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A CHARACTERIZATION OF THE EXISTENCE OF A SOUSLIN LINE

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The main purpose of this paper is to show that there exists a Souslin line if and only if there exists a countable chain condition space which is not weak-separable but has a generic π-base. If I is the closure of the isolated points in a space X, then X is said to be weak-separable if a first category set is dense in $X - I$. A π-base $\underline{\underline{B}}$ is said to be generic if, whenever a member of $\underline{\underline{B}}$ is included in the disjoint union of members of $\underline{\underline{B}}$, it is included in one of them.

Miller [*4*], showed that the equivalence of the existence of a Souslin line and the existence of a Souslin tree. Furthermore Tall [*7*] obtained a topological condition of the existence of a Souslin line without mentioning the orderability. In this paper we shall give a simpler equivalence condition for the existence of a Souslin line.

A *Souslin line* is a linearly ordered countable chain condition topological space which is not separable. A *tree* is a partial ordered set in which for each element, the set of its predecessors is well ordered. A tree T is said to be *Souslin* if T is uncountable for which every chain and every antichain are countable.

It is well known that the existence of a Souslin line can not be proved or refuted from the usual axioms of set theory; see Solovay and Tennenbaum [*6*].

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DEFINITION 1. Let X be a topological space. A *π-base* $\underline{\underline{B}}$ for X is a collection of open sets of X such that for each non-empty open set there exists a member of $\underline{\underline{B}}$ which is included in it. A π-base $\underline{\underline{B}}$ is said to be *generic* if, whenever a member of $\underline{\underline{B}}$ is included in the disjoint union of $\underline{\underline{B}}$, it is included in one of them.

Next we shall give a new class of topological spaces.

DEFINITION 2. A topological space X is said to be *weak-separable* if a first category set is dense in $X - I$, where I is the closure of the isolated points in X.

Clearly, separable space is weak-separable. But the converse is not true; see the example on p.

LEMMA 1. *Let $I = (u, v)$ be a separable open interval without isolated points of a linearly ordered topological space; then I has a generic π-base for I.*

Proof. Let $A = \{a_k : k < \omega_0\}$ be a dense subset of I. By induction on ω_0, we will construct $\underline{\underline{B}}_m$ of a finite family of disjoint intervals and a finite subset C_m of A for each $m < \omega_0$. First let $\underline{\underline{B}}_0 = \{I\}$ and $C_0 = \{a_0\}$. Assume we have constructed $\underline{\underline{B}}_m = \{(x_n, y_n) : n < m_0\}$ and C_m, where x_n, y_n are elements of $A \cup \{u, v\}$ for each $n < m_0$. For each $n < m_0$, let k_n be the least k such that a_k is in (x_n, y_n). Then put

$$C_{m+1} = \{a_{k_n} : n < m_0\}$$

and

$$\underline{\underline{B}}_{m+1} = \{(x_n, a_{k_n}) : n < m_0\} \cup \{(a_{k_n}, y_n) : n < m_0\} .$$

Then easily we can prove that $\bigcup_{m<\omega_0} \underline{\underline{B}}_m$ is a generic π-base for I.

THEOREM 1. *A Souslin line is a countable chain condition space which is not weak-separable but has a generic π-base.*

Proof. Let S be a Souslin line. S has a countable chain condition by definition. Since S is linear and has countable chain condition, the isolated points of S are countable, and every nowhere dense subset of S is separable. Thus, since S is not separable, S is not weak-separable.

Next let $\underline{\underline{I}} = \{\{p\} : p \text{ is an isolated point of } S\}$, and $\underline{\underline{S}} = \{I_n : n < \omega_0\}$ be a maximal family of separable open intervals of S. Then, by Lemma 1, there exists a generic π-base $\underline{\underline{B}}(I_n)$ for I_n for each $n < \omega_0$. Next, by induction on ω_1, we will construct $\underline{\underline{B}}_\alpha$ of pairwise disjoint open intervals of S and countable subsets C_α and D_α of S for each $\alpha < \omega_1$. First, let $C_0 = \emptyset$ and let $\underline{\underline{B}}_0 = \{(x_n, y_n) : n < \omega_0\}$ be a maximal family of pairwise disjoint open intervals of S such that each member of $\underline{\underline{B}}_0$ is disjoint from $(\cup \underline{\underline{I}}) \cup (\cup \underline{\underline{S}})$. Furthermore for each $n < \omega_0$, take $z_n \in (x_n, y_n)$, and put $D_0 = \{x_n, y_n, z_n : n < \omega_0\}$. Next assume that we have constructed C_β, D_β and $\underline{\underline{B}}_\beta$ for each $\beta < \alpha$. Let $C_\alpha = \bigcup_{\beta<\alpha} D_\beta$ and $\underline{\underline{B}}_\alpha = \{(x_n, y_n) : n < \omega_0\}$ be a maximal family of pairwise disjoint open intervals of S such that each member of $\underline{\underline{B}}_\alpha$ is disjoint from $(\cup \underline{\underline{I}}) \cup (\cup \underline{\underline{S}}) \cup C_\alpha$. Furthermore, for each $n < \omega_0$, take $z_n \in (x_n, y_n)$, and put $D_\alpha = \{x_n, y_n, z_n : n < \omega_0\}$.

