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Polarized endomorphisms of uniruled varieties

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With an appendix by Y. Fujimoto and N. Nakayama

Dedicated to Professor M. Miyanishi on the occasion of his seventieth birthday

Abstract

We show that polarized endomorphisms of rationally connected threefolds with at worst terminal singularities are equivariantly built up from those on $\mathbb{Q}$-Fano threefolds, Gorenstein log del Pezzo surfaces and $\mathbb{P}^1$. Similar results are obtained for polarized endomorphisms of uniruled threefolds and fourfolds. As a consequence, we show that every smooth Fano threefold with a polarized endomorphism of degree greater than one is rational.

1. Introduction

We work over the field $\mathbb{C}$ of complex numbers. We study *polarized* endomorphisms $f : X \to X$ of varieties X, i.e., those f with $f^*H \sim qH$ for some $q > 0$ and some ample line bundle H. Every surjective endomorphism of a projective variety of Picard number one is polarized. If $f = [F_0 : F_1 : \cdots : F_n] : \mathbb{P}^n \to \mathbb{P}^n$ is a surjective morphism and $X \subset \mathbb{P}^n$ a f-stable subvariety, then $f^*H \sim qH$ and hence $f|X : X \to X$ is polarized; here $H \subset X$ is a hyperplane and $q = \deg(F_i)$. If A is an abelian variety and $m_A : A \to A$ the multiplication map by an integer $m \neq 0$, then $m_A^*H \sim m^2H$ and hence m_A is polarized; here $H = L + (-1)^*L$ with L an ample divisor, or H is any ample divisor with $(-1)^*H \sim H$. One can also construct polarized endomorphisms on quotients of $\mathbb{P}^n$ or A. So there are many examples of polarized endomorphisms f. See [Zha06] for the many conjectures on such f.

From the arithmetical point of view, given a polarized endomorphism $f : X \to X$ of degree $q^{\dim X}$ and defined over $\mathbb{Q}$, one can define a unique height function $h_f : X(\overline{\mathbb{Q}}) \to \mathbb{R}$ such that $h_f(f(x)) = qf(x)$. Further, x is f-preperiodic if and only if $h_f(x) = 0$; see [Zha06, § 4] for more details.

In [NZ07b], it is proved that a normal variety X with a non-isomorphic polarized endomorphism f either has only canonical singularities with $K_X \sim_{\mathbb{Q}} 0$ (and further is a quotient of an abelian variety when $\dim X \leqslant 3$), or is uniruled so that f descends to a polarized endomorphism f_Y of the non-uniruled base variety Y (so $K_Y \sim_{\mathbb{Q}} 0$) of a specially chosen maximal rationally connected fibration $X \dashrightarrow Y$. By the induction on dimension and since Y has a dense set of f_Y-periodic points $y_0, y_1, \ldots$ (cf. [Fak03, Theorem 5.1]), the study of polarized endomorphisms is then reduced to that of rationally connected varieties Γ_{y_i} as fibres of the graph $\Gamma = \Gamma(X/Y)$ (cf. [NZ07b, Remark 4.3]).

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The study of non-isomorphic endomorphisms of singular varieties (like Γ_{y_i} above) is very important from the dynamics point of view, but is very hard even in dimension two and especially for rational surfaces; see [Fav], [Nak08] (about 150 pages).

In this paper, we consider polarized endomorphisms of rationally connected varieties (or more generally of uniruled varieties) of dimension greater than or equal to three. Theorems 1.1–1.4 below and Theorems 3.2–3.4 in § 3, are our main results.

Theorem 1.1. *Let X be a $\mathbb{Q}$-factorial n-fold, with $n \in \{3, 4\}$, having only log terminal singularities and a polarized endomorphism f of degree $q^n > 1$. Let $X = X_0 \dashrightarrow X_1 \cdots \dashrightarrow X_r$ be a composite of divisorial contractions and flips. Replacing f by its positive power, we have the following.*

(1) *The dominant rational maps $g_i : X_i \dashrightarrow X_i$ $(0 \leqslant i \leqslant r)$ (with $g_0 = f$) induced from f, are all holomorphic.*
(2) *Let $\pi : X_r \to Y$ be an extremal contraction with $\dim Y \leqslant 2$. Then g_r is polarized and it descends to a polarized endomorphism $h : Y \to Y$ of degree $q^{\dim Y}$ with $\pi \circ g_r = h \circ \pi$.*

The result above reduces the study of (X, f) to (X_r, g_r) where the latter is easier to be dealt with since X_r has a fibration structure preserved by g_r. *The existence of such a fibration $\pi : X_r \to Y$ is guaranteed* when X is uniruled by the recent development in MMP (Minimal Model Program). The relation between the two pairs is very close because f^{-1}, as seen in Theorem 3.2, preserves the maximal subset of X where the birational map $X \dashrightarrow X_r$ is not holomorphic.

Theorem 1.2. *Let X be a $\mathbb{Q}$-factorial threefold having only terminal singularities and a polarized endomorphism of degree $q^3 > 1$. Suppose that X is rationally connected. Then we have the following.*

(1) *There is an $s > 0$ such that $(f^s)^*{}_{|N^1(X)} = q^s\,\mathrm{id}$. We then call such f^s cohomologically a scalar.*
(2) *Either X is rational or $-K_X$ is big.*
(3) *There are only finitely many irreducible divisors $M_i \subset X$ with the Iitaka D-dimension $\kappa(X, M_i) = 0$.*

Theorem 1.2(3) above apparently does not hold for $X = S \times \mathbb{P}^1$, where S is any rational surface with infinitely many (-1)-curves and hence S has no endomorphisms of degree greater than one by [Nak02, Proposition 10]; the blowup of nine general points of $\mathbb{P}^2$ is such S as observed by Nagata.

Theorem 1.2(1) above strengthens (in our situation) Serre's result [Ser60] on a conjecture of Weil (in the projective case): (*Serre*) if f is a polarized endomorphism of degree $q^{\dim X} > 1$ of a smooth variety X then every eigenvalue of $f^*|N^1(X)$ has the same modulus q.

The proof of Theorem 1.3 below is done without using the classification of smooth Fano threefolds. This result has been reproved in [Zha08a] where f is assumed to be only of degree greater than one but not necessarily polarized.

Theorem 1.3. *Let X be a smooth Fano threefold with a polarized endomorphism f of degree greater than one. Then X is rational.*

A klt (Kawamata log terminal) $\mathbb{Q}$-Fano variety has only finitely many extremal rays. A similar phenomenon occurs in the quasi-polarized case (cf. 2.1).

Theorem 1.4. *Let X be a $\mathbb{Q}$-factorial rationally connected threefold having only Gorenstein terminal singularities and a quasi-polarized endomorphism of degree greater than one. Then X has only finitely many K_X-negative extremal rays.*

As explained in Remark 1.5 below, the building blocks of polarized endomorphisms on rationally connected varieties should be those on $\mathbb{Q}$-Fano varieties of Picard number one.

Remark 1.5. (1) The Y in Theorem 1.1 is $\mathbb{Q}$-factorial and has at worst log terminal singularities; see [Nak04].

(2) Suppose that the X in Theorem 1.1 is rationally connected. Then Y is also rationally connected. Suppose further that X has at worst terminal singularities and $(\dim X, \dim Y) = (3, 2)$. Then Y has at worst Du Val singularities by [MP08, Theorem 1.2.7]. So there is a composition $Y \to \hat{Y}$ of divisorial contractions and an extremal contraction $\hat{Y} \to B$ such that either $\dim B = 0$ and $\hat{Y}$ is a Gorenstein log del Pezzo surface of Picard number one, or $\dim B = 1$ and $\hat{Y} \to B \cong \mathbb{P}^1$ is a $\mathbb{P}^1$-fibration with all fibres irreducible. After replacing f by its power, h descends to polarized endomorphisms $\hat{h} : \hat{Y} \to \hat{Y}$, and $k : B \to B$ (of degree $q^{\dim B}$); see Theorems 2.7.

(3) By [Fak03, Theorem 5.1], there are dense subsets $Y_0 \subset Y$ (for the Y in Theorem 1.1) and $B_0 \subset B$ (when $\dim B = 1$) such that for every $y \in Y_0$ (respectively $b \in B_0$) and for some $r(y) > 0$ (respectively $r(b) > 0$), $g^{r(y)}|(X_r)_y$ (respectively $\hat{h}^{r(b)}|\hat{Y}_b$) is a well-defined polarized endomorphism of the Fano fibre.

