$$(\Delta_X.V)^T_{loc} = q_{s*}(c^{\mathbb{P}}_{2,\mathbb{P}_t}(\mathcal{G}'.) \cap [\mathbb{P}]) \in A_0(W_t).$$

From the definition of $\mathcal{E}'.$, $\mathcal{G}'.$, the snake lemma and the exact sequence (66), we get the exact sequence

$$0 \to q^*g^*(\mathcal{F}.) \to \mathcal{E}'. \to \mathcal{G}'. \to 0,$$

from which we deduce:

$$c^{\mathbb{P}}_{2,\mathbb{P}_t}(\mathcal{E}'.) = c^{\mathbb{P}}_{2,\mathbb{P}_t}(\mathcal{G}'.) + c^{\mathbb{P}}_{2,\mathbb{P}_t}(q^*g^*\mathcal{F}.) + c_1(\mathcal{G}'.)c^{\mathbb{P}}_{1,\mathbb{P}_t}(q^*g^*\mathcal{F}.).$$

Therefore, by q-push-forward,

$$(\Delta_X.V)^S_{loc} = (\Delta_X.V)^T_{loc} + c^W_{2,W_t}(g^*\mathcal{F}.) \cap q_*([\mathbb{P}]) + c^W_{1,W_t}(g^*\mathcal{F}.) \cap q_*(c_1(\mathcal{G}'.) \cap [\mathbb{P}]).$$

Hence,

$$\begin{aligned}\deg(\Delta_X.V)^S_{loc} = \deg(\Delta_X.V)^T_{loc} + \deg c^X_{2,X_t}(f^*\Omega^1_{T/S}) \cap g_*q_*([\mathbb{P}]) +\\ + \deg c^X_{1,X_t}(f^*\Omega^1_{T/S}) \cap g_*q_*(c_1(\mathcal{G}'.) \cap [\mathbb{P}]).\end{aligned}$$

On the other hand, as f is flat, we have:

$$f_{s*}(c^X_{2,X_t}(f^*\Omega^1_{T/S}) \cap g_*q_*[\mathbb{P}]) = c^T_{2,T_t}(\Omega^1_{T/S}) \cap f_*g_*q_*[\mathbb{P}] = 0 \in A_0(t),$$

$$\begin{aligned}&f_{s*}(c^X_{1,X_t}(f^*\Omega^1_{T/S}) \cap g_*q_*(c_1(\mathcal{G}'.) \cap [\mathbb{P}]))\\ &\quad = c^T_{1,T_t}(\Omega^1_{T/S}) \cap f_*g_*q_*(c_1(\mathcal{G}'.) \cap [\mathbb{P}]) \in A_0(t).\end{aligned}$$

Write $f_*g_*q_*(c_1(\mathcal{G}'.) \cap [\mathbb{P}]) = \alpha[T] \in A_1(T)$. Hence,

$$\deg(\Delta_X.V)^S_{loc} - \deg(\Delta_X.V)^T_{loc} = d_{T/S}\ \alpha.$$

In order to compute α, we work over the geometric generic fibers. Denote by an over-line the geometric generic fibers. Then, $\overline{f}_*\overline{g}_*\overline{q}_*(c_1(\overline{\mathcal{G}}.) \cap [\mathbb{P}]) = \alpha[\overline{g}] \in Z_0(\overline{g})$. Over $\overline{\mathbb{P}}$, $\mathcal{G}'.$ is a resolution of the locally free sheaf ξ' given by the exact sequence

$$0 \to \xi' \to \overline{q}^*(\Omega^1_{\overline{X}/\overline{K}} \oplus \mathcal{O}_{\overline{X}}) \to \mathcal{O}_{\overline{\mathbb{P}}}(1) \to 0.$$

Furthermore, the usual intersection theory [7] for the regular embedding $\overline{X} \to \overline{X} \times \overline{X}$ implies that

$$\overline{q}_*(c_1(\xi') \cap [\overline{\mathbb{P}}]) = (\overline{X}.\overline{V}) \in A_0(\overline{W}).$$

This finishes the proof. □

Formula (64) follows from Lemma 9.3, Corollary 8.7 and the relation $1 + d_{T/S} = n + sw_{L/K}(1)$.

Proof of formula (65). We assume now that $\tau \neq 1$. Denote \$ the set of singular points in X_t. Decompose fix(σ) into $Y \cup F$, where Y is a Cartier divisor over Γ_σ defined locally by the greatest common divisor of all functions in the ideal of fix(σ), and F is the residual scheme to Y in fix(σ). The main difference with Section 8 is that Y is a vertical divisor. Indeed, the ideal sheaf of fix(σ) in Γ_σ contains $\sigma(\pi) - \pi = \tau(\pi) - \pi = \gamma\pi^j$, where π is a uniformizing element of B, $j = j(\tau) = v(\tau(\pi) - \pi)$, and $\gamma \in B^*$ (see Section 2). Therefore, fix(σ) is a scheme over Spec(B/π^j).

Lemma 5.6 implies that F is regularly embedded in Γ_σ. For any closed point x of X, let $l(x)$ be the multiplicity of F at x. It is also the algebraic multiplicity of Γ_σ

https://doi.org/10.1023/A:1001822419774 Published online by Cambridge University Press

along F at x ([7] example 4.3.5–c)). We will compute the multiplicity $l(x)$ of any closed point of X fixed by σ.

LEMMA 9.4. *All components of Y occur with multiplicity $i \leqslant j$. Moreover, no two components of multiplicity j meet.*

Proof. Let C be a component of Y of multiplicity i. Let $\mathcal{I}$ and $\mathcal{J}$ be the ideal sheaves of Y and C in Γ_σ. Then $(\sigma(\pi) - \pi) = (\pi^j) \subset \mathcal{I} \subset \mathcal{J}^i$. Choose a smooth point x of X_t contained in C. The equation of C at the local ring of X at x is π , so $i \leqslant j$.

For the second point, let x be a closed point of X that is singular in X_t. The completion of the local ring of X at x is isomorphic to $A = B[[t, \epsilon]]/(t\epsilon - \pi)$, and t and ϵ give the branches of X_t through x. If these branches occur in Y with multiplicity j each, then $(\sigma(t) - t, \sigma(\epsilon) - \epsilon) \subset (t^j\epsilon^j) = (\pi^j)$. Therefore, from the relation

$$\sigma(\pi) - \pi = \gamma\pi^j = \sigma(\epsilon)(\sigma(t) - t) + t(\sigma(\epsilon) - \epsilon),$$

we get that $\gamma\pi^j \in \pi^j\mathfrak{m}$, where $\mathfrak{m}$ is the maximal ideal of A. Contradiction. $\square$

Let C be a component of Y which occurs with multiplicity $i \leqslant j - 1$. Recall from the last section that $(\sigma - 1)$ defines a map $\mathcal{O}_{nC} \to \mathcal{O}_{nC}$ which vanishes if and only if $n \leqslant i$. Therefore, we get a map

$$(\sigma - 1) : \mathcal{O}_{(i+1)C} \to \mathcal{J}^i/\mathcal{J}^{i+1} = \mathcal{O}_C(-iC),$$

where $\mathcal{J}$ is the ideal sheaf of C in X. Let $\mathcal{I}$ be the ideal sheaf of fix(σ) in X. Under the condition $i \leqslant j - 1$, we still have *Claim* 2: $(\sigma - 1)$ vanishes on $\mathcal{J}/\mathcal{J}^{i+1}$ (cf. [3] lemma (3.5)). We get a k-derivation $D_{\sigma,C} : \mathcal{O}_C \to \mathcal{O}_C(-iC)$. We denote also by $D_{\sigma,C} : \Omega^1_{C/k} \to \mathcal{O}_C(-iC)$ the induced $\mathcal{O}_C$-linear map and κ its cokernel. For any closed point x of C, define $\mathrm{ord}_x(D_{\sigma,C}) = \mathrm{Leng}_{\mathcal{O}_{C,x}}(\kappa_x)$.