CLAIM 1. $S = \overline{\cup \underline{\underline{I}}} \cup \overline{\cup \underline{\underline{S}}} \cup \big(\bigcup_{\alpha<\omega_1} \overline{C_\alpha}\big)$.

To prove this we assume $p \notin \overline{\cup \underline{\underline{I}}} \cup \overline{\cup \underline{\underline{S}}} \cup \big(\bigcup_{\alpha<\omega_1} \overline{C_\alpha}\big)$. Then for each $\alpha < \omega_1$, there exists a member I_α of $\underline{\underline{B}}_\alpha$ such that $p \in I_\alpha$. Since $I_{\alpha+1} \subset I_\alpha$ and by the construction, there exists a non-empty open interval J_α included in $I_\alpha - I_{\alpha+1}$ for each $\alpha < \omega_1$. Hence $\{J_\alpha : \alpha < \omega_1\}$ is an uncountable family of pairwise disjoint open intervals, which is a contradiction.

CLAIM 2. $\underline{\underline{B}} = \cup\{\underline{\underline{B}}(I_n) : I_n \in \underline{\underline{S}}\} \cup \underline{\underline{I}} \cup \cup\{\underline{\underline{B}}_\alpha : \alpha < \omega_1\}$ is a π-base for S.

To prove this let (x, y) be a non-empty interval of S. If (x, y)

has an isolated point p then $\{p\} \subset (x, y)$ and $\{p\} \in I$. If (x, y) has no isolated point and $(x, y) \cap I_n \neq \emptyset$ for some $I_n \in \underline{\underline{S}}$, then there exists an $I \in \underline{\underline{B}}(I_n)$ such that $I \subset (x, y) \cap I_n$, because $\underline{\underline{B}}(I_n)$ is a π-base for I_n. Let (x, y) has no isolated point and $(x, y) \cap (\cup \underline{\underline{S}}) = \emptyset$. Take $a, b \in S$ such that $x < a < b < y$, then there exists an $\alpha < \omega_1$ such that $a, b \in \overline{C_\alpha}$. By the maximality of $\underline{\underline{S}}$, $(a, b) - \overline{C_\alpha} \neq \emptyset$. Then by the construction of $\underline{\underline{B}}_\alpha$, there exists an $I \in \underline{\underline{B}}_\alpha$ such that $I \subset (a, b) - \overline{C_\alpha}$. Hence $I \subset (x, y)$. Thus $\underline{\underline{B}}$ is a π-base for S.

Also one can easily prove the genericity of $\underline{\underline{B}}$.

This completes the proof.

REMARK. If there exists a Souslin line S, then one can construct a connected Souslin line S^*; see Kunen [2]. Since a connected linearly ordered space is locally connected, all connected open subsets of S^* are generic π-bases for S^*.

THEOREM 2. *If X is a countable chain condition space which is not weak-separable but has a generic π-base $\underline{\underline{G}}$, then $\underline{\underline{G}}$, partially ordered by reverse inclusion, contains a Souslin tree.*

Proof. Since X is a countable chain condition we can assume without loss of generality that X has no isolated point. By induction, for each $\alpha < \omega_1$, we choose $T_\alpha \subset \underline{\underline{G}}$ as follows. Let T_0 be a maximal family of infinite pairwise disjoint members of $\underline{\underline{G}}$. Having chosen $\{T_\beta : \beta < \alpha\}$, define $\underline{\underline{G}}_\alpha = \{g \in \underline{\underline{G}} : g \subset \bigcap_{\beta<\alpha} (\cup T_\beta)\}$ and T'_α to be a maximal family of infinite pairwise disjoint members of $\underline{\underline{G}}_\alpha$. For $g' \in T'_\alpha$, let $\underline{\underline{G}}_\alpha(g') = \{g \in \underline{\underline{G}}_\alpha : g \subset g'\}$ and $T_\alpha(g')$ is a maximal family of infinite pairwise disjoint members of $\underline{\underline{G}}_\alpha(g')$. Finally let $T_\alpha = \{T_\alpha(g') : g' \in T'_\alpha\}$. Since $\underline{\underline{G}}$ is generic, $T = \bigcup_{\alpha<\omega_1} T_\alpha$, ordered by reverse inclusion, is a tree, and X is a countable chain condition, and by construction, each chain and each antichain are countable. To see that T