The difficulty 1.6. In Theorem 1.1, if $X \to X_1$ is a divisorial contraction, one can descend a polarized endomorphism f on X to an endomorphism on X_1, but the latter may not be polarized any more because the pushforward of a nef divisor may not be nef in dimension greater than or equal to three (the first difficulty). If $X \dashrightarrow X_1$ is a flip, then in order to descend f on X to some holomorphic f_1 on X_1, one has to show that a power of f preserves the center of the flipping contraction (the second difficulty). The second difficulty is taken care by Lemma 2.10 where the polarizedness is essentially used.

A key argument in the proof of Theorem 1.1(2) is to show that a power of f is cohomologically a scalar unless Y is a surface with torsion K_Y (this case will not happen when X is rationally connected); see Lemma 3.11.

The question below is the generalization of Theorem 1.3 and the famous conjecture: every smooth Fano n-fold of *Picard number one* with a non-isomorphic surjective endomorphism, is $\mathbb{P}^n$ (for its affirmative solution when $n = 3$, see Amerik–Rovinsky–Van de Ven [ARV99] and Hwang–Mok [HM03]).

Question 1.7. Let X be a smooth Fano n-fold with a non-isomorphic polarized endomorphism. Is X rational?

Remark 1.8. A recent preprint of Kollár and Xu [KX] showed that one can descend the endomorphism $\mathbb{P}^n \to \mathbb{P}^n$ ($[X_0, \ldots, X_n] \to [X_0^m, \ldots, X_n^m]$; $m \geqslant 2$) to some quotient $X := \mathbb{P}^n/G$ (with G finite) so that X has only terminal singularities but X is irrational, invoking a famous prime power order group action of David Saltman on Noether's problem. Thus one cannot remove the smoothness assumption in Theorem 1.3 and Question 1.7.

However, we will show in Theorem 3.3 that every rationally connected $\mathbb{Q}$-factorial threefold X with only terminal singularities, is rational, provided that X has a non-isomorphic polarized endomorphism and an extremal contraction $X \to Y$ with $\dim Y \in \{1, 2\}$. The terminal singularity assumption there is used to deduce the Gorenstein-ness of Y (when $\dim Y = 2$), making use of [MP08, Theorem 1.2.7].

It would be interesting if one could determine whether the 'terminal singularity' assumption can further be weakened to the 'log canonical singularity' in order to deduce the rationality as above.

See also [Zha08a] for the generalization of Theorem 3.3 to non-polarized endomorphisms.

For the recent development on endomorphisms of algebraic varieties, we refer to Amerik–Rovinsky–Van de Ven [ARV99], Fujimoto–Nakayama [FN08], Hwang–Mok [HM03], Hwang–Nakayama [HN08], Zhang [Zha06], as well as [NZ07a, Zha].

2. Preliminary results

2.1 Conventions

Every endomorphism in this paper is assumed to be surjective.

For a projective variety X, an endomorphism $f : X \to X$ is *polarized* or *polarized by* H (respectively *quasi-polarized* or *quasi-polarized by* H) if $f^*H \sim_{\mathbb{Q}} qH$ for some $q > 0$ and some ample (respectively nef and big) line bundle H. If f is polarized or quasi-polarized then so is its induced endomorphism on the normalization of X.

On a projective variety X, denote by $N^1(X)$ (respectively $N_1(X)$) the usual $\mathbb{R}$-vector space of $\mathbb{R}$-Cartier $\mathbb{R}$-divisors (respectively 1-cycles with coefficients in $\mathbb{R}$) modulo numerical equivalence, in terms of the perfect pairing $N^1(X) \times N_1(X) \to \mathbb{R}$. The Picard number $\rho(X)$ equals $\dim_{\mathbb{R}} N^1(X) = \dim_{\mathbb{R}} N_1(X)$. The *nef cone* $\mathrm{Nef}(X)$ is the closure in $N^1(X)$ of the ample cone, and is dual to the closed cone $\overline{\mathrm{NE}}(X) \subset N_1(X)$ generated by effective 1-cycles (Kleiman's ampleness criterion).

Denote by $S(X)$ the set of $\mathbb{Q}$-Cartier prime divisors D with $D_{|D}$ non-pseudo-effective; see [Nak04, ch. II, § 5] for the relevant material.

For a normal projective surface S, a Weil divisor is *numerically equivalent to zero* if so is its Mumford pullback to a smooth model of S. Denote by $\mathrm{Weil}(S)$ the set of $\mathbb{R}$-divisors (divisor = Weil divisor) modulo this numerical equivalence. We can also define the intersection of two Weil divisors by Mumford pulling back them to a smooth model and then taking the usual intersection.

A Weil divisor is *nef* if its intersection with every curve is non-negative. A Weil divisor D on a normal projective variety is *big* if $D = A + E$ for an ample line bundle A and an effective Weil $\mathbb{R}$-divisor E (see [Nak04, ch. II, 3.15, 3.16]).

Let $f : X \to X$ be an endomorphism and $\sigma_V : V \to X$ and $\sigma_Y : X \to Y$ morphisms. We say that f *lifts* to an endomorphism $f_V : V \to V$ if $f \circ \sigma_V = \sigma_V \circ f_V$; f *descends* to an endomorphism f_Y if $\sigma_Y \circ f = f_Y \circ \sigma_Y$.

A normal projective variety X is *Q-abelian* in the sense of [NZ07b] if $X = A/G$ with A an abelian variety and G a finite group acting freely in codimension one, or equivalently X has an abelian variety as an étale in codimension-one cover.

For a normal projective variety X, we refer to [KMM87] or [KM98] for the definition of $\mathbb{Q}$-*factoriality* and *terminal singularity* or *log terminal singularity*. An *extremal contraction* $X \to Y$ is always assumed to be K_X-negative.

We do not distinguish a Cartier divisor with its corresponding line bundle.

Lemma 2.2. *Let X be a normal projective n-fold and $f : X \to X$ an endomorphism such that $f^*H \equiv qH$ for some $q > 0$ and a nef and big line bundle H. Then we have the following.*

(1) *The above q is an integer. There is a nef and big line bundle H' such that $H' \equiv H$ and $f^*H' \sim_{\mathbb{Q}} qH'$. Hence f is quasi-polarized. Furthermore, $\deg(f) = q^n$.*

(2) *Every eigenvalue of $f^*|N^1(X)$ has modulus q.*

(3) *Suppose that* $\sigma : X \to Y$ *is a fibred space (with connected fibres) and* f *descends to an endomorphism* $h : Y \to Y$. *Then* $\deg(h) = q^{\dim Y}$. *Every eigenvalue of* $h^*|N^1(Y)$ *has modulus* q.

Proof. Parts (1) and (2) are just [NZ07b, Lemmas 2.1 and 2.3].

Set $d := \deg(h)$ and $\dim Y = k$. Then $f^*X_y \equiv dX_y$ for a general fibre X_y over $y \in Y$. Now part (3) follows from the fact that $\sigma^*N^1(Y)$ is a f^*-stable subspace of $N^1(X)$ and the calculation,

$$q^n H^{n-k}.X_y = f^*H^{n-k}.f^*X_y = q^{n-k}dH^{n-k}.X_y > 0. \qquad \square$$

2.3 Pullback of cycles

We will consider pullbacks of cycles by finite surjective morphisms. Let X be a normal projective variety. We now define a numerical equivalence, $\equiv$, for cycles in the Chow group $\mathrm{CH}_r(X)$ of r-cycles modulo rational equivalence. An r-cycle is called *numerically equivalent to zero*, denoted as $C \equiv 0$, if $H_1 \ldots H_r.C = 0$ for all Cartier divisors H_i.

If C is a non-zero effective r-cycle then C is not numerically equivalent to zero since $H^r.C > 0$ for an ample line bundle H. Denote by $[C]$ the equivalence class of all r-cycles numerically equivalent to C. Denote by $N_r(X)$ the set $\{[C] \mid C \text{ is an } r\text{-cycle with coefficients in } \mathbb{R}\}$. The usual product of an r-cycle with s line bundles naturally extends to

$$N^1(X) \times \cdots \times N^1(X) \times N_r(X) \longrightarrow N_{r-s}(X).$$

Let $f : X \to X$ be a surjective endomorphism of degree d, so f is a finite morphism. For an r-dimensional subvariety C, write $f^{-1}C = \bigcup_i C_i$ and define $f^*[C] := \sum_i e_i[C_i]$ with $e_i > 0$ chosen such that $\sum_i e_i\delta_i = d$ for $\delta_i := \deg(C_i/C)$. Then

$$f_*f^*[C] = d[C].$$

If C, C_i are not in $\mathrm{Sing}\, X$, then for the usual f^*-pullback f^*C of the cycle C, we have $[f^*C] = f^*[C]$ by having the right choice of e_i. By the linearity of the intersection form, we can linearly extend the definition to $f^*[C]$ for an arbitrary r-cycle C. Then the usual projection formula gives

$$f^*L_1 \ldots f^*L_r.f^*[C] = d\,(L_1 \ldots L_r.C).$$

Note that $f^* : N^1(X) \to N^1(X)$ is an isomorphism. With this, $[C] \to f^*[C]$ (or simply f^*C by the abuse of notation) gives a well defined map

$$f^* : N_r(X) \longrightarrow N_r(X).$$

The projection formula above implies the following in $N_{r-s}(X)$

$$f^*(L_1 \ldots L_s.C) \equiv f^*L_1 \ldots f^*L_s.f^*C.$$

Lemma 2.4. *Let* X *be a normal projective* n*-fold and* $f : X \to X$ *an endomorphism of degree* q^n *for some* $q > 0$. *Suppose that every eigenvalue of* $f^*|N^1(X)$ *has modulus* q. *Then we have the following.*

(1) *If* D *is an* r*-cycle such that* $0 \neq [D] \in N_r(X)$ *and* $f^*D \equiv aD$ *then* $|a| = q^{n-r}$.