LEMMA 9.5. *Let x be a closed point of X fixed by σ.*

(i) *If x is smooth in X_t , then x can be on Y. In this case, let C be the component of Y containing x and let i be its multiplicity in Y. Put $i = 0$ if x doesn't belong to Y.*

$$l(x) = \begin{cases} jm & \text{if } i = 0, \\ (j - i)\mathrm{ord}_x(D_{\sigma,C}) & \text{if } 0 < i < j, \\ 0 & \text{if } i = j, \end{cases}$$

where if $i = 0$, m is the multiplicity of x as a fixed point of σ acting on X_t.

(ii) *If x is singular in X_t and σ switches the branches of X_t through x. Then, x doesn't belong to Y, and $l(x) = 2j - 1$.*

(iii) *If x is singular in X_t and σ fixes the branches of X_t through x. We have two cases:*

(a) *The branches of X_t through x are contained in two different components C_1 and C_2 of X_t. Let i_1 and i_2 be their multiplicities in Y. We choose C_1 and C_2 such that*

$0 \leqslant i_1 \leqslant i_2$. *Then* , $i_1 \leqslant j-1$ *by Lemma* 9.4, *and*

$$l(x) = \begin{cases} j(s_1+s_2-2)+1 & \text{if } i_1=i_2=0, \\ j(s_1-i_2-1)+1+(j-i_2)(\mathrm{ord}_x(D_{\sigma,C_2})-1) & \text{if } 0=i_1<i_2<j, \\ j(s_1-i_2-1)+1 & \text{if } i_1=0, i_2=j, \\ (j-i_1)(\mathrm{ord}_x(D_{\sigma,C_1})-i_2-1)+ \\ \quad +(j-i_2)(\mathrm{ord}_x(D_{\sigma,C_2})-i_1-1)+1 & \text{if } 0<i_1\leqslant i_2<j, \\ (j-i_1)(\mathrm{ord}_x(D_{\sigma,C_1})-i_2-1)+1 & \text{if } 0<i_1<i_2=j, \end{cases}$$

where s_1 *and* s_2 *are the multiplicities of* x *as a fixed point of* σ *acting respectively on* C_1 *and* C_2 *(defined when* $i_1=0$ *or* $i_2=0$ *).*

(b) *The branches of* X_t *through* x *are contained in the same component* C *of* X_t. *Let* i *be its multiplicity in* Y. *Then,* $0 \leqslant i \leqslant j-1$ *by Lemma* 9. 4, *and*

$$l(x) = \begin{cases} j(s-1)+1 & \text{if } i=0, \\ (j-i)(\mathrm{ord}_x(D_{\sigma,C})-1)+1 & \text{if } 0<i<j, \end{cases}$$

where if $i=0$, s *is the multiplicity of* x *as a fixed point of* σ *acting on* C.

Proof. Lemma B.1 will be frequently used in this proof without any indication.

(i) Assume that x is smooth in X_t. The completion of the local ring of X at x is isomorphic to $B[[t]]$. The ideal of $\mathrm{fix}(\sigma)$ at x is given by

$$(\sigma(\pi)-\pi, \sigma(t)-t) = (\tau(\pi)-\pi, \sigma(t)-t) = (\pi^j, \sigma(t)-t).$$

The local equation defining the divisor Y is the greatest common divisor of π^j and $\sigma(t)-t$. It is π^i with $i \leqslant j$. If $i>0$, then x belongs to Y. Let C be the component of Y on which x lies. Its multiplicity in Y is i. The ideal of F at x is generated by π^{j-i} and $(\sigma(t)-t)/\pi^i$. If $i=j$, then $l(x)=0$. If $0<i<j$, then

$$l(x) = \mathrm{Leng}(B[[t]]/(\pi^{j-i}, (\sigma(t)-t)/\pi^i)) = (j-i)\mathrm{Leng}(B[[t]]/((\sigma(t)-t)/\pi^i, \pi)).$$

On the other hand, we can define in this case the derivation $D_{\sigma,C}$, and we have

$$\mathrm{ord}_x(D_{\sigma,C}) = \mathrm{Leng}\left(B[[t]]/\left(\pi, \frac{\sigma(t)-t}{\pi^i}\right)\right).$$

Finally, if $i=0$, then

$$l(x) = \mathrm{Leng}(B[[t]]/(\pi^j, (\sigma(t)-t)) = j\mathrm{Leng}(k[[t]]/(\sigma(t)-t)) = jm.$$

Assume that x is singular in X_t. The completion of the local ring of X at x is isomorphic to $A = B[[t,\epsilon]]/(t\epsilon-\pi)$. Let $\mathfrak{m}$ be its maximal ideal. The automorphism σ induces an automorphism on A also denoted σ. As in the proof of Lemma 8.1, one proves that there exist two units α and β in A, with $\alpha\beta = 1 + \gamma\pi^{j-1}$, such that

(ii) if σ fixes the branches through x, then $\begin{cases} \sigma(t) &= \alpha t \\ \sigma(\epsilon) &= \beta\epsilon \end{cases}$

(iii) if σ switches the branches through x, then $\begin{cases} \sigma(t) &= \alpha\epsilon \\ \sigma(\epsilon) &= \beta t \end{cases}$

(ii) The ideal of fix(σ) at x is $I = (\sigma(t) - t, \sigma(\epsilon) - \epsilon) = (\alpha\epsilon - t, \beta t - \epsilon)$, which is also the ideal generated by $(\alpha\epsilon - t)$ and $(\alpha\beta t - \alpha\epsilon) = (t - \alpha\epsilon + \gamma\pi^{j-1}t)$. Then, $I = (\alpha\epsilon - t, \pi^{j-1}t)$. The greatest common divisor of $\alpha\epsilon - t$ and $\pi^{j-1}t$ is 1, because α is invertible in A. It follows that x cannot be on Y and

$$\begin{aligned} l(x) &= \mathrm{Leng}(A/(\alpha\epsilon - t, \pi^{j-1}t)) \\ &= 1 + \mathrm{Leng}(A/(\alpha\epsilon - t, \pi^{j-1})) \\ &= 1 + (j-1)\mathrm{Leng}(A/(\alpha\epsilon - t, \pi)) \\ &= 2j - 1. \end{aligned}$$

(iii) The ideal I is generated by $(\alpha - 1)t$ and $(\beta - 1)\epsilon$. Define

$$\begin{aligned} n_1 &= \mathrm{ord}_t(\beta - 1), \\ n_2 &= \mathrm{ord}_\epsilon(\alpha - 1). \end{aligned}$$

Put $\gcd(t(\alpha - 1), \epsilon(\beta - 1)) = t^{i_1}\epsilon^{i_2}$, and assume that $i_1 \leqslant i_2$ (otherwise exchange t and ϵ).

