which is exactly what it would be using the theorem.

But the theorem is true for $n = 1$ and $n = 2$, so it is true for $n = 3$, and hence similarly for all positive integral values of n.

Applications. (i) *Expansion for* $2 \cos n\,\theta$ *in terms of* $2 \cos \theta$. Let $x = \cos\theta + i \sin\theta$, $y = \cos\theta - i\sin\theta$; then $x + y = 2\cos\theta$ and $xy = 1$. Also $x^n = \cos n\,\theta + i \sin n\,\theta$; $y^n = \cos n\,\theta - i \sin n\,\theta$ so that $x^n + y^n = 2 \cos n\,\theta$. Substituting these in the theorem, we obtain

$$2 \cos n\,\theta = \sum_{r=0}^{N} (-)^r \frac{n}{n-r} \binom{n-r}{r} (2\cos\theta)^{n-2r}.$$

(ii) *To find the equation whose roots are the n-th powers of those of* $ax^2 + bx + c = 0$. Let the roots of the given equation be α and β. Then the required equation is $x^2 - (\alpha^n + \beta^n)\,x + (\alpha\beta)^n = 0$

$$i.e.\quad x^2 - \left[\sum_{r=0}^{N} (-)^r \frac{n}{n-r} \binom{n-r}{r} (\alpha+\beta)^{n-2r} (\alpha\beta)^r\right] x + (\alpha\beta)^n = 0$$

or, since $\alpha + \beta = -\,b/a$ and $\alpha\beta = c/a$,

$$a^n x^2 - \left[\sum_{r=0}^{N} (-)^{n-r} \frac{n}{n-r} \binom{n-r}{r} a^r b^{n-2r} c^r\right] x + c^n = 0.$$

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Dirichlet's Integrals.

By L. J. MORDELL, F.R.S.

The integrals referred to here are those given by the following *Theorem*:

$$\text{Let } I = \int\!\!\int\!\!\int f(x+y+z)\, x^{l-1}\, y^{m-1}\, z^{n-1}\, dx\, dy\, dz,$$

where the integration is extended over

$$x \geqq 0,\ y \geqq 0,\ z \geqq 0, \quad a \leqq x + y + z \leqq b,$$

and $l > 0$, $m > 0$, $n > 0$, $\quad 0 \leqq a \leqq b < \infty$.

$$\text{Let } J = \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n)} \int_a^b t^{l+m+n-1} f(t)\, dt.$$

Then $I = J$ when either,

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(1) $f(t)$ is continuous for $a \leqq t \leqq b$ (and this includes the case when $f(t)$ is bounded and has only a finite number of discontinuities), or (2) $f(t)$ is bounded and monotone for $a \leqq t \leqq b$ (and this includes the case when $f(t)$ is of bounded variation in $a \leqq t \leqq b$, but the same proof gives both).

Probably most people have been not quite content with the usual proof obtained by changing the variables. Either the procedure seems to require more rigour than that usually given, or seems less simple when rigorous as, for example, in Whittaker and Watson's *Modern Analysis*. There is also the restriction that $f(t)$ should be continuous. My feeling of dissatisfaction led me to the following proof. I do not claim any great novelty for it. It is in idea only the old proofs given by Todhunter and Williamson but put in a modern form. But still it may be welcome to many.

Lemma :

It is easily proved that the result is true when $f(t) \equiv 1$ on integrating successively for x, y, z.

Now divide the interval $a \leqq t \leqq b$ into sub-intervals by the points $a = t_0, < t_1, < t_2, \ldots < t_r = b$. Denote by M_s, m_s the upper and lower bounds of $f(t)$ in $t_{s-1} \leqq t \leqq t_s$.

Then since I exists as a multiple integral,

$$I = \sum_{s=1}^{r} \iiint f(x+y+z)\, x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz,$$

the typical integral now being taken over

$$t_{s-1} \leqq x+y+z \leqq t_s, \qquad x \geqq 0,\ y \geqq 0,\ z \geqq 0.$$

Hence by the lemma,

$$\sum_{s=1}^{r} m_s \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n)} \int_{t_{s-1}}^{t_s} t^{l+m+n-1}\, dt \leqq I \leqq \sum_{s=1}^{r} M_s \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n)} \int_{t_{s-1}}^{t_s} t^{l+m+n-1}\, dt.$$

Since J exists as a simple integral

$$J = \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n)} \sum_{s=1}^{r} \int_{t_{s-1}}^{t_s} t^{l+m+n-1} f(t)\, dt.$$

Write $$D = \sum_{s=1}^{r} \int_{t_{s-1}}^{t_s} t^{l+m+n-1} \left[f(t) - m_s \,(\text{or } M_s) \right] dt.$$

To show $I = J$, we need only prove now that

$$D \to 0 \text{ when } t_s - t_{s-1} \to 0 \text{ for } s = 1, 2, \ldots r \text{ and } r \to \infty.$$

Now $$|D| < \sum_{s=1}^{r} (M_s - m_s)\,(t_s^{l+m+n} - t_{s-1}^{l+m+n}) \,/\, (l+m+n).$$

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If $f(t)$ is continuous, take the subdivisions so fine that $M_s - m_s < \epsilon$. Then

$$|D| < \epsilon\,(b^{l+m+n} - a^{l+m+n}) / (l + m + n),$$
$$\to 0 \text{ as } \epsilon \to 0.$$

If $f(t)$ is monotone or of bounded variation, take the subdivisions so fine that $t_s^{l+m+n} - t_{s-1}^{l+m+n} < \epsilon$. Then

$$(l + m + n)\,|D| < \epsilon \sum_{s=1}^{r} (M_s - m_s),$$
$$< \epsilon\,|f(b-0) - f(a+0)|, \text{ or } \epsilon K, \text{ say},$$

according as $f(t)$ is monotone or of bounded variation,

$$\to 0 \text{ as } \epsilon \to 0.$$

Hence the theorem is proved. The result clearly also holds for the limiting cases $a = 0$ or $b = \infty$ when $f(t)$ is not bounded in $a \leqq t \leqq b$ if I, J then exist as improper or infinite integrals, and $f(t)$ satisfies (1) or (2) in every closed sub-interval of $0 < t < \infty$.

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On Desargues Theorem

By J. H. M. WEDDERBURN.

The usual proofs of Desargues Theorem employ either metrical or analytical methods of projection from a point outside the plane; and if it is attempted to translate the analytical proof by the von Stuadt-Reye methods, the result is very long and there is trouble with coincidences. It is the object of this note to give a short geometrical proof which, in addition to the usual axioms of incidence and extension, uses only the assumption that a projectivity which leaves three points on a line unchanged also leaves all points on it unchanged. Degenerate cases are excluded as having no interest.

LEMMA 1. *If the triangles ABC, $A'B'C'$ are in perspective from O, and if B lies on $A'C'$, B' on AC, then the triangles are coaxial.*

Let $X = (BC,\ B'C')$, $Y = (ACB',\ A'C'B)$, $Z = (AB,\ A'B')$, $D = (B'C',\ OAA')$, $F = (YZ,\ OBB')$, $E = (YZ,\ BC)$, $E' = (YZ,\ B'C'D)$; then $YZFE$ projects from B into $YAB'C$, which projects into