(2) *Suppose that* S *is a* k*-dimensional subvariety of* X *with* $f^{-1}(S) = S$ *as set. Then* $f^*S \equiv q^{n-k}S$ *and* $\deg(f : S \to S) = q^k$.

(3) *For the* S *in part (2), there is a Cartier* $\mathbb{R}$*-divisor* M *on* X *such that* $M_S := M_{|S}$ *is a non-zero element in* $\mathrm{Nef}(S)$ *and* $f_{|S}^*M_S \equiv qM_S$ *in* $N^1(S)$.

(4) *If* $\rho(X) \leqslant 2$, *then* $(f^2)^*|N^1(X) = q^2\,\mathrm{id}$.

Proof. To prove part (4), we may assume that $(f^2)^*E_i \equiv a_i E_i$ for the extremal rays E_i ($1 \leqslant i \leqslant \rho(X)$) in $\mathrm{Nef}(X)$. Thus $a_i = |a_i| = q^2$ by the assumption, done!

Part (2) follows from part (1) and our definition of pullback.

To prove part (1), choose a basis $L_1, \ldots, L_\rho$ with $\rho = \rho(X)$ such that $f^*|N^1(X)$ is lower triangular. Hence $f^*L_i = qu(i)L_i +$ lower term with $|u(i)| = 1$. Since $[D] \neq 0$, for some $s > 0$, the cycle $L_s.D$ is not numerically equivalent to zero. We choose s to be minimal. Now

$$f^*(L_s.D) \equiv f^*L_s.f^*D = (qu(s)L_s + \text{lower term}).aD = aqu(s)(L_s.D).$$

Similarly, we can find $C := L_s.L_{s_1} \ldots L_{s_{r-2}}.D \in N_1(X)$ which is not numerically equivalent to zero, such that $f^*C \equiv bC$ with

$$b = aq^{r-1} \prod_{i=0}^{r-2} u(s_i), \quad (s_0 := s).$$

Since $N_1(X)$ is dual to $N^1(X)$, the eigenvalue b of $f^*|N_1(X)$ satisfies $|b| = q^{n-1}$. So $|a| = q^{n-r}$ as claimed.

To prove part (3), let $N^1(X)_{|S} \subseteq N^1(S)$ (respectively $\mathrm{Nef}(X)_{|S} \subseteq \mathrm{Nef}(S)$) be the image of $\iota^* : N^1(X) \to N^1(S)$ (respectively of the restriction of this ι^* to $\mathrm{Nef}(X)$) with $\iota : S \to X$ the closed embedding. Let $\overline{N}$ be the closure of $\mathrm{Nef}(X)_{|S}$ in $N^1(S)$. Then $\overline{N}$ spans the subspace $N^1(X)_{|S}$ of $N^1(S)$. Let λ be the spectral radius of $f^*|\overline{N}$. By the generalized Perron–Frobenius theorem in [Bir67], $f^*(M_S) \equiv \lambda(M_S)$ for a non-zero nef divisor $M_S := M_{|S}$ in $\overline{N}$ (with M a Cartier $\mathbb{R}$-divisor on X). Write $M|S = a_t L_t|S +$ lower term, with t the smallest (and $a_t \neq 0$). Then

$$\lambda a_t L_t|S + \text{lower term} = \lambda M|S = f^*(M|S) = a_t qu(t) L_t|S + \text{lower term}.$$

By the minimality of t, we have $\lambda a_t = a_t qu(t)$ and $\lambda = |\lambda| = q$. $\square$

Lemma 2.5. *Let X be a normal projective surface and $f : X \to X$ an endomorphism of degree $q^2 > 1$. Suppose that $f^*M \equiv qM$ for a non-zero nef Weil divisor. Then every eigenvalue of $f^*|\mathrm{Weil}(X)$ has modulus q.*

Proof. Let λ be the spectral radius of $f^*|\mathrm{Weil}(X)$. Then $f^*L \equiv \lambda L$ for a non-zero nef $\mathbb{R}$-divisor L. Now $q^2 L.M = f^*L.f^*M = \lambda q L.M$. Hence either $L.M > 0$ and $\lambda = q$, or $L.M = 0$. In the latter case, $M \equiv cL$ by the Hodge index theorem (on a resolution of X) and again we have $\lambda = q$.

Similarly, let μ be the spectral radius of $(f^*)^{-1}|\mathrm{Weil}(X)$ so that $(f^*)^{-1}H \equiv \mu H$ for a non-zero nef $\mathbb{R}$-divisor H. Then $f^*H \equiv \mu^{-1}H$. By the argument above, we have $\mu^{-1} = q$. The lemma follows. $\square$

Here is an easy polarizedness criterion for ruled normal surfaces.

Lemma 2.6. *Let X be a normal projective surface and $X \to B$ a $\mathbb{P}^1$-fibration. Suppose that $f : X \to X$ is an endomorphism of degree q^2 and $f^*H \equiv qH$ for a non-zero nef $\mathbb{R}$-divisor H and an integer $q > 1$. Then there is an $s > 0$ such that $(f^s)^*|\mathrm{Weil}(X) = q^s\mathrm{id}$. Hence f is polarized.*

Proof. Note that a basis of $\mathrm{Weil}(X)$ consists of some negative curves $C_1, \ldots, C_r$ in fibres, a general fibre and a multiple section. Contract C_i's to get a Moishezon normal surface Y with $\mathrm{Weil}(Y) = \mathbb{R}E_1 + \mathbb{R}E_2$ for two extremal rays $\mathbb{R}_{\geqslant 0}E_i$ of the cone $\overline{\mathrm{NE}}(X)$. By [Nak02, Proposition 10] or as in the proof of Lemma 2.9, replacing f by its power, we may assume that $f^{-1}(C_i) = C_i$ for all i.

So f descends to an endomorphism $f_Y : Y \to Y$ and we may assume that $f^*E_i \equiv e_i E_i$ for some $e_i > 0$ after replacing f by f^2.

Write $f^*C_i = a_i C_i$ with $a_i > 0$. Then $f^*|\mathrm{Weil}(X) = \mathrm{diag}[a_1, \ldots, a_r, e_1, e_2]$ with respect to the basis: $C_1, \ldots, C_r$ and the pullbacks of E_1, E_2. Now the first assertion follows from Lemma 2.5 while the second follows from the first as in Note 1 of Theorem 2.7. This proves the lemma. □

Nakayama's [Nak08, Example 4.8] (Version of January 2008) produces many examples of polarized f on abelian surfaces with non-scalar $f^*|N^1(X)$. The result below shows that this happens only on abelian surfaces and their quotients.

THEOREM 2.7. *Let X be a normal projective surface. Suppose that $f : X \to X$ is an endomorphism such that $f^*P \equiv qP$ for some $q > 1$ and some big Weil $\mathbb{Q}$-divisor P. Then we have the following.*

(1) *The above f is polarized of degree q^2.*
(2) *There is an $s > 0$ such that $(f^s)^*|\mathrm{Weil}(X) = q^s\,\mathrm{id}$ unless X is Q-abelian with $\mathrm{rank}\mathrm{Weil}(X) \in \{3, 4\}$.*

Proof. Let $P = P' + N'$ be the Zariski decomposition. Then P' is a nef and big Weil $\mathbb{Q}$-divisor. The uniqueness of such decomposition and $f^*P \equiv qP$ imply $f^*P' \equiv qP'$ and $f^*N' \equiv qN'$. Replacing P by P', we may assume that P is already a nef and big Weil $\mathbb{R}$-divisor. So $\deg(f) = (f^*P)^2/P^2 = q^2$.