(**a**) The local equation of C_1 at x is $t = 0$, and the one of C_2 is $\epsilon = 0$.

Claim: $i_1 = n_1$ and $i_2 = \inf(n_2, j)$.

Proof. We have the relation:

$$(\alpha - 1) = \alpha - \alpha\beta + \gamma\pi^{j-1} = \alpha(1 - \beta) + \gamma\pi^{j-1}. \tag{67}$$

Then, $\mathrm{ord}_t(\alpha - 1) \geqslant n_1$ if $n_1 \leqslant j - 1$, and $\mathrm{ord}_t(\alpha - 1) = j - 1$ otherwise. The first case gives $i_1 = n_1$. The second implies that $i_1 = j$. But $i_1 \leqslant i_2$, hence $i_1 \leqslant j - 1$ by Lemma 9.4. This case cannot occur. For the second relation, use the relation:

$$\beta - 1 = \beta(1 - \alpha) + \gamma\pi^{j-1}. \tag{68}$$

□

With these notation,

$$l(x) = \mathrm{Leng}\left(A/\left(\frac{\alpha - 1}{t^{i_1-1}\epsilon^{i_2}}, \frac{\beta - 1}{t^{i_1}\epsilon^{i_2-1}}\right)\right).$$

We distinguish the following cases:

https://doi.org/10.1023/A:1001822419774 Published online by Cambridge University Press

- $i_1 = 0 = i_2$:

$$\begin{aligned} l(x) &= \mathrm{Leng}(A/(t(\alpha-1), \epsilon(\beta-1))) \\ &= 1 + \mathrm{Leng}(A/(t, \beta-1)) + \mathrm{Leng}(A/(\alpha-1, \epsilon)) + \mathrm{Leng}(A/(\alpha-1, \beta-1)) \\ &= 1 + (s_1-1) + (s_2-1) + \mathrm{Leng}(A/(\alpha-1, \beta-1)). \end{aligned}$$

From (67), we get $(\alpha-1, \beta-1) = (\beta-1, \pi^{j-1})$. Hence,

$$\begin{aligned} \mathrm{Leng}(A/(\alpha-1, \beta-1)) &= \mathrm{Leng}(A/(\beta-1, \pi^{j-1})) \\ &= (j-1)(\mathrm{Leng}(A/(t, \beta-1)) + \mathrm{Leng}(A/(\epsilon, \beta-1))) \\ &= (j-1)(s_1-1+s_2-1). \end{aligned}$$

We explain the last equality: there is nothing to be proved if $j=1$. Assume that $j>1$, then from (67) and (68), we get $(\epsilon, \beta-1) = (\epsilon, \alpha-1)$. Hence,

$$\mathrm{Leng}(A/(\epsilon, \beta-1)) = \mathrm{Leng}(A/(\epsilon, \alpha-1)) = s_2 - 1.$$

On the other hand, we have $\mathrm{Leng}(A/(t, \beta-1)) = s_1 - 1$.

- $i_1 = 0$ and $0 < i_2 < j$: from the Claim, $i_2 = n_2$ and ϵ^{n_2} divides $(\beta-1)$ by (68). Therefore,

$$\begin{aligned} l(x) &= \mathrm{Leng}\left(A/\left(t\frac{\alpha-1}{\epsilon^{n_2}}, \epsilon\frac{\beta-1}{\epsilon^{n_2}}\right)\right) \\ &= 1 + \mathrm{Leng}\left(A/\left(\epsilon, \frac{\alpha-1}{\epsilon^{n_2}}\right)\right) + \mathrm{Leng}\left(A/\left(\frac{\beta-1}{\epsilon^{n_2}}, t\right)\right) + \\ &\quad + \mathrm{Leng}\left(A/\left(\frac{\beta-1}{\epsilon^{n_2}}, \frac{\alpha-1}{\epsilon^{n_2}}\right)\right). \end{aligned}$$

Using (67), we get

$$\begin{aligned} l(x) &= 1 + \mathrm{Leng}\left(A/\left(\epsilon, \frac{\alpha-1}{\epsilon^{n_2}}\right)\right) + \mathrm{Leng}\left(A/\left(\frac{\beta-1}{\epsilon^{n_2}}, t\right)\right) + \\ &\quad + \mathrm{Leng}\left(A/\left(\frac{\beta-1}{\epsilon^{n_2}}, t^{j-1}\epsilon^{j-1-n_2}\right)\right) \\ &= 1 + \mathrm{Leng}\left(A/\left(\epsilon, \frac{\alpha-1}{\epsilon^{n_2}}\right)\right) + j\mathrm{Leng}\left(A/\left(\frac{\beta-1}{\epsilon^{n_2}}, t\right)\right) + \\ &\quad + (j-1-n_2)\mathrm{Leng}\left(A/\left(\epsilon, \frac{\beta-1}{\epsilon^{n_2}}\right)\right) \\ &= 1 + (j-n_2)\mathrm{Leng}\left(A/\left(\epsilon, \frac{\alpha-1}{\epsilon^{n_2}}\right)\right) + j(\mathrm{Leng}(A/(\beta-1, t)) - n_2). \end{aligned}$$

We explain the last equality. If $j-1-n_2 = 0$, there is nothing to be proved. Otherwise, using (67) and (68), $(\epsilon, (\beta-1)/\epsilon^{n_2}) = (\epsilon, (\alpha-1)/\epsilon^{n_2})$. The Lemma follows

https://doi.org/10.1023/A:1001822419774 Published online by Cambridge University Press

because of the relations $n_2 = i_2$ and

$$\mathrm{ord}_x(D_{\sigma,C_2}) - 1 = \mathrm{Leng}\left(A/\left(\epsilon, \frac{\alpha-1}{\epsilon^{n_2}}\right)\right) \quad \text{and} \quad s_1 - 1 = \mathrm{Leng}(A/(t, \beta-1)).$$

- $i_1 = 0$ and $i_2 = j$: from the Claim, $n_2 \geqslant j$ and ϵ^{j-1} divides $(\beta - 1)$ by (68). Therefore,

$$\begin{aligned} l(x) &= \mathrm{Leng}\left(A/\left(t\frac{\alpha-1}{\epsilon^j}, \frac{\beta-1}{\epsilon^{j-1}}\right)\right) \\ &= \mathrm{Leng}\left(A/\left(t, \frac{\beta-1}{\epsilon^{j-1}}\right)\right) + \mathrm{Leng}\left(A/\left(\frac{\alpha-1}{\epsilon^j}, \frac{\beta-1}{\epsilon^{j-1}}\right)\right). \end{aligned}$$

  Using the relation (68), we get $((\alpha-1)/\epsilon^j, (\beta-1)/\epsilon^{j-1}) = ((\alpha-1)/\epsilon^j, t^{j-1})$. Then,

$$\begin{aligned} l(x) &= \mathrm{Leng}\left(A/\left(t, \frac{\beta-1}{\epsilon^{j-1}}\right)\right) + \mathrm{Leng}\left(A/\left(t^{j-1}, \frac{\alpha-1}{\epsilon^j}\right)\right) \\ &= \mathrm{Leng}\left(A/\left(t, \frac{\beta-1}{\epsilon^{j-1}}\right)\right) + \mathrm{Leng}\left(A/\left(t^{j-1}, \frac{\alpha-1}{\epsilon^{j-1}}\right)\right) - (j-1) \\ &= j\mathrm{Leng}\left(A/\left(t, \frac{\beta-1}{\epsilon^{j-1}}\right)\right) - (j-1) = j(s_1 - j - 1) + 1. \end{aligned}$$

  The last relation follows from (67) and (68).

