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PRIMARY GROUPS WHOSE BASIC SUBGROUP DECOMPOSITIONS CAN BE LIFTED

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ABSTRACT. A primary group G is said to be a l.i.b. group if every idempotent endomorphism of every basic subgroup of G can be extended to an endomorphism of G. We establish the following characterization: A primary group is a l.i.b. group if and only if it is the direct sum of a torsion complete group and a divisible group. The technique used consists of a close analysis of certain subgroups of Prüfer-like primary groups.

All groups considered in this article are primary abelian groups for a fixed prime number p. We give a complete characterization of the groups which satisfy the following condition: Every idempotent endomorphism of every basic subgroup extends to an endomorphism of the group. We show that these groups are the direct sum of a divisible group with a torsion complete group.

0. **Introduction.** Basic subgroups of abelian groups, being direct sums of cyclic groups, have an abundant supply of idempotent endomorphisms. It is therefore to be expected that a group satisfying the condition in the opening paragraph, will have a rich structure. It turns out that this is in fact the case. Torsion complete groups form one of the best known classes of primary groups. L. Fuchs devoted several sections of [4] to their properties. Let us call a group satisfying the property that all idempotent endomorphisms of its basic subgroups can be extended to endomorphisms of itself, a l.i.b. group. (l.i.b. is used to suggest: lifting idempotents of basics). Let G be an l.i.b. group then, it is not difficult to see that if $G^1 = 0$, every decomposition of a basic subgroup B of G, say $B = B_1 \oplus B_2$, induces a decomposition of G, $G = G_1 \oplus G_2$, where $B_i \subset G_i$, $i = 1, 2$. Furthermore G_i is the closure of B_i in the p-adic topology of G. In [6] J. Irwin and T. Koyama considered precisely this property and showed that such a group is torsion complete. ([6] Theorem 3). Thus in order to characterize l.i.b. groups we need only to show that their reduced part has no elements of infinite height and that a group is l.i.b. if and only if its reduced part is an l.i.b. group. The crux of the matter lies in the fact that reduced

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primary groups with non-zero elements of infinite height contain, as pure subgroups, certain Prüfer-like groups. In the first section we define and study some useful properties of these modified Prüfer groups. Then in the second part we apply these results to the problem at hand.

1. **Modified Prüfer groups.** The standard Prüfer group is described in [3] as the group $A=\langle a_0, a_1, \ldots, a_n \cdots\rangle$ with defining relations, $pa_0=0$, $pa_1=a_0, \ldots, p^n a_n=a_0, \ldots$. This is also called the Prüfer group of length $\omega+1$. For our purposes we need a somewhat more flexible set of defining relations.

DEFINITION 1.1. Let $k\in N$ and $\{n_i\}_{i=1}^{\infty}$ be a strictly increasing sequence of positive integers. The group G with generators $\{\bar{x}_i\}_{i=0}^{\infty}$ and defining relations: $p^k\bar{x}_0=0$, and $\bar{x}_0=p^{n_i}\bar{x}_i$, $i=1,2,\ldots$ is called the $(k, \{n_i\})$-Prüfer group. The class of these groups is called the class of modified Prüfer groups.

Modified Prüfer groups, although they do not appear explicitly under this name in the literature of primary abelian groups, are no doubt well-known to every specialist in the field. They are so-to-speak part of the folklore of the theory. We give next two lemmas concerning them whose proofs are straightforward and need not be reproduced here.

LEMMA 1.2. *Let G be a $(k, \{n_i\})$-Prüfer group. Then $G^1=\langle\bar{x}_0\rangle$ and $A=\bigoplus_{i=1}^{\infty}\langle\bar{x}_i-p^{n_{i+1}-n_i}\bar{x}_{i+1}\rangle$ is a basic subgroup of G.*

LEMMA 1.3. *Let G and A be as in Lemma 1.2. Then the ith Ulm–Kaplansky invariant of G is the same as that of A for $1\le i<\omega$, that is:*

$$f_j(G)=f_j(A)\begin{cases}0 & \text{if} \quad j\neq n_{i-1}, \\ 1 & \text{if} \quad j=n_i-1\end{cases} \qquad i=1,2,\ldots.$$

and

$$f_{\omega+i}(G)=0, \qquad 0\le i<k-1, \quad \text{while} \quad f_{\omega+k-1}(G)=1.$$

It is easy to see that the parameters $k, \{n_i\}$ are a complete set of invariants of the $(k,\{n_i\})$-Prüfer groups. We show next that the class of modified Prüfer groups is in a way a "pure core class" for reduced groups with non-zero elements of infinite height. This fact is hinted at without too much precision in [1] p. 203.