Note 1. If $(f^s)^*H' \equiv q^s H'$ for an ample line bundle H' on X then f is polarized. Indeed, If we set $H := \sum_{i=0}^{s-1} (f^i)^*H/q^i$, then H is an ample $\mathbb{Q}$-divisor with $f^*H \equiv qH$, and we apply Lemma 2.2.

CLAIM 1.

(1) Every eigenvalue of $f^*|\mathrm{Weil}(X)$ has modulus q.
(2) If $(f^s)^*|\mathrm{Weil}(X)$ is scalar for some $s > 0$, then it is q^s id.

Claim 1(1) follows from Lemma 2.5 while Claim 1(2) follows from (1).
Claim 2 below is from Claim 1 and the proof of Lemma 2.4(4).

CLAIM 2. If $\rho := \dim_{\mathbb{R}} \mathrm{Weil}(X) \leqslant 2$, then $(f^2)^*|\mathrm{Weil}(X) = q^2$ id.

By [Nak02, Proposition 10] or as in the proof of Lemma 2.9, the set $S'(X)$ of negative curves on X is finite and f^{-1} induces a bijection of $S'(X)$. We may assume that $f|S'(X) = \mathrm{id}$ after replacing f by its power. Let $X \to Y$ be the composition of contractions of negative curves $C_1, \ldots, C_r$ (with r maximum) intersecting the canonical divisor negatively. Then Y is a relatively minimal Moishezon normal surface in the sense of [Sa87]. Further f descends to an endomorphism $f_Y : Y \to Y$.

Case 1. K_Y is not pseudo-effective. Then either $\mathrm{rank}\mathrm{Weil}(Y) = 2$ and there is a $\mathbb{P}^1$-fibration $Y \to B$, or $\mathrm{Weil}(Y) = \mathbb{R}[-K_Y]$ with $-K_Y$ numerically ample; see [Sa87, Theorem 3.2]. With f replaced by its square, we may assume that $f_Y^*|\mathrm{Weil}(Y) = q$ id (use Claim 1, and see the proof of Lemma 2.4(4)). Thus $f^*|\mathrm{Weil}(X) = q$ id with respect to the basis consisting of $C_1, \ldots, C_r$ and the pullback of a basis of $\mathrm{Weil}(Y)$; see Claim 1. So the theorem is true in this case.

Case 2. K_Y is pseudo-effective (and hence nef by the minimality). So K_X is also pseudo-effective. It is well known then that the ramification divisor $R_f = 0$ and hence f is étale in codimension one. Further, $K_X = f^*K_X$ and hence $K_X^2 = 0$ since $\deg(f) > 1$. If $C \in S'(X)$ is a negative curve on X then $f^*C = qC$ by Claim 1, and because of the extra assumption $f|S'(X) = \mathrm{id}$; f is ramified along C. Thus $S'(X) = \emptyset$. So $X = Y$ and K_X is nef. Also P is numerically ample. The proof is completed by the following claim.

CLAIM 3. X is Q-abelian. So $\mathrm{rank}\,\mathrm{Weil}(X) \leqslant 4$, X is $\mathbb{Q}$-factorial, and f is polarized by P which is $\mathbb{Q}$-Cartier.

Since $q^2 P.K_X = f^*P.f^*K_X = qP.K_X$, we have $P.K_X = 0$. The Hodge index theorem (applied to a resolution of X) implies that $K_X \equiv 0$ in $\mathrm{Weil}(X)$. Thus the claim follows from [Nak08, Theorem 7.1.1]. □

LEMMA 2.8. *Let X be a normal projective n-fold and $f : X \to X$ a quasi-polarized endomorphism of degree $q^n > 0$. Then we have the following.*

(1) *Suppose that $V \to X$ is a birational morphism and f lifts to an endomorphism $f_V : V \to V$. Then f_V is also quasi-polarized.*
(2) *Let $X \dashrightarrow W$ be a birational map with W being $\mathbb{Q}$-factorial, such that the dominant rational map $f_W : W \dashrightarrow W$ induced from f, is holomorphic. Then $f_W^* H_W \sim_{\mathbb{Q}} qH_W$ for some big line bundle H_W and every eigenvalue of $f_W^*|N^1(W)$ has modulus q.*

Proof. By the definition, there is a nef and big line bundle H on X such that $f^*H \sim_{\mathbb{Q}} qH$. Part (1) holds because f_V is quasi-polarized by the pullback H_V of H.

To prove part (2), let V be the normalization of the graph $\Gamma_{X/W}$. Then f lifts to a quasi-polarized endomorphism f_V of V. For the first assertion, we take H_W to be (a multiple of) the direct image of H_V (consider pullback to V of H_W and use Lemma 2.2(2) and the argument in Note 1 of Theorem 2.7). The second follows from Lemma 2.2, since $N^1(W)$ can be regarded as a subspace of $N^1(V)$ with the action f_W^* and f_V^* compatible. □

LEMMA 2.9. *Let V and X be normal projective n-folds with X being $\mathbb{Q}$-factorial, and $\tau : V \dashrightarrow X$ a birational map. Suppose an endomorphism $f : X \to X$ of degree greater than one, lifts to a quasi-polarized endomorphism $f_V : V \to V$. Then the set $S(X)$ of prime divisors D on X with $D_{|D}$ not pseudo-effective, is a finite set. Further, $f^{-1}(S(X)) = S(X)$, so $f^r|S(X) = \mathrm{id}$ for some $r > 0$.*

Proof. Replacing V by the normalization of the graph of $\tau : V \dashrightarrow X$ and using Lemma 2.8, we may assume that τ is already holomorphic. By the assumption, there is a nef and big line bundle H such that $f_V^*H \sim qH$ and hence $\deg(f) = \deg(f_V) = q^n > 1$. Note that f^* and $f_* = q^n(f^*)^{-1}$ are automorphisms on both $N^1(X)$ and $N_1(X)$.

Step 1. If $D \in S(X)$ then $D' := f(D) \in S(X)$. Indeed, $f^*D' \equiv cD$ with $c > 0$ because $f_*(f^*D')$ is parallel to f_*D. Since $f^*(D'_{|D'}) \equiv cD_{|D}$ is not pseudo-effective, $D' \in S(X)$.

Step 2. If $D' := f(D) \in S(X)$ then $D \in S(X)$. This is because $f^*D' \equiv cD$ as in Step 1 and hence $cD_{|D} \equiv f^*(D'_{|D'})$ is not pseudo-effective.

Step 3. If $f(D_1) = D' = f(D_2)$ for $D_1 \in S(X)$, then $D_1 = D_2$. Indeed, $f_*D_1 \equiv ef_*D_2$ for some $e > 0$. So $D_1 \equiv eD_2$. Since $eD_{2|D_1} \equiv D_{1|D_1}$ is not pseudo-effective, $D_1 = D_2$.

Step 4. It follows then that $f^{-1}(S(X)) = S(X)$, and f and f^{-1} act bijectively on $S(X)$.

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Step 5. Let $(H^{n-1})^{\perp}$ be the set of prime divisors F with $F.H^{n-1} = 0$. Then it is a finite set. Indeed, writing $H = A + E$ with A an ample Cartier $\mathbb{Q}$-divisor and E an effective Cartier $\mathbb{Q}$-divisor, then the set above is contained in the support of E.

Step 6. There is a finite set Σ, such that $f^{i(D)}(D) \in \Sigma$ with some $i(D) \geqslant 0$ for every $D \in S(X)$. This will imply the lemma (see [Nak02, Proposition 10]). We take Σ to be the union of the set of prime divisors in Sing X and the ramification divisor R_f of f, and the set of prime divisors on X whose strict transform on V is in $(H^{n-1})^{\perp}$.