- $0 < i_1 \leqslant i_2 < j$: from the Claim, $n_1 = i_1$, $n_2 = i_2$, and by (67) and (68), t^{i_1} divides $(\alpha - 1)$ and ϵ^{i_2} divides $(\beta - 1)$. Therefore,

$$\begin{aligned} l(x) &= \mathrm{Leng}\left(A/\left(\frac{\alpha-1}{t^{i_1-1}\epsilon^{i_2}}, \frac{\beta-1}{t^{i_1}\epsilon^{i_2-1}}\right)\right) \\ &= 1 + \mathrm{Leng}\left(A/\left(t, \frac{\beta-1}{t^{i_1}\epsilon^{i_2}}\right)\right) + \mathrm{Leng}\left(A/\left(\epsilon, \frac{\alpha-1}{t^{i_1}\epsilon^{i_2}}\right)\right) + \\ &\quad + \mathrm{Leng}\left(A/\left(\frac{\alpha-1}{t^{i_1}\epsilon^{i_2}}, \frac{\beta-1}{t^{i_1}\epsilon^{i_2}}\right)\right). \end{aligned}$$

  Using (67) and (68), we get that

$$\left(\frac{\alpha-1}{t^{i_1}\epsilon^{i_2}}, \frac{\beta-1}{t^{i_1}\epsilon^{i_2}}\right) = \left(t^{j-i_1-1}\epsilon^{j-i_2-1}, \frac{\beta-1}{t^{i_1}\epsilon^{i_2}}\right).$$

https://doi.org/10.1023/A:1001822419774 Published online by Cambridge University Press

Therefore,

$$\begin{aligned}
&\operatorname{Leng}\left(A/\left(\frac{\alpha-1}{t^{i_1}\epsilon^{i_2}},\frac{\beta-1}{t^{i_1}\epsilon^{i_2}}\right)\right)\\
&\quad=(j-i_1-1)\operatorname{Leng}\left(A/\left(t,\frac{\beta-1}{t^{i_1}\epsilon^{i_2}}\right)\right)+\\
&\qquad+(j-i_2-1)\operatorname{Leng}\left(A/\left(\epsilon,\frac{\beta-1}{t^{i_1}\epsilon^{i_2}}\right)\right)\\
&\quad=(j-i_1-1)\operatorname{Leng}\left(A/\left(t,\frac{\beta-1}{t^{i_1}\epsilon^{i_2}}\right)\right)+\\
&\qquad+(j-i_2-1)\operatorname{Leng}\left(A/\left(\epsilon,\frac{\alpha-1}{t^{i_1}\epsilon^{i_2}}\right)\right).
\end{aligned}$$

The second equality follows from (67) and (68). The result follows from:

$$\begin{aligned}
\operatorname{Leng}\left(A/\left(t,\frac{\beta-1}{t^{i_1}\epsilon^{i_2}}\right)\right)&=\operatorname{ord}_x(D_{\sigma,C_1})-1-i_2,\\
\operatorname{Leng}\left(A/\left(\epsilon,\frac{\alpha-1}{t^{i_1}\epsilon^{i_2}}\right)\right)&=\operatorname{ord}_x(D_{\sigma,C_2})-1-i_1.
\end{aligned}$$

- $0<i_1<i_2=j$: from the Claim, $n_1=i_1$, $n_2\geqslant j$, and by (67) and (68), t^{i_1} divides $(\alpha-1)$ and ϵ^{j-1} divides $(\beta-1)$. Therefore,

$$\begin{aligned}
l(x)&=\operatorname{Leng}\left(A/\left(\frac{\alpha-1}{t^{i_1-1}\epsilon^{j}},\frac{\beta-1}{t^{i_1}\epsilon^{j-1}}\right)\right)\\
&=\operatorname{Leng}\left(A/\left(t,\frac{\beta-1}{t^{i_1}\epsilon^{j-1}}\right)\right)-\operatorname{Leng}\left(A/\left(\epsilon,\frac{\beta-1}{t^{i_1}\epsilon^{j-1}}\right)\right)+\\
&\quad+\operatorname{Leng}\left(A/\left(\frac{\alpha-1}{t^{i_1}\epsilon^{j-1}},\frac{\beta-1}{t^{i_1}\epsilon^{j-1}}\right)\right)\\
&=\operatorname{Leng}\left(A/\left(t,\frac{\beta-1}{t^{i_1}\epsilon^{j-1}}\right)\right)-\operatorname{Leng}\left(A/\left(\epsilon,\frac{\beta-1}{t^{i_1}\epsilon^{j-1}}\right)\right)+\\
&\quad+\operatorname{Leng}\left(A/\left(t^{j-1-i_1},\frac{\beta-1}{t^{i_1}\epsilon^{j-1}}\right)\right).
\end{aligned}$$

The last equality follows from (67), which implies that

$$\left(\frac{\alpha-1}{t^{i_1}\epsilon^{j-1}},\frac{\beta-1}{t^{i_1}\epsilon^{j-1}}\right)=\left(t^{j-1-i_1},\frac{\beta-1}{t^{i_1}\epsilon^{j-1}}\right).$$

As ϵ divides $\dfrac{\alpha-1}{t^{i_1}\epsilon^{j-1}}$, and using (68), we deduce that

$$\left(\epsilon,\frac{\beta-1}{t^{i_1}\epsilon^{j-1}}\right)=(\epsilon,t^{j-1-i_1}).$$

https://doi.org/10.1023/A:1001822419774 Published online by Cambridge University Press

We conclude that

$$l(x) = -(j - 1 - i_1) + (j - i_1)(\mathrm{ord}_x(D_{\sigma,C_1}) - j).$$

(**b**) In this case, $i_1 = i_2 = i < j$, and the local equation of C at x is $\epsilon t = 0$. We get from the Claim that $n_1 = n_2 = i$, and from (67) and (68) that t^i divides $(\alpha - 1)$ and ϵ^i divides $(\beta - 1)$ because $i < j$. Then,

$$\begin{aligned}
&\mathrm{Leng}\left(A/\left(\frac{t(\alpha-1)}{t^i\epsilon^i}, \frac{\epsilon(\beta-1)}{t^i\epsilon^i}, t\epsilon\right)\right)\\
&\quad = \mathrm{Leng}\left(A/\left(t, \frac{\epsilon(\beta-1)}{t^i\epsilon^i}\right)\right) + \mathrm{Leng}\left(A/\left(\frac{(\alpha-1)}{t^i\epsilon^i}, \epsilon, \frac{\epsilon(\beta-1)}{t^i\epsilon^i}\right)\right)\\
&\quad = \mathrm{Leng}\left(A/\left(t, \frac{\epsilon(\beta-1)}{t^i\epsilon^i}\right)\right) + \mathrm{Leng}\left(A/\left(\frac{(\alpha-1)}{t^i\epsilon^i}, \epsilon\right)\right)\\
&\quad = \mathrm{Leng}\left(A/\left(t, \frac{\epsilon(\beta-1)}{t^i}\right)\right) + \mathrm{Leng}\left(A/\left(\frac{t(\alpha-1)}{\epsilon^i}, \epsilon\right)\right) - 2i - 1.
\end{aligned}$$