THEOREM 1.4. *Let G be a reduced group, G^1 its subgroup of elements of infinite height. Then $G^1\neq 0$ if and only if G contains a pure subgroup isomorphic to a modified Prüfer group.*

Proof. Let $x\in G^1[p]$, $x\neq 0$. Write $G^1[p]=\langle x\rangle\oplus S$ and let H be an S-high subgroup of G containing x. Since $S\subset G^1$, H is a pure subgroup of G and $H^1[p]=\langle x\rangle$. By Proposition 26.2 p. 115 of [3], $\langle x\rangle$ can be inbedded in a countable pure subgroup K of H. Then $K^1 = \langle y \rangle$ for some $y \in K$ such that: $\langle y \rangle[p] = \langle x \rangle$. Let $0(y) = p^k$. Then the Ulm–Kaplansky invariants $f_\alpha(K)$ are zero for $\omega \le \alpha < \omega + k - 1$, while $f_{\omega+k-1}(K) = 1$. Furthermore since K is reduced and has non-zero elements of infinite height. There exists a strictly increasing sequence $\{n_i\}_{i=1}^\infty$ such that

$$f_{n_i-1}(K) \neq 0$$

Let $M = L \oplus B$, where L is a $(k, \{n_i\})$-Prüfer group and B is a direct sum of cyclic groups such that: $f_j(B) = f_j(K)$, if $j+1 \notin \{n_i\}_{i=1}^\infty$ and $f_{n_i-1}(B) + 1 = f_{n_i-1}(K)$, for $i = 1, \ldots$. The groups M and K are countable and have the same Ulm–Kaplansky invariants for every ordinal up to $\omega + k - 1$. Furthermore their length are the same. Therefore M is isomorphic to K by Ulm's theorem and K contains a pure subgroup isomorphic to the modified Prüfer group L. The converse is evident.

2. **Characterization of l.i.b. groups.** Recall that a group G is said to be an l.i.b. group if every idempotent endomorphism of every basic subgroup of G extends to an endomorphism of G.

Lemma 2.1. *Summands of* l.i.b. *groups are* l.i.b. *groups.*

Lemma 2.2. *Let G be a reduced* l.i.b. *group and B a basic subgroup of G. Then every decomposition $B = B_1 \oplus B_2$ induces a decomposition $G = G_1 \oplus G_2$ where $B_i \subset G_i$, $i = 1, 2$. Furthermore G_i is uniquely determined by B_i, $i = 1, 2$.*

Proof. Let $f: B \to B$ be the projection on B_2 induced by the given decomposition of B. Then f extends to h: $G \to G$. Since $(f^2 - f)(B) = 0$, $(h^2 - h)(B) = 0$ and $(h^2 - h)(G)$ is a divisible subgroup of G. Therefore $(h^2 - h)(G) = 0$. Let $G_1 = \ker h$, $G_2 = h(G)$ then $G = G_1 \oplus G_2$ and $B_i \subset G_i$, $i = 1, 2$. Now G_i/B_i is the divisible part of G/B_i, $i = 1, 2$. Therefore G_i is uniquely determined by B_i, $i = 1, 2$.

We show next that reduced l.i.b. groups have no non-zero elements of infinite height. We need two more lemmas:

Lemma 2.3. *Let M, N be subgroups of a group G such that $M + N = G$ and $M^1 = N^1 = M \cap N$. The following holds*:

(1) *If A and B are respectively pure subgroups of M and N, then $A + B$ is a pure subgroup of G.*

(2) *If A and B are basic subgroups respectively of M and N, then $A + B$ is a basic subgroup of G.*

Proof. Let $p^k g \in A + B$, where $g \in G$, $k \in N$. Then $g = m + n$ for some $m \in M$ and $n \in N$. Thus, $p^k(m+n) = a + b$, for some $a \in A$ and $b \in B$. Therefore, $p^k m - a = b - p^k n \in M \cap N = M^1 = N^1$. Hence, there exists $m' \in M$ and $n' \in N$, such that: $p^k m - a = p^k m'$ and $b - p^k n = p^k n'$. That is: $a = p^k(m - m')$, $b = p^k(n + n')$. By the purity of A and B respectively in M and N, there exists $a' \in A$ and $b' \in B$ such that: $a = p^k a'$, $b = p^k b'$. Thus, $A + B$ is seen to be pure in G. Now if A and B are respectively basic subgroups of M and N, then, first: $A \cap B = 0$. Indeed, $A \cap M^1 = 0$ and $B \cap N^1 = 0$, and $A \cap B \subset M \cap N = M^1 = N^1$. Therefore, $A \oplus B$ is a direct sum of cyclic groups which is pure in G. It remains to show that $G/(A \oplus B)$ is divisible. Let $g = m + n \in M + N = G$. Since M/A is divisible, there exists $m' \in M$ such that: $m + A = pm' + A$, similarly there exists $n' \in N$ such that: $n + B = pn' + B$. Therefore $g + (A \oplus B) = p(m' + n' + (A \oplus B))$ and $G/(A \oplus B)$ is divisible.