To finish Step 6, we only need to consider those $D \in S(X)$ where $D_i := f^{i-1}(D)$ is not in Σ for all $i \geqslant 1$. Write $f^*D_{i+1} = a_i D_i$ with $a_i \in \mathbb{Z}_{>0}$. Let $D'_i \subset V$ be the strict transform of D_i. Then $f_V^* D'_{i+1} \equiv a_i D'_i$ in $N_{n-1}(V)$. Hence

$$q^n H^{n-1}.D'_{i+1} = f_V^* H^{n-1}.f_V^* D'_{i+1} = q^{n-1} a_i H^{n-1}.D'_i,$$
$$1 \leqslant H^{n-1}.D'_{i+1} = \frac{a_i}{q} \cdots \frac{a_1}{q} H^{n-1}.D'_1.$$

Thus $a_{i_0} \geqslant q$ for infinitely many i_0. Hence D_{i_0} is in R_f and hence in Σ. This completes Step 6 and also the proof of the lemma. □

Lemma 2.10. *Let V and X be projective n-folds, $\tau : V \to X$ a birational morphism, $\Delta = \Delta_X \subset X$ a Zariski-closed subset and $f : X \to X$ an endomorphism of degree $q^n > 1$. Assume the four conditions below.*

(1) *The above f lifts to an endomorphism $f_V : V \to V$ quasi-polarized by a nef and big line bundle H so that $f^*H \sim qH$.*
(2) *We have $f^{-1}(\Delta(i)) = \Delta(i)$ for every irreducible component $\Delta(i)$ of Δ (but we only need $f^{-1}(\Delta) = \Delta$ in the proof).*
(3) *The above $\tau : V \to X$ is isomorphic over $X \backslash \Delta$.*
(4) *For every subvariety $Z \subset V$ not contained in $\tau^{-1}(\Delta)$, the restriction $H_{|Z}$ is nef and big (and hence $\deg(f|Z : Z \to Z) = q^{\dim Z}$).*

Let $A \subset X$ be a positive-dimensional subvariety such that $f^{-j}f^j(A) = A$ for all $j \geqslant 0$. Then either $M(A) := \{f^i(A) \mid i \geqslant 0\}$ is a finite set, or $f^{i_0}(A) \subseteq \Delta$ for some i_0 (and hence for all $i \geqslant i_0$).

Proof. We shall prove by induction on the codimension of A in X.

Set $k := \dim A$, $A_1 := A$ and $A_i := f^{i-1}(A)$ $(i \geqslant 1)$. Denote by Σ or $\Sigma(V, X, \Delta, f)$ the set of prime divisors in Δ, Sing X and the ramification divisor R_f of f. This Σ is a finite set.

Claim 1. A_i is contained in the union $U(\Sigma)$ of prime divisors in Σ for infinitely many i; so if $\dim A = \dim X - 1$, our $M(A)$ is finite and the lemma holds.

Suppose the contrary that Claim 1 is false. Replacing A by some A_{i_0}, we may assume that A_j is not contained in $U(\Sigma)$ for all $j \geqslant 1$. Set $b_j := \deg(f : A_j \to A_{j+1})$. Write $f^*A_{j+1} = a_j A_j$ as cycles with $a_j = q^n/b_j \in \mathbb{Z}_{>0}$ now. Let $A'_j \subset V$ be the strict transform of A_j. Now $f_V^* A'_{j+1} = a_j A'_j$ as cycles, and

$$q^n H^k.A'_{j+1} = f_V^* H^k.f_V^* A'_{j+1} = q^k a_j H^k.A'_j,$$
$$1 \leqslant H^k.A'_{j+1} = \frac{a_j}{q^{n-k}} \cdots \frac{a_1}{q^{n-k}} H^k.A'_1.$$

Thus $a_{j_0} \geqslant q^{n-k}$ for infinitely many j_0. So A_{j_0} is contained in R_f and hence also in $U(\Sigma)$ for infinitely many j_0. Thus Claim 1 is true.

We may assume that $|M(A)| = \infty$ and $k \leqslant n-2$. Let B be the Zariski-closure of the union of those A_{i_0} contained in $U(\Sigma)$. Then $\dim B \in \{k+1, \ldots, n-1\}$, and $f^{-j} f^j(B) = B$ for all $j \geqslant 0$. Choose $r \geqslant 1$ such that $B' := f^r(B), f(B'), f^2(B'), \ldots$ all have the same number of irreducible components. Let X_1 be an irreducible component of B' of maximal dimension. Then $\dim X_1 \in \{k+1, \ldots, n-1\}$ and $f^{-j} f^j(X_1) = X_1$ for all $j \geqslant 0$. Note also that X_1 contains infinitely many A_{i_1}. If $f^j(X_1) \subseteq \Delta$ for some $j \geqslant 0$, then $A_{i_1+j} \subseteq \Delta$ and we are done. Thus we may assume that $\Delta \cap f^j(X_1) \subset f^j(X_1)$ for all $j \geqslant 0$ and hence $M(X_1) < \infty$ by the inductive assumption with codimension. We may assume that $f^{-1}(X_1) = X_1$, after replacing f with its power and X_1 with its image of some f^j.

Let $V_1 \subset V$ be the strict transform of X_1. Then all four conditions in the lemma are satisfied by $(V_1, H|V_1, X_1, \Delta|X_1, f|X_1, A_{i_1})$. Since the codimension of A_{i_1} in X_1 is smaller than that of A in X, by the induction, either $M(A_{i_1})$ and hence $M(A)$ are finite or $A_{j_0} \subseteq \Delta|X_1 \subseteq \Delta$ for some j_0. This completes the proof of the lemma. □

LEMMA 2.11. *Let X be a projective variety and $f : X \to X$ a surjective endomorphism. Let $R_C := \mathbb{R}_{\geqslant 0}[C] \subset \overline{\mathrm{NE}}(X)$ be an extremal ray (not necessarily K_X-negative). Then we have the following.*

(1) *The ray $R_{f(C)}$ is an extremal ray.*
(2) *If $f(C_1) = C$, then R_{C_1} is an extremal ray.*
(3) *Denote by Σ_C the set of curves whose classes are in R_C. Then $f(\Sigma_C) = \Sigma_{f(C)}$.*
(4) *If R_{C_1} is extremal then $\Sigma_{C_1} = f^{-1}(\Sigma_{f(C_1)}) := \{D \mid f(D) \in \Sigma_{f(C_1)}\}$.*

Proof. Note that $f^* : N^1(X) \to N^1(X)$ and $f_* : N_1(X) \to N_1(X)$ are isomorphisms.

To prove part (1), suppose $z_1 + z_2 \equiv f_* C$ for $z_i \in \overline{\mathrm{NE}}(X)$. Write $z_i = f_* z_i'$ for $z_i' \in \overline{\mathrm{NE}}(X)$. Then $f_*(z_1' + z_2' - C) \equiv 0$ and hence $z_1' + z_2' \equiv C$. Thus $z_i' \equiv a_i C$ for some $a_i \geqslant 0$ by the assumption on C, whence $z_i = f_* z_i' \equiv a_i f_* C \in R_{f(C)}$.

The proofs of parts (2) to (4) are also easy. □

LEMMA 2.12. *Let X be a normal projective variety with at worst log terminal singularities, and $f : X \to X$ a surjective endomorphism. Suppose that $R_{C_i} = \mathbb{R}_{\geqslant 0}[C_i]$ $(i = 1, 2)$, with $C_2 = f(C_1)$, are K_X-negative extremal rays and $\pi_i : X \to Y_i$ the corresponding contractions. Then there is a finite surjective morphism $h : Y_1 \to Y_2$ such that $\pi_2 \circ f = h \circ \pi_1$.*

Proof. Let $X \to Y \xrightarrow{h} Y_2$ be the Stein factorization of $\pi_2 \circ f : X \to X \to Y_2$. By Lemma 2.11, the map $X \to Y$ is just $\pi_1 : X \to Y_1$. □

The result below is crucial and used in proving Theorem 3.2. It was first proved by the author when $\dim Y \leqslant 2$ or $\rho(Y) \leqslant 2$, and has been extended and simplified by Fujimoto and Nakayama to the current form below. See appendix for its proof.

THEOREM 2.13. *Let X be a normal projective variety defined over an algebraically closed field of characteristic zero such that X has only log-terminal singularities. Let $R \subset \overline{\mathrm{NE}}(X)$ be an extremal ray such that $K_X R < 0$ and the associated contraction morphism cont_R is a fibration to a lower-dimensional variety. Then, for any surjective endomorphism $f\colon X \to X$, there exists a positive integer k such that $(f^k)_*(R) = R$ for the automorphism $(f^k)_*\colon \mathsf{N}_1(X) \xrightarrow{\simeq} \mathsf{N}_1(X)$ induced from the iteration $f^k = f \circ \cdots \circ f$.*

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3. Proof of theorems

In this section we prove the theorems in the Introduction and three theorems below. Theorem 3.2 below includes Theorem 1.1 as a special case, while Theorem 3.4 implies 1.4 because a result of Benveniste says that a Gorenstein terminal threefold has no flips.

Remark 3.1. All X_i, Y in Theorem 3.2 are again $\mathbb{Q}$-factorial and have at worst log terminal singularities by MMP (see e.g. [Nak04]).