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is uncountable, we will show that $X - \operatorname{int}\left(\bigcap_{\beta<\alpha} \bigcup T_\beta\right)$ has a dense first category set for each $\alpha < \omega_1$. Then, since X is not weak-separable, $\underline{\underline{G}}_\alpha \neq \emptyset$ and thus $T_\alpha \neq \emptyset$. Hence T is uncountable. Clearly $X - \operatorname{int}\left(\bigcup T_0\right) = X - \left(\bigcup T_0\right)$ is nowhere dense. Assume that $X - \operatorname{int}\left(\bigcap_{\beta<\alpha} \bigcup T_\beta\right)$ has a dense first category set. By the maximality of T'_α, $\operatorname{int}\left(\bigcap_{\beta<\alpha} \bigcup T_\beta\right) - \left(\bigcup T'_\alpha\right)$ is nowhere dense. Also $g' - \left(\bigcup T_\alpha(g')\right)$ is nowhere dense for each $g' \in T_\alpha$. Since

$$X - \left(\operatorname{int}\left(\bigcap_{\beta<\alpha+1} \bigcup T_\beta\right)\right) = \left(X - \left(\operatorname{int}\left(\bigcap_{\beta<\alpha} \bigcup T_\beta\right)\right)\right)$$
$$\cup \left(\operatorname{int}\left(\bigcap_{\beta<\alpha} \bigcup T_\beta\right) - \left(\bigcup T'_\alpha\right)\right) \cup \bigcup\left\{g' - \left(\bigcup T_\alpha(g')\right) : g' \in T'_\alpha\right\}$$

and X is a countable chain condition, $X - \left(\operatorname{int}\left(\bigcap_{\beta<\alpha+1} \bigcup T_\beta\right)\right)$ has a dense first category set. Assume that α is limit and

$$X - \left(\operatorname{int}\left(\bigcap_{\beta<\gamma} \bigcup T_\beta\right)\right) = \overline{\bigcup_{n<\omega_0} F_n(\gamma)}$$

for each $\gamma < \alpha$, where $F_n(\gamma)$ is a nowhere dense set of X for each $n < \omega_0$. Then clearly

$$X - \left(\operatorname{int}\left(\bigcap_{\beta<\alpha} \bigcup T_\beta\right)\right) = \overline{\bigcup_{\gamma<\alpha} \left(X - \left(\bigcap_{\beta<\gamma} \bigcup T_\beta\right)\right)} = \overline{\bigcup_{\gamma<\alpha} \bigcup_{n<\omega_0} F_n(\gamma)} .$$

Hence $X - \left(\operatorname{int}\left(\bigcap_{\beta<\alpha} \bigcup T_\beta\right)\right)$ has a dense first category set. This completes the proof.

The following is a consequence of the above theorems.

COROLLARY. *The following conditions are equivalent:*

(1) *there exists a Souslin line;*

(2) *there exists a connected Souslin line;*

(3) *there exists a Souslin tree;*

(4) *there exists a countable chain condition space which is not weak-separable but has a generic* π*-base;*

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(5) *there exists a locally connected countable chain condition space which is not weak-separable;*

(6) *there exists a countable chain condition space* X *which is not weak-separable such that*

$$\{x \in X : X \text{ is locally connected at } x\}$$

is dense in X *;*

(7) *there exists a countable chain condition space which is not weak-separable but has a* π*-base consisting of connected open sets.*

Proof. The equivalence of (1) and (2) is due to Kunen [*2*], and the equivalence of (1) and (3) is due to Miller [*4*]. The implication of (1) → (4) follows from Theorem 1, but the implication of (2) → (4) follows only from the remark of Theorem 1. The implication of (4) → (3) follows from Theorem 2. The implication of (5) → (6) → (7) → (4) is clear. (2) → (5) also follows from Theorem 1 and its remark.

REMARK. Since the existence of a Souslin line is independent of the Zermelo-Frankel set theory with the Axiom of Choice, the existence of a locally connected countable chain condition space, which is not weak-separable, is also independent of the Zermelo-Frankel set theory with the Axiom of Choice by the corollary. But there exists a locally connected countable chain condition space which is not separable; see the next example.

EXAMPLE. Let κ be a cardinal such that $\kappa > 2^{\aleph_0}$. Then I^{κ} is a countable chain condition by Engelking [*1*, 2.3.17], where I denotes the unit interval of the real line. Moreover I^{κ} is not separable; see Pondiczery [5] and Marczewski [*3*]. Furthermore, since I is connected and locally connected, I^{κ} is locally connected. But I^{κ} is weak-separable.

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