The left-hand side of this equation is s if $i = 0$, and $\mathrm{ord}_x(D_{\sigma,C})$ if $0 < i < j$. The Lemma follows from this relation and the formula

$$l(x) = 1 + (j - i)\left(\mathrm{Leng}\left(A/\left(t, \frac{\beta-1}{t^i\epsilon^i}\right)\right) + \mathrm{Leng}\left(A/\left(\epsilon, \frac{\alpha-1}{t^i\epsilon^i}\right)\right)\right)$$

proved previously. $\square$

LEMMA 9.6. *Let* $[Y] = \sum_C i_C[C]$ *be the decomposition of* Y *into its irreducible components. Then,*

$$(\Delta_X.\Gamma_\sigma)_{loc} = \sum_C i_C((C.C) + 2\chi(\mathcal{O}_C)) - (Y.Y) + \sum_{x\in F} l(x).$$

Proof. Let ω be the dualizing sheaf of X over S. By Proposition 5.8., we have

$$\begin{aligned}
(\Delta_X.\Gamma_\sigma)_{loc} &= -(\omega + Y.Y) + \sum_{x\in F} l(x)\\
&= -\sum_C i_C(\omega.C) - (Y.Y) + \sum_{x\in F} l(x)\\
&= \sum_C i_C((C.C) + 2\chi(\mathcal{O}_C)) - (Y.Y) + \sum_{x\in F} l(x).
\end{aligned}$$

$\square$

Let C be an irreducible component of X_t stabilized by σ. Define $\delta(C)$ to be the number of σ-fixed nodes of C with σ-fixed branches. If $i_C = 0$, then σ induces a

https://doi.org/10.1023/A:1001822419774 Published online by Cambridge University Press

non-trivial automorphism over C. Define, for any point x in C, the number $m_C(x)$ to be

- the multiplicity of x as a fixed point of σ acting on C, if x is smooth in C or a node and σ fixes the branches of C through it,
- 0 otherwise.

LEMMA 9.7. *We have:*

$$(\Delta_X.\Gamma_\sigma)_{loc} = j\left(\sum_{C;i_C=0}^{*}\left[\sum_{x\in C} m_C(x) + \delta(C)\right] + 2\sum_{C;i_C>0}^{*}[\chi(\mathcal{O}_C)+\delta(C)]\right) - (2j-1)\sum_{x\in\$} l_x,$$

where $\sum^{*}$ *denotes the sum over all components* C *of* X_t *stabilized by* σ *, and* l_x *for* $x\in\$$ *is defined in* (58).

Proof. A painful computation based on Lemma 9.5 gives:

$$\begin{aligned}\sum_{x\in X_t} l(x) = j\sum_{C;i_C=0}^{*}\ \sum_{x\in C} m_C(x) + \sum_{C;i_C>0}^{*}(j-i_C)\sum_{x\in C}\mathrm{ord}_x(D_{\sigma,C})\\ -(2j-1)\sum_{x\in\$} l_x + \sum_C^{*}(j+i)\delta(C) + \sum_C i_C(j-1)(C.C)\\ +\sum_{C\neq C'} i_C i_{C'}(C.C').\end{aligned}\tag{69}$$

For any C with $0<i_C<j$, Lemma 8.6 gives:

$$\sum_{x\in C}\mathrm{ord}_x(D_{\sigma,C}) = -i_C(C.C) + 2\chi(\mathcal{O}_C) + \delta'(C),$$

where $\delta'(C)$ is the total number of nodes in C. Remark that as $i_C>0$, then $\delta(C)=\delta'(C)$. Therefore, (69) becomes:

$$\begin{aligned}\sum_{x\in X_t} l(x) = j\left(\sum_{C;i_C=0}^{*}\left[\sum_{x\in C} m_C(x)+\delta(C)\right] + 2\sum_{C;i_C>0}^{*}[\chi(\mathcal{O}_C)+\delta(C)]\right)\\ -2\sum_C i_C\chi(\mathcal{O}_C) - \sum_C i_C(C.C) + (Y.Y) - (2j-1)\sum_{x\in\$} l_x\ .\end{aligned}$$

The Lemma follows from this equation and Lemma 9.6. □

Let $\mathcal{X}$ be the normalization of X_t, and σ be the automorphisms of $\mathcal{X}$ extending σ over X_t. Let G_σ be the graph of σ in $\mathcal{X}\times_k\mathcal{X}$.

https://doi.org/10.1023/A:1001822419774 Published online by Cambridge University Press

LEMMA 9.8. *The geometric intersection number of G_σ with the diagonal in $\mathcal{X} \times_k \mathcal{X}$ is given by:*

$$(\Delta_{\mathcal{X}}.G_\sigma) = \sum_{C;i_C=0}^{*} \left[\sum_{x \in C} m_C(x) + \delta(C) \right] + 2 \sum_{C;i_C>0}^{*} [\chi(\mathcal{O}_C) + \delta(C)].$$

Proof. Let C be a component of X_t and $\widetilde{C}$ be the connected (smooth) component of $\mathcal{X}$ above it. By the self–intersection formula for curves, we have:

$$(\Delta_{\mathcal{X}}.G_\sigma) = \sum_{\widetilde{C}}^{1} \sum_{x \in \widetilde{C}} m_{\widetilde{C}}(x) + 2 \sum_{\widetilde{C}}^{2} \chi(\mathcal{O}_{\widetilde{C}}),$$

where $\sum^1$ is the sum over all connected components of $\mathcal{X}$ stabilized by σ but not fixed, and $\sum^2$ is the sum over all components fixed (point by point) of $\mathcal{X}$. For any component C with $i_C > 0$, we have $\chi(\mathcal{O}_{\widetilde{C}}) = \chi(\mathcal{O}_C) + \delta(C)$. Indeed, C has only nodes as singularities and $\delta(C)$ is the total number of this nodes. Let C be a stabilized component such that $i_C = 0$, and let x be a fixed node of C, and x_1 and x_2 be the points of $\widetilde{C}$ above it. If σ does not fix the branches through x, then σ exchanges x_1 and x_2 and by definition $m_C(x) = 0$. If σ fixes the branches through x, then, with the notation of the proof of Lemma 9.5,

$$\begin{aligned} m_C(x) &= \mathrm{Leng}(k[[t,\epsilon]]/(t(\alpha-1), \epsilon(\beta-1), t\epsilon)) \\ &= \mathrm{Leng}(k[[t,\epsilon]]/(t, \epsilon(\beta-1)) + \mathrm{Leng}(k[[t,\epsilon]]/(\epsilon, \alpha-1)) \\ &= m_{\widetilde{C}}(x_1) + m_{\widetilde{C}}(x_2) - 1. \end{aligned}$$