Theorem 3.2. *Let X be a $\mathbb{Q}$-factorial n-fold, with $n \in \{3, 4\}$, having only log terminal singularities and a polarized endomorphism f of degree $q^n > 1$. Let $X = X_0 \dashrightarrow X_1 \cdots \dashrightarrow X_r$ be a composite of K-negative divisorial contractions and flips. Replacing f by its positive power,* (I) *and* (II) *hold.*

(I) *The dominant rational maps $g_i : X_i \dashrightarrow X_i$ $(0 \leqslant i \leqslant r)$ (with $g_0 = f$) induced from f, are all holomorphic. Further, g_i^{-1} preserves each irreducible component of the exceptional locus of $X_i \to X_{i+1}$ (when it is divisorial) or of the flipping contraction $X_i \to Z_i$ (when $X_i \dashrightarrow X_{i+1} = X_i^+$ is a flip).*

(II) *Let $\pi : W = X_r \to Y$ be the contraction of a K_W-negative extremal ray $\mathbb{R}_{\geqslant 0}[C]$, with $\dim Y \leqslant n - 1$. Then $g := g_r$ descends to a surjective endomorphism $h : Y \to Y$ of degree $q^{\dim Y}$ such that*

$$\pi \circ g = h \circ \pi.$$

For all $0 \leqslant i \leqslant r$, all eigenvalues of $g_i^*|N^1(X_i)$ and $h^*|N^1(Y)$ are of modulus q; there are big line bundles H_{X_i} and H_Y satisfying

$$g_i^* H_{X_i} \sim q H_{X_i}, \quad h^* H_Y \sim q H_Y.$$

Suppose further that either $\dim Y \leqslant 2$ or $\rho(Y) = 1$. Then H_W and H_Y can be chosen to be ample and g and h are polarized.

The contraction π below exists by the MMP for threefolds.

Theorem 3.3. *Let X be a $\mathbb{Q}$-factorial rationally connected threefold having at worst terminal singularities and a polarized endomorphism of degree greater than one. Let $X \dashrightarrow W$ be a composite of K-negative divisorial contractions and flips, and $\pi : W \to Y$ an extremal contraction of non-birational type. Suppose either $\dim Y \geqslant 1$, or $\dim Y = 0$ and W is smooth. Then X is rational.*

Theorem 3.4. *Let X be a $\mathbb{Q}$-factorial rationally connected threefold having only terminal singularities. Suppose either X has a quasi-polarized endomorphism of degree greater than one, or the set $S(X)$ as in 2.1 is finite. Then X has only finitely many K_X-negative extremal rays which are not of flip type.*

We start with some preparations for the proof of Theorem 3.2.

Proposition 3.5. *Let X be a $\mathbb{Q}$-factorial n-fold with $n \in \{3, 4\}$, having at worst log terminal singularities and a polarized endomorphism $f : X \to X$ of degree $q^n > 1$. Let $X = X_0 \dashrightarrow X_1 \cdots \dashrightarrow X_r$ be a composite of K-negative divisorial contractions and flips. Suppose that for each $0 \leqslant j \leqslant r$, the dominant rational map $f_j : X_j \dashrightarrow X_j$ induced from f, is holomorphic and f_j^{-1} preserves each irreducible component of the exceptional locus of $X_j \to X_{j+1}$ (when it is divisorial) or of the flipping contraction $X_j \to Y_j$ (when $X_j \dashrightarrow X_{j+1} = X_j^+$ is a flip). Let S' be*

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a surface on some X_i with $(f_i{}^v)(S') = S'$ for some $v > 0$. Then the endomorphism $f_S : S \to S$ induced from $f_i{}^v|S'$, is polarized of degree q^{2v}. Here S is the normalization of S'.

Proof. We may assume that $v = 1$ after replacing f by its power; see Note 1 of Theorem 2.7. By the assumption, $f^*H_X \sim qH_X$ for a very ample line bundle H_X, and $\deg(f) = q^n$. By Lemmas 2.8 and 2.4, $\deg(f_S : S \to S) = q^2$. To show the polarizedness of f_S, we only need to show the assertion of the existence of a big Weil divisor as an eigenvector of f_S^*; see Theorem 2.7.

We shall prove this assertion by ascending induction on the index i of X_i. When $X_i = X$, S is polarized by the pullback of H_X via the morphism $S \to S' \subset X$.

If $X_{i-1} \to X_i$ is birational over S' with $S'_{i-1} \subset X_{i-1}$ the strict transform of S' and S_{i-1} the normalization of S'_{i-1}, then the polarizedness of S_{i-1} (by the inductive assumption) gives rise to a big Weil divisor P_S on S with $f_S^*P_S \equiv qP_S$ (using Lemma 2.5 and the proof of Lemma 2.8). We are done.

Thus, we have only to consider the two cases below (where $n = 4$).

Case 1. $X_{i-1} \to X_i$ is a divisorial contraction so that S' is the image of a prime divisor Z' on X_{i-1} (being necessarily the support of the whole exceptional divisor $X_{i-1} \to X_i$). By the assumption, $f_{i-1}^{-1}(Z') = Z'$ and hence $f^{-1}(Z'_X) = Z'_X$ where $Z'_X \subset X$ is the (birational) strict transform of Z'. The normalization Z of Z'_X has an endomorphism f_Z (induced from $f|Z'_X$) polarized by H_Z (the pullback of H_X) so that $f_Z^*H_Z \sim qH_Z$. $Z' \to S'$ induces $\sigma : Z \to S$ (with general fibre $\mathbb{P}^1$) so that f_S is the descent of f_Z. By [Nak07, the proof of Proposition 4.17], the intersection sheaf $H_S := I_{Z/S}(H_Z, H_Z)$ is an integral Weil divisor satisfying $f_S^*H_S \sim qH_S$. Further, $H_S = (\sigma|H_Z)_*(H_{Z|H_Z})$ and hence is big by the ampleness of H_Z. We are done again.

Case 2. $X_{i-1} \dashrightarrow X_i = X_{i-1}^+$ is a flip and S' is an irreducible component of the exceptional locus of the flipping contraction $X_i \to Y_{i-1}$. We have $f_i^{-1}(S') = S'$ by the assumption on the flipping contraction $X_{i-1} \to Y_{i-1}$. Note that the assumption of Lemma 2.4 is satisfied by (X_i, f_i) (see Lemma 2.8). In particular, $f_i^*M|S' \equiv qM|S'$ for a non-zero nef Cartier $\mathbb{R}$-divisor $M|S'$ in $N^1(X_i)|S' \subset N^1(S')$. We divide into two subcases.

Case 2a. S' is mapped to a curve B' on Y_{i-1}. Then we have an induced map $S \to B$ with general fibre $\mathbb{P}^1$. Here B the normalization of B'. Thus f_S is polarized by Lemmas 2.6 and 2.4.

Case 2b. S' is mapped to a point on Y_{i-1}. Note that $\rho(X_i/Y_{i-1}) = 1$ since $\rho(X_{i-1}/Y_{i-1}) = 1$ and $\rho(X_{i-1}) = \rho(X_i)$. So for any ample Cartier divisor A on X_i, there is a $b \neq 0$ such that $A - bM$ is the pullback of some divisor by $X_i \to Y_{i-1}$. Thus $A|S' \equiv bM|S'$ in $N^1(S')$. Hence $f_i^*A|S' \equiv qA|S'$ in $N^1(S')$. Thus f_S is polarized by an ample line bundle A_S (the pullback of $A|S'$). $\square$

LEMMA 3.6. *Let X be a $\mathbb{Q}$-factorial projective variety with at worst log terminal singularities, $f : X \to X$ a surjective endomorphism, and $X \dashrightarrow X^+$ a flip with $\pi : X \to Y$ the corresponding flipping contraction of an extremal ray $R_C := \mathbb{R}_{\geqslant 0}[C]$. Suppose that $R_{f(C)} = R_C$. Then the dominant rational map $f^+ : X^+ \dashrightarrow X^+$ induced from f, is holomorphic. Both f and f^+ descend to one and the same endomorphism of Y.*

Proof. We note that

$$X = \operatorname{Proj} \bigoplus_{m \geqslant 0} \mathcal{O}_Y(-mK_Y), \quad X^+ = \operatorname{Proj} \bigoplus_{m \geqslant 0} \mathcal{O}_Y(mK_Y)$$

and there is a natural birational morphism $\pi^+ : X^+ \to Y$. By the assumption and Lemma 2.12, $f : X_1 = X \to X_2 = X$ descends to an endomorphism $h : Y_1 = Y \to Y_2 = Y$ with $\pi_2 \circ f = h \circ \pi_1$. Here $\pi_i : X_i \to Y_i$ are identical to $\pi : X \to Y$. Set $Z := X_2^+ \times_{Y_2} Y_1$. Then the projection $Z \to Y_1$ is a small birational morphism with $\rho(Z/Y_1) = 1$, and it is identical to either $X_1 \to Y_1$ or $X_1^+ = X^+ \to Y_1$, noting that $-K_X$ and K_{X^+} are relatively ample over Y. Now we have only to consider and rule out the case $Z = X_1$. Set $W := X_2^+ \times_{Y_2} X_2$. Since the composite $X_1 = Z \to X_2^+ \to Y_2$ is identical to that of $Z \to Y_1 \to Y_2$ and hence to that of $X_1 \to X_2 \to Y_2$, there is a morphism $\sigma : X_1 \to W$ such that $X_1 = Z \to X_2^+$ factors as $X_1 \to W \to X_2^+$, and $X_1 \to X_2$ factors as $X_1 \to W \to X_2$. So the projection $W \to X_2$ is birational (because so is $X_2^+ \to Y_2$) and finite (because so is $X_1 \to X_2$), whence it is an isomorphism. Thus the birational map $X_2 \to X_2^+$ is a well defined morphism as the composite $X_2 \to W \to X_2^+$. This is absurd. Therefore, $Z = X_1^+$ and the lemma is true. □