LEMMA 2.4. *Let G be a $(k, \{n_i\})$-Prüfer group. Let N, the set of natural numbers, be partitioned into two infinite sets I and J. Let $M = \langle \bar{x}_0, \{\bar{x}_i\}_{i \in I}\rangle$ and $N = \langle \bar{x}_0, \{\bar{x}_j\}_{j \in J}\rangle$. Then $G = M + N$, $M^1 = N^1 = M \cap N = \langle \bar{x}_0 \rangle$.*

Proof. M and N are simply modified Prüfer groups corresponding respectively to $(k, \{n_i\}_{i \in I})$ and $(k, \{n_j\}_{j \in J})$. Therefore, $M^1 = N^1 = \langle \bar{x}_0 \rangle$. Clearly, $G = M + N$. Now, $M \cap N = \langle \bar{x}_0 \rangle$ follows from the fact that: $\sum_{i=0}^{m} a_i \bar{x}_i = 0$, implies that $a_i = c_i p^{n_i}$, for $i = 1, \ldots, m$, and therefore, $a_i \bar{x}_i = c_i p^{n_i} x_i = c_i \bar{x}_0$.

PROPOSITION 2.5. *Let G be a reduced* l.i.b. *group. Then $G^1 = 0$.*

Proof. Suppose $G^1 \neq 0$. From Theorem 1.4, G contains a pure subgroup K isomorphic to a modified Prüfer group. From Lemma 2.4, K contains two subgroups M and N such that $M + N = K$ and $K^1 = M^1 = N^1$. Let A and B be, respectively, basic subgroups of M and N. By Lemma 2.3, $A \oplus B$ is a basic subgroup of K. Extend $A \oplus B$ by a subgroup C such that $A \oplus B \oplus C$ is a basic subgroup of G. Now apply Lemma 2.2 to obtain a decomposition: $G = G_A \oplus G_B \oplus G_C$ and G_X/X is the divisible subgroup of G/X for $X \in \{A, B, C\}$. However, M/A is divisible and so is N/B. Therefore, $M \subset G_A$ and $N \subset G_B$. This contradicts the fact that $M \cap N \neq 0$. Therefore $G^1 = 0$.

THEOREM 2.6. *A primary group G is an* l.i.b. *group if and only if G is the direct sum of a divisible and a torsion complete group.*

Proof. Let G be an l.i.b. group, $G = D \oplus R$, where D is the divisible part of G and R is reduced. By Lemma 2.1 R is an l.i.b. group. From Proposition 2.5, $R^1 = 0$. From Lemma 2.2, if B is a basic subgroup of R and $B = B_1 \oplus B_2$, $R = R_1 \oplus R_2$, where R_i/B_i is the divisible part of R/B_i, $i = 1, 2$. But $(R/B_i)/(R_i/B_i) \simeq R/R_i$ is a group without non-zero elements of infinite height. Therefore, R_i is the closure of B_i $(i = 1, 2)$ in the p-adic topology of R. By Theorem 3 of [6], it follows that R is torsion complete. Conversely, let $G = D \oplus R$ where D is divisible and R is torsion complete. Let B be a basic subgroup of G. Then $B \cap D = 0$ and there exists K, a D-high subgroup of G containing B. By the absolute direct summand property of divisible groups

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$G = D \oplus K$. Clearly K is torsion complete. Thus every decomposition of B extends to a decompositon of K and thus every idempotent endomorphism of B extends to an endomorphism of G.

REMARKS. Theorem 3 of [6] is proved with a great deal of computations. Our Theorem 2.6 can be proved differently via the result on quasi-complete groups of [5]. Indeed a reduced l.i.b. group is easily seen to satisfy the property that the closure in the p-adic topology of a pure subgroup is a summand of the group. This fact can be established using Proposition 2.5 and Lemma 2.2. Knowing this, one can use Proposition 74.9 of [4] to show that the reduced part of an l.i.b. group is torsion-complete. This approach seems free of cumbersome computations, however this is not the case. Indeed the proof of Proposition 74.9 of [4] is indirectly based on Proposition 74.3 in which not so easy calculations deriving from the statement in the sixth line of the proof, have been omitted both in [4] and in the original [5].

In conclusion, although l.i.b. groups have been characterized when they are primary, we think that it would be interesting to characterize them in general using the notion of p-basic subgroups. We have briefly considered these groups previously in connection with quasi-p-pure-injective groups in [2] p. 580.

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