The Lemma follows. □

Proof of formula (65): Lemmas 9.7 and 9.8, and the geometric Lefschetz fixed point formula for $\mathcal{X}$ imply that

$$(\Delta_X.\Gamma_\sigma)_{loc} = j\mathrm{tr}(\sigma)|H^*_{et}(\mathcal{X}, \mathbb{Q}_l) - (2j-1) \sum_{x \in \$} l_x. \tag{70}$$

On the other hand, we have

$$\mathrm{tr}(\sigma)|H^*_{et}(\mathcal{X}, \mathbb{Q}_l) = \mathrm{tr}(\sigma)|H^*_{et}(X_t, \mathbb{Q}_l) + \sum_{x \in \$} l_x. \tag{71}$$

If $j = 1$ (i.e. $\tau \notin P(L/K)$), Equations (70) and (71) are enough to get (65). In general, we need a third relation. For this purpose, we introduce the notation M, $\overline{T}$, $\overline{X} = X \times_T \overline{T}$ and $a : X_{\overline{s}} \to \overline{X}$ as in the proof of (62). We fix a lifting $\overline{\tau} \in G(\overline{K}/K)$ of τ, and let $\overline{\sigma} = \sigma \times_\tau \overline{\tau}$ acts over $X_{\overline{s}}$ and $\overline{X}$. Then $\overline{\sigma}$ acts over the cohomology groups $H^m(\overline{X}, R^n a_* \mathbb{Q}_l)$. The latter are given by Equation (63), and the action of $\overline{\sigma}$ turns to be the same as the one of σ over respectively M and $H^m(X_t, \mathbb{Q}_l)$. Therefore, the Leray spectral sequence for a implies that

$$\mathrm{tr}(\sigma)|H^*_{et}(X_t, \mathbb{Q}_l) = \mathrm{tr}(\overline{\sigma})|H^*_{et}(X_{\overline{g}}, \mathbb{Q}_l) + \sum_{x\in \mathcal{S}} l_x.$$

Hence, $\mathrm{tr}(\overline{\sigma})|H^*_{et}(X_{\overline{g}}, \mathbb{Q}_l)$ does not depend on the choice of the lifting $\overline{\tau}$, and the above formula can be written as

$$\mathrm{tr}(\sigma)|H^*_{et}(X_t, \mathbb{Q}_l) = \mathrm{tr}(\sigma)|H^*_{et}(X_{\overline{g}}, \mathbb{Q}_l) + \sum_{x\in \mathcal{S}} l_x. \tag{72}$$

Formula (65) is a consequence of (70), (71) and (72). $\square$

10. Lefschetz Formula: The General Case

Let X be an arithmetic surface over S and σ be a non-trivial S-automorphism of X.

Step (1). Let C be the generic fiber of X. By the semi-stable reduction theorem [2], we can find a finite Galois extension L of K of degree n and Galois group G such that:

(1) $C \times_K L$ admits a semi-stable regular model V over $T = \mathrm{Spec}(B)$ (where B is the integral closure of R in L);
(2) the automorphism σ over $C \times_K L$ extends to a T-automorphism over V equally denoted σ;
(3) the G-action over $C \times_K L$ extends to an action of G over V by S-automorphisms.

There exists an arithmetic surface W over T, equipped with two birational morphisms π and ρ

$$\begin{tikzcd}
& W \arrow[dl, "\rho"'] \arrow[dr, "\pi"] & & \\
V & & X \times_S T \arrow[r] \arrow[d] & X \arrow[d] \\
& & T \arrow[r] & S
\end{tikzcd}$$

such that the automorphism σ and the group action of G over V lift to W. In other words, the morphism $W \to X$ is a base change (see Definition 7.16).

Step (2). We consider all surfaces over S. By Proposition 7.20, the base change $W \to X$ satisfies the weak projection formula:

$$n(\Delta_X.\Gamma^X_\sigma)_{loc} = \sum_{\tau\in G}(\Delta_W.\Gamma^W_{\sigma\tau})_{loc} - \sum_{\tau\in G}\mathrm{tr}(\sigma\tau)|H^*_{et}(W_s, \mathbb{Q}_l) + n\mathrm{tr}(\sigma)|H^*_{et}(X_s, \mathbb{Q}_l).$$

https://doi.org/10.1023/A:1001822419774 Published online by Cambridge University Press

Theorem 7.1, applied to the birational map ρ, gives:

$$(\Delta_W.\Gamma^W_{\sigma\tau})_{loc} = (\Delta_V.\Gamma^V_{\sigma\tau})_{loc} - \mathrm{tr}(\sigma\tau)|H^*_{et}(V_s, \mathbb{Q}_l) + \mathrm{tr}(\sigma\tau)|H^*_{et}(W_s, \mathbb{Q}_l).$$

Let t be the closed point of T. As V_t and V_s have the same reduced scheme structure, we get from the last two relations:

$$n(\Delta_X.\Gamma^X_\sigma)_{loc} = \sum_{\tau\in G}(\Delta_V.\Gamma^V_{\sigma\tau})_{loc} - \sum_{\tau\in G}\mathrm{tr}(\sigma\tau)|H^*_{et}(V_t, \mathbb{Q}_l) + n\mathrm{tr}(\sigma)|H^*_{et}(X_s, \mathbb{Q}_l).$$

By equations (64) and (65), we get:

$$\begin{aligned} n(\Delta_X.\Gamma^X_\sigma)_{loc} &= n\mathrm{tr}(\sigma)|H^*_{et}(X_s, \mathbb{Q}_l) - (\mathrm{sw}_{L/K}(1) + n)\mathrm{tr}(\sigma)|H^*_{et}(X_{\overline{\eta}}, \mathbb{Q}_l) \\ &\quad - \sum_{\tau\in G,\ \tau\neq 1} \mathrm{sw}_{L/K}(\tau)\mathrm{tr}(\sigma\tau)|H^*_{et}(X_{\overline{\eta}}, \mathbb{Q}_l) \\ &= -\sum_{\tau\in G}\mathrm{sw}_{L/K}(\tau)\mathrm{tr}(\sigma\tau)|H^*_{et}(X_{\overline{\eta}}, \mathbb{Q}_l) + n(\mathrm{tr}(\sigma)|H^*_{et}(X_s, \mathbb{Q}_l) \\ &\quad - \mathrm{tr}(\sigma)|H^*_{et}(X_{\overline{\eta}}, \mathbb{Q}_l)). \end{aligned}$$

Theorem 1.1 follows using equation (5). □

Remark 10.1. K. Kato, S. Saito and T. Saito conjectured the Lefschetz fixed point formula in a different formulation ([10] conjecture (1.5)). As they have already noticed (loc. cit. second paragraph in page 53), their conjecture can be reformulated, in their notation, as follows:

$$\chi(X^\sigma, Li^*\Omega^1_{X/S} \to \mathcal{I}/\mathcal{I}^2) = \mathrm{tr}(\sigma)|\underline{\mathrm{dt}}(R\Gamma(X_s, R\Phi\mathbb{Q}_l)), \tag{73}$$

where $X^\sigma = \mathrm{fix}(\sigma)$ is the scheme of fixed points, i is defined in the diagram below and $\mathcal{I}$ is the ideal sheaf of the closed immersion $X^\sigma \to \Gamma_\sigma$. The morphism $Li^*\Omega^1_{X/S} \to \mathcal{I}/\mathcal{I}^2$ induces a quasi-isomorphism on the generic fiber of X^σ. So, the left hand side of formula (73) is well defined.