Lemma 3.7. *With the hypotheses and notation in Lemma 2.10, assume further that X is $\mathbb{Q}$-factorial with at worst log terminal singularities and $\sigma : X \to X_1$ is a divisorial contraction of an extremal ray $\mathbb{R}_{\geqslant 0}[\ell]$ with E the exceptional locus (necessarily an irreducible divisor). Then we have the following.*

(1) *There is an $s > 0$ such that $(f^s)^{-1}(E) = E$.*
(2) *The dominant rational map $g : X_1 \cdots\to X_1$ induced from f^s, is holomorphic, after s is replaced by a larger one.*
(3) *Let $\Delta_1 \subset X_1$ be the image of $\Delta \cup E$. Then $g^{-1}(\Delta_1) = \Delta_1$.*
(4) *Let V_1 be the normalization of the graph of $V \cdots\to X_1$, and $H_1 \subset V_1$ the pullback of H on V. Then g lifts to an endomorphism $g_1 : V_1 \to V_1$ such that $(V_1 \supset H_1,\ g_1,\ X_1 \supset \Delta_1, g)$ satisfies all four conditions in Lemma 2.10.*

Proof. Part (1) follows from Lemma 2.9 since $E \in S(X)$, while parts (3) and (4) follow from (2). Now part (2) follows from the proof of Theorem 2.13 applied to $N^1(X)|E \subset N^1(E)$ and the extremal curve ℓ in the closed cone of curves on E (dual to the cone $\mathrm{Nef}(X)|E$). □

Lemma 3.8. *With the hypotheses and notation in Lemma 2.10, assume further the following.*

(1) *If $T' \subset X$ is a surface with $f^t(T') = T'$ for some $t > 0$, then the endomorphism of the normalization T of T' induced from $f^t{}_{|T'}$, is polarized.*
(2) *We have $\dim \Delta \leqslant 2$.*
(3) *The X has at worst log terminal singularities and $X \cdots\to X^+$ is a flip with $\pi : X \to Y$ the corresponding flipping contraction of an extremal ray $R_C := \mathbb{R}_{\geqslant 0}[C]$.*
(4) *The union U_C of curves in the set Σ_C in Lemma 2.11 is of dimension less than or equal to two.*

Then we have the following assertions.

(1) *There is an $s > 0$ such that $R_{f^s(C)} = R_C$ and $(f^s)^{-1}(U_C(i)) = U_C(i)$ for every irreducible component $U_C(i)$ of U_C.*
(2) *The dominant rational map $g : X^+ \cdots\to X^+$ induced from f^s is holomorphic.*
(3) *Let $\Delta^+ = \Delta(X^+) \subset X^+$ be the set consisting of the exceptional locus of the flipping contraction $\pi^+ : X^+ \to Y$ (i.e., $(\pi^+)^{-1}(\pi(U_C))$) and the total transform of $\Delta \subset X$. Then $g^{-1}(\Delta^+(i)) = \Delta^+(i)$ for every irreducible component $\Delta^+(i)$ of Δ^+.*

(4) *Let V^+ be the normalization of the graph of $V \cdots\to X^+$, and $H^+ \subset V^+$ the pullback of H on V. Then g lifts to an endomorphism $g_{V^+} : V^+ \to V^+$ such that $(V^+ \supset H^+,\ g_{V^+},\ X^+ \supset \Delta^+, g)$ satisfies all four conditions in Lemma 2.10.*

Proof. Note that the assertion (2) follows from assertion (1) and Lemma 3.6, while assertions (3) and (4) follow from assertions (1) and (2). It remains to prove assertion (1). By Lemma 2.11, we have only to show that $f^u(C)$ and $f^v(C)$ (and hence $f^{u-v}(C)$ and C) are parallel for some $u > v$.

By Lemma 2.11, $f^{-j}f^j(U_C) = U_C$ for all $j \geqslant 0$. Choose $r' \geqslant 0$ such that $U' := f^{r'}(U_C), f(U'), f^2(U'), \ldots$ all have the same number of irreducible components. Then $f^{-j}f^j(U'(k)) = U'(k)$ for every irreducible component $U'(k)$ of U'. By Lemma 2.10, either $M(U'(k))$ is finite and $S' := f^{j_1}(U'(k)) = f^{j_2}(U'(k))$ for some $j_2 > j_1 > 1$, or $f^{j_1}(U'(k))$ is contained in an irreducible component $\Delta(1)$ of Δ for infinitely many j_1. We divide into two cases.

Case 1. $\dim U'(k) = 2$. Since $\dim \Delta(1) \leqslant 2$ we may assume that $M(U'(k))$ is always finite and $(f^m)^{-1}(S') = S'$ for $m = j_2 - j_1$. Take a two-dimensional irreducible component S of U_C such that $f^r(S) = S'$, where $r := r' + j_1$. Note that f^{-m} permutes irreducible components of $f^{-r}(S')$. Hence some f^{-t} with $t \in m\mathbb{N}$ stabilizes all of these components. Especially, $f^{\pm t}(S) = S$. Replacing f by f^t, we may assume that $f^{\pm}(S) = S$. We may also assume that $C \subset S$. If the flipping contraction $\pi : X \to Y$ maps S to a point P, then $f(C)$ is parallel to C because $\pi(f(C)) = P$, so assertion (1) is true. Suppose π induces a (necessarily $\mathbb{P}^1$) fibration $S \to B$ onto a curve. Let $\widetilde{S} \to S$ be the normalization. Then f induces a finite morphism $\widetilde{f} : \widetilde{S} \to \widetilde{S}$ which is polarized by our assumption, so $\widetilde{f}^*|\mathrm{Weil}(\widetilde{S}) = q\,\mathrm{id}$ after replacing f by its power (see Lemmas 2.6, 2.4 and 2.8). Thus $f(C)$ is parallel to C. Hence assertion (1) is true in Case(1).

Case 2. $\dim U'(k) = 1$. We may assume that $U'(k) = f^{r'}(C)$. We only need to consider the situation where $f^{j_1}(U'(k)) \subset \Delta(1)$ and $\dim \Delta(1) = 2$. Relabel $f^{r'+j_1}(C)$ as C, we have $C \subset S := \Delta(1)$. By the hypotheses, $f^{\pm}(S) = S$. Set $C_v := f^v(C)$ $(v \geqslant 0)$. By the choice of r', we have $f^{-j}f^j(C_v) = C_v$ for all $j \geqslant 0$. Let $\widetilde{S} \to S$ be the normalization and $\Theta \subset \widetilde{S}$ the union of the conductor and the ramification divisor R_h of the finite morphism $h : \widetilde{S} \to \widetilde{S}$ induced from f. If C_v has preimage in Θ for infinitely many v then C_v and $C_{v'}$ (and hence $C_{v-v'}$ and C) are parallel for some $v > v'$ because Θ has only finitely many components, so assertion (1) is true. Thus we may assume that no C_v is contained in Θ for all $v \geqslant 0$. Let $D_v \subset \widetilde{S}$ be the (birational) preimage of C_v. Then $h^{-j}h^j(D_v) = D_v$ for all $j \geqslant 0$. The extra assumption implies $h^*D_{v+1} = D_v$. By Lemmas 2.4 and 2.8, we have $\deg(h) = q^2$. Now $q^2 D_{v+1}.D_{w+1} = h^*D_{v+1}.h^*D_{w+1}$ and

$$D_{v+1}.D_{w+1} = \frac{1}{q^2} D_v.D_w = \cdots = \frac{1}{q^{2b}} D_{v+1-b}.D_{w+1-b}.$$

On the other hand, $D_i.D_j \in (1/d)\mathbb{Z}$ with d the determinant of the intersection matrix for the exceptional divisor of a resolution of S. Thus $D_i.D_{i+1} = D_i^2 = 0$ for $i >> 0$. This and the Hodge index theorem applied to the resolution of S, imply that D_i and D_{i+1} are parallel. So C_i and C_{i+1} (and hence C and $f(C)$) are parallel. Therefore, assertion (1) is true in Case(2). This completes the proof of the lemma. □

3.9 Proof of Theorem 3.2(I)

By the assumption, $f^*H_X \sim qH_X$ for an ample line bundle H_X. We will inductively define $\Delta_i \subset X_i$, $\tau_i : V_i \to X_i$, $g_{V_i} : V_i \to V_i$, $g_i : X_i \to X_i$, and big and semi-ample line bundle H_{V_i} with

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$g_{V_i}^* H_{V_i} \sim qH_{V_i}$. Define H_{X_i} to be (a large multiple of) the direct image of H_{V_i}, so $g_i^* H_{X_i} \sim qH_{X_i}$ using Lemma 2.8. Since X_i is $\mathbb{Q}$-factorial by MMP, H_{X_i} is a big line bundle. Consider the following.