Clearly, the right-hand sides of equations (1) and (73) are opposite. We prove the same property for the left hand sides as follows. By a sequence of blow-ups of Γ_σ at closed points, we get a regular surface $\widetilde{\Gamma}_\sigma$ such that the inverse image $\widetilde{X}^\sigma$ of X^σ is a Cartier divisor.

$$\begin{tikzcd} \widetilde{X}^\sigma \arrow[r] \arrow[d, "h"] \arrow[dd, bend right, "j"'] & \widetilde{\Gamma}_\sigma \arrow[d] \\ X^\sigma \arrow[r] \arrow[d, "i"] & \Gamma_\sigma \arrow[d] \\ \Delta_X \arrow[r] & X \times_S X \end{tikzcd}$$

Let $\mathcal{J}$ be the ideal sheaf of the closed immersion $\widetilde{X}^\sigma \to \widetilde{\Gamma}_\sigma$. I claim that $Rh_*\mathcal{O}_{\widetilde{X}^\sigma} = \mathcal{O}_{X^\sigma}$ and $Rh_*\mathcal{J}/\mathcal{J}^2 = \mathcal{I}/\mathcal{I}^2$. These relations are proved by considering a blow–up of a regular surface at a closed point. We deduce that

$$\chi(\widetilde{X}^{\sigma}, Lj^{*}\Omega^{1}_{X/S} \to \mathcal{J}/\mathcal{J}^{2}) = \chi(X^{\sigma}, Li^{*}\Omega^{1}_{X/S} \to \mathcal{I}/\mathcal{I}^{2}).$$

Since the rank of the complex $Lj^{*}\Omega^{1}_{X/S} \to \mathcal{J}/\mathcal{J}^{2}$ over the curve $\widetilde{X}^{\sigma}$ is zero, the refined Riemann–Roch formula ([7] example 18.3.12) gives

$$\begin{aligned}\chi(\widetilde{X}^{\sigma}, Lj^{*}\Omega^{1}_{X/S} \to \mathcal{J}/\mathcal{J}^{2}) &= \deg(c^{\widetilde{X}^{\sigma}}_{1,\widetilde{X}^{\sigma}_{s}}(Lj^{*}\Omega^{1}_{X/S} \to \mathcal{J}/\mathcal{J}^{2}) \cap [\widetilde{X}^{\sigma}]) \\ &= -\deg(\Delta_X.\widetilde{\Gamma}_{\sigma}) = -\deg(\Delta_X.\Gamma_{\sigma}).\end{aligned}$$

We used the excess formula of Theorem 4.7.

Remark 10.2. The Lefschetz fixed point formula (1) holds for a normal surface and a non–trivial automorphism, if we define the Lefschetz number as in remark 7.6. This follows from the definition and the Lefschetz fixed point formula for a desingularisation of the normal surface.

Finally, we give the proof of Lemma 1.2 and Corollary 1.3. Lemma 1.2 is a consequence of Proposition 5.8. Then, by Theorem 1.1,

$$\begin{aligned}a_G(\sigma) = -(\Delta_X.\Gamma_{\sigma})_{loc} &= \mathrm{tr}(\sigma)|\underline{\mathrm{sw}}(H^{*}_{et}(X_{\overline{\eta}}, \mathbb{Q}_l)) - \mathrm{tr}(\sigma)|H^{*}_{et}(X_s, \mathbb{Q}_l) \\ &\quad + \mathrm{tr}(\sigma)|H^{*}_{et}(X_{\overline{\eta}}, \mathbb{Q}_l); \qquad \forall \sigma \in G - \{1\}.\end{aligned}$$

Therefore,

$$\begin{aligned}a_G(\sigma) &= \mathrm{tr}(\sigma)|\underline{\mathrm{sw}}(H^{*}_{et}(X_{\overline{\eta}}, \mathbb{Q}_l)) - \mathrm{tr}(\sigma)|H^{*}_{et}(X_s, \mathbb{Q}_l) + \mathrm{tr}(\sigma)|H^{*}_{et}(X_{\overline{\eta}}, \mathbb{Q}_l) \\ &\quad + n r_G(\sigma); \qquad \forall \sigma \in G,\end{aligned}$$

for some integer n, where r_G is the character of the regular representation of G. But we know by [19] proposition 7, that $|G|a_G$ is the character of a linear representation of G. Therefore, a_G is the character of a $\mathbb{Q}_l$-rational representation of G.

A. Intersection Numbers Over Normal Surfaces

Let X be a normal surface over S (i.e. an integral normal scheme of dimension 2, proper and flat over S). The object of this appendix is to recall Mumford's definition of the intersection number of a vertical Weil divisor with any Weil divisor over X ([14] II (b)).

Assume first that X is regular. By [6] (Exposé X, example 1.1.), one can define the intersection number of any divisor D with a vertical divisor E.

Let X be a normal surface. Fix $\pi : X' \to X$ a resolution of singularities of X, and let $(E_i)_{1 \leqslant i \leqslant r}$ be the irreducible reduced components of the exceptional fibers of π. Let A be an irreducible effective Weil divisor over X. Mumford define the pull-back of A by the formula

$$\pi^*A = A' + \sum_{i=1}^{r} r_i E_i,$$

where A' is the strict transform of A in X', and the r_i are the rationals defined by the equations $(\pi^*A.E_i) = 0$ for $i = 1, \dots, r$. In other words, the r_i are defined by the linear system:

$$\sum_{j=1}^{r} r_j(E_j.E_i) = -(A'.E_i).$$

As $\det(E_i.E_j) \neq 0$, the r_i are well defined. We extend this definition by linearity to any divisor.

(1) If A is the divisor of a rational function f over X, then π^*A is the divisor of the same function over X'. Indeed, $(\mathrm{div}_{X'}(f).E_i) = 0$ for any i. Therefore, π^* passes to rational equivalence.

(2) Let A and B be two Weil divisors over X such that one of them is vertical. Then, we can compute the rational number $(\pi^*A.\pi^*B)$. It does not depend on the resolution we choose. Indeed, it is enough to compare these numbers for two resolutions X' and X'' such that X'' is obtained by blowing-up a closed point in X'. The computation is easy in this case. We define the intersection number of A and B by $(A.B) = (\pi^*A.\pi^*B)$.

These intersection numbers have the same properties as in the regular case. Namely, let A , B and C be Weil divisors over X such that A is vertical.

(3) If B is the divisor of a rational function, then $(A.B) = 0$. This follows from (1).

(4) $(A.B + C) = (A.B) + (A.C)$.

(5) If B is also vertical then, $(A.B) = (B.A)$.

(6) For any Weil divisor D over X', we have the projection formula

$$(\pi^*A.D)_{X'} = (A.\pi_*D)_X.$$

Indeed, the divisor $D - \pi^*\pi_*D$ is supported over the exceptional fibers.

Let $f : X \to Y$ be a dominant map between normal surfaces over S. Choose resolutions X' of X and Y' of Y over which f lifts:

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "\pi"'] & Y' \arrow[d, "\rho"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Define the pull–back map $f^* : Z_1(Y) \to Z_1(X)$ by the formula $f^* = \pi_* f'^* \rho^*$, where f'^* is the refined Gysin associated to the l.c.i. map $f' : X' \to Y'$ (see Subsection 7.2). Notice that f'^* is defined on the cycle level because it is refined and f' is dominant.

(7) The pull–back f^* passes to rational equivalence. This follows from the same statement for ρ^* proved in (1). It is clear that f^* coincides with the flat pull-back (defined in [7]) if f is flat.

(8) Assume that f is finite of degree d, and let A and B be two Weil divisors over Y with one vertical. Then

$$(f^*A.f^*B) = d(A.B).$$

Indeed,

$$(f^*A.f^*B) = (\pi_* f'^*\rho^*A.\pi_* f'^*\rho^*B) = (f'^*\rho^*A.\pi^*\pi_* f'^*\rho^*B) = (\rho^*A.f'_*\pi^*\pi_* f'^*\rho^*B).$$

Remark that $\pi^*\pi_* f'^*\rho^*B = f'^*\rho^*B + D$, where D is supported over the exceptional fibers of π. Then, $f'_*\pi^*\pi_* f'^*\rho^*B = f'_* f'^*\rho^*B + f'_*D$. As f' is finite over the complement of the exceptional fibers of ρ, then $f'_* f'^*\rho^*B = d\rho^*B + C$, where C is supported over the exceptional fibers of ρ. Hence, $f'_*\pi^*\pi_* f'^*\rho^*B = d\rho^*B + C + f'_*D$. Finally, f'_*D is supported over the exceptional fibers of ρ because $\rho_* f'_*D = f_*\pi_*D = 0$. Therefore, $(\rho^*A.f'_*\pi^*\pi_* f'^*\rho^*B) = (\rho^*A.d\rho^*B) = d(A.B)$.

B. Additivity of Colength

Let A be a Noetherian commutative ring. An ideal I of A has finite colength $\operatorname{col}(I)$ if A/I has finite length, and $\operatorname{col}(I) = \operatorname{Leng}_A(A/I)$. If I is generated by $a_1, \ldots, a_n$, we say that the sequence $(a_1, \ldots, a_n)$ has finite colength if the ideal I has finite colength, and we define $\operatorname{col}(a_1, \ldots, a_n) = \operatorname{col}(I)$.

LEMMA B.1 *let a and b be two elements of A such that a is not a zero-divisor in $A/(b)$, and let I be an ideal. If two of* $\operatorname{col}(aI + bA)$, $\operatorname{col}(I + bA)$ *and* $\operatorname{col}(a, b)$ *are finite, then so is the third and*

$$\operatorname{col}(aI + bA) = \operatorname{col}(a, b) + \operatorname{col}(I + bA).$$

In particular, let a_1, a_2 and b be elements of A such that a_1 or a_2 is not a zero-divisor in $A/(b)$. If two of $\operatorname{col}(a_1, b)$, $\operatorname{col}(a_2, b)$ *and* $\operatorname{col}(a_1 a_2, b)$ *are finite, then so is the third and*

$$\operatorname{col}(a_1 a_2, b) = \operatorname{col}(a_1, b) + \operatorname{col}(a_2, b).$$

Proof. Let $\overline{A} = A/(b)$ and $\overline{I} = (I + bA)/(b)$. The Lemma is a consequence of the exact sequence

$$0 \longrightarrow \overline{A}/\overline{I} \xrightarrow{.a} \overline{A}/a\overline{I} \longrightarrow \overline{A}/(a) \longrightarrow 0.$$

The multiplication by a is injective because a in not a zero-divisor in $\overline{A}$. □

If A is a local regular ring of dimension 2, then (a, b) is of finite colength if and only if (a, b) is a system of parameters. As A is regular, a system of parameters is a regular sequence and conversely. Therefore, a is not a zero-divisor in $A/(b)$ if and only if $\mathrm{col}(a, b)$ is finite.

Acknowledgements

A part of this work was prepared during my visits to the Institut des Hautes Études Scientifiques (France) and the Max-Planck-Institut für Mathematik (Germany). I would like to express my gratitude to these institutions for their hospitality and their generous support. I thank Takeshi Saito for drawing my attention to the relationship between the Lefschetz fixed point formula (1) and his conjecture with K. Kato and S. Saito in [10]. I am grateful to Jean-Benoit Bost for his encouragement and very interesting conversations, and to Boas Erez, Klaus Künnemann, Shouwu Zhang and the referee for their comments on an earlier version of this paper.

References

1. Artin, M.: Some numerical criteria for contractability of curves on algebraic surfaces, *Amer. J. Math.* **84** (1962), 485–496.
2. Artin, M. and Winters, G.: Degenerate fibres and stable reduction of curves, *Topology* **10** (1971), 373–383.
3. Bloch, S.: Cycles on arithmetic schemes and Euler characteristics of curves, *Proc. Sympos. Pure Math.* **46** (1987), 421–450.
4. Bloch, S.: De Rham cohomology and conductors of curves, *Duke Math. J.* **54** (1987), 295–308.
5. Deligne, P.: Cohomologie à support propre et construction du foncteur $f^!$, appendix to Hartshorne, *Residues and Duality*, Lecture Notes in Math. 20, Springer-Verlag, New York, 1966, pp. 404–421.
6. Deligne, P. and Katz, N.: *Groupes de monodromie en géométrie algébrique*, S.G.A. 7 II, Lecture Notes in Math. 340, Springer-Verlag, New York, 1973.
7. Fulton, W.: *Intersection Theory*, Springer-Verlag, Berlin, 1984.
8. Kato, K.: Generalized class field theory, *Proc. I.C.M. Kyoto, Japan* (1990), 419–428.
9. Kato, K.: Class field theory, $\mathcal{D}$-modules, and ramification on higher dimensional schemes, part I, *Amer. J. Math.* **116** (1994), 757–784.
10. Kato, K., Saito, S. and Saito, T.: Artin characters for algebraic surfaces, *Amer. J. Math.* **110** (1987), 49–76.
11. Lipman, J.: Rational singularities, *Publ. Math. I.H.E.S.* **36** (1969), 195–279.
12. Milne, J.: *Étale Cohomology*, Princeton Univ. Press, New Jersey, 1980.
13. Moret-Bailly, L.: Groupes de Picard et problèmes de Skolem I, *Ann. Sci. Ecole Norm. Sup.* **22** (1989), 161–179.
14. Mumford, D.: The topology of normal singularities of an algebraic surface and a criterion for simplicity, *Publ. Math. I.H.E.S.* **9** (1961), 5–22.
15. Roberts, P.: Local Chern characters and intersection multiplicities, *Proc. Sympos. Pure Math.* **46** (1987), 389–400.
16. Saito, T.: Conductor, discriminant, and the Noether formula of an arithmetic surface, *Duke Math. J.* **57** (1988), 151–173.
17. Saito, T.: Self–intersection, 0–cycles and coherent sheaves on arithmetic schemes, *Duke Math. J.* **57** (1988), 555–578.
18. Serre, J.-P.: *Corps locaux*, Publ. Inst. Math. Univ. Nancago VIII, Hermann, Paris, 1968.

19. Serre, J.-P.: Sur la rationalité des représentations d'Artin, *Ann. of Math.* **72** (1960), 406–420.
20. Serre, J.-P.: *Représentations linéaires des groupes finis*, Hermann, Paris, 1971.