Property(i): Theorem 3.2(I) holds for $X_0 \dashrightarrow \cdots \dashrightarrow X_i$. The quadruple $(V_i,\ g_{V_i},\ X_i \supset \Delta_i,\ g_i)$ satisfies the four conditions in Lemma 2.10. The divisor H_{V_i} is big and semi-ample. We have $\dim \Delta_i \leqslant 2$.

The last inequality follows from the fact that for a divisorial contraction $\sigma : W \to Z$ between n-folds with exceptional divisor $E_{W/Z}$, one has $\dim \sigma(E_{W/Z}) \leqslant n-2$, and for a flip $W \dashrightarrow W^+$ with $W \to Z$ and $W^+ \to Z$ the flipping contractions, one has $\dim E_{W'/Z} \leqslant n-2$ for both $W' = W, W^+$.

We prove Property(i) $(0 \leqslant i \leqslant r)$ by induction. Set

$$V_0 = X_0, \quad \Delta_0 = \emptyset, \quad H_{V_0} := H_X, \quad g_{V_0} = g_0 = f.$$

Then Property(0) holds. Suppose Property(i) holds for $i \leqslant t$. If $X_t \to X_{t+1}$ is a divisorial contraction, then we just apply Lemma 3.7.

When $X_t \dashrightarrow X_{t+1} = X_t^+$ is a flip, we apply Lemma 3.8 and set $\Delta_{t+1} := \Delta(X_t^+)$ so that Property(t+1) holds. Indeed, the first condition in Lemma 3.8 is satisfied, thanks to Proposition 3.5. This proves Theorem 3.2(I).

3.10 Proof of Theorem 3.2(II)

By Theorem 2.13, replacing f by its power, we may assume that $g(C)$ is parallel to C in $N_1(W)$ so that $g : W \to W$ descends to a finite morphism $h : Y \to Y$; see Lemma 2.12. Set $H_W := H_{X_r}$, a big effective line bundle with $g^* H_W \sim qH_W$. Now Theorem 3.2 follows from the following lemma.

Lemma 3.11.

(1) *We have* $\deg h = q^{\dim Y}$.

(2) *All eigenvalues of* $g_i^* | N^1(X_i)$ *and* $h^* | N^1(Y)$ *are of modulus q; the intersection sheaf* $H_Y := I_{V_r/Y}(H_{V_r}^s)$ *(with* $s = 1 + \dim V_r - \dim Y$*) is a big* $\mathbb{Q}$*-Cartier integral divisor such that* $h^* H_Y \sim_{\mathbb{Q}} qH_Y$*; so h is polarized when* $\dim Y \leqslant 2$.

(3) *If h is polarized, then* $g : W \to W$ *is polarized of degree* $q^{\dim W}$.

(4) *Suppose that* $h^* | N^1(Y) = q\ \mathrm{id}$. *Replacing f by its power, we have*

$$g_i^* | N^1(X_i) = q\ \mathrm{id} \quad (0 \leqslant i \leqslant r).$$

Hence h and g_i *are all polarized (see Lemma 2.2).*

Proof. (1) Assertion (1) follows from Lemma 2.2 and the proof of Lemma 2.8.

(2) The first part follows from Lemmas 2.2 and 2.8. We use the birational morphism $V_r \to X_r = W$ and the big and semi-ample line bundle H_{V_r} in Theorem 3.2(I). Replacing H_{V_r} by its large multiple, we may assume that $Bs|H_{V_r}| = \emptyset$. Thus the second part is true as in Proposition 3.5, since $I_{V_r/Y}(H_{V_r}^s) = \tau_*(H_{V_r}|V')$, where τ is the restriction to $V' := H_1 \cap \cdots \cap H_{s-1}$ of the composite $V_r \to W \to Y$, with H_i general members in $|H_{V_r}|$. The last part follows from Theorem 2.7.

(3) We may assume $h^* L \sim qL$ for an ample line bundle L on Y (using part (1)). The big divisor H_W is π-ample since $N_1(W/Y)$ is generated by the class $[C]$. Thus $H := H_W + t\pi^* L$ is ample for $t >> 0$ (see [KM98, Proposition 1.45]) and $g^* H \sim qH$, so g is polarized.

(4) Assertion (4) is true because $N^1(X_i)$ is spanned by the pullbacks of the big divisor H_W in 3.2(I), the divisors (lying below those divisors in $S(V_j)$, $j \geqslant i$) contracted by $X_j \dashrightarrow W$ and the divisors in $\pi^* N^1(Y)$, noting that a flip $X_k \dashrightarrow X_{k+1}$ induces an isomorphism $N^1(X_k) \cong N^1(X_{k+1})$ (see Lemmas 2.9, 2.8 and 2.2). This proves Lemma 3.11 and also Theorem 3.2. $\square$

3.12 Proof of Theorem 3.3

By Theorem 3.2, f (replaced by its power) induces a polarized endomorphism $g: W \to W$ of degree $q^3 > 1$. Note that W is also rationally connected and $\mathbb{Q}$-factorial with at worst terminal singularities. So K_W is not nef. If the Picard number $\rho(W) = 1$, then $-K_W$ is ample, and hence $W \cong \mathbb{P}^3$ (so X is rational) provided that W is smooth, because every smooth Fano threefold of Picard number one having an endomorphism of degree greater than one, is $\mathbb{P}^3$; see [ARV99, HM03].

Thus, we only need to consider the extremal contraction $\pi: W \to Y$ with $\dim Y = 1, 2$. Our Y is rational. Note that $\operatorname{Sing} W$ and hence its image in Y are finite sets, so a general fibre $W_y \subset W$ over $y \in Y$ is smooth.

We apply Theorem 3.2. Hence each $U \in \{X, W, Y\}$ has an endomorphism $f_U: U \to U$ polarized by an ample line bundle H_U and with $\deg(f_U) = q^{\dim U} > 1$. Here $f_W = g$ and $f_Y = h$ in notation of Theorem 3.2.

A polarized endomorphism of degree greater than one has a dense set of periodic points [Fak03, Theorem 5.1]. Let y_0 be a general point with $h(y_0) = y_0$ (after replacing f by its power). Then the fibre $W_0 := W_{y_0} \subset W$ over $y_0 \in Y$ has an endomorphism $g_0 := g|W_0 : W_0 \to W_0$ polarized by the ample line bundle $H_0 := H_W|W_0$ so that $g_0^* H_0 \sim q H_0$ and $\deg g_0 = q^{\dim W_0} > 1$. Our W_0 is a smooth Fano variety with $\dim W_0 = \dim W - \dim Y$.

Suppose that $\dim Y = 1$. Then W_0 is a del Pezzo surface with a polarized endomorphism of degree $q^2 > 1$. Thus $K_{W_0}^2 = 6, 8, 9$ (see [FN08, Theorem 1.1] or [Zha02, Theorem 3]; [Miy83, p. 73]). The case $K_{W_0}^2 = 7$ does not occur because $\rho(W/Y) = 1$. Thus, W (and hence X) are rational (see, e.g., [Isk97, § 2.2]).

Therefore, we may assume that $\dim Y = 2$. Then $\pi: W \to Y$ is a conic bundle. Further, π is dominated by another conic bundle $\pi': W' \to Y'$ with W', Y' smooth, with $\rho(W'/Y') = 1$ and with birational morphisms $\sigma_w: W' \to W$ and $\sigma_y: Y' \to Y$ satisfying $\pi \circ \sigma_w = \sigma_y \circ \pi'$ (cf. [Miy83, the proof of Theorem 4.8]).

Let D' be the discriminant of π'. If $D' = \emptyset$, then π' is a $\mathbb{P}^1$-bundle in the Zariski topology which is locally trivial for the Brauer group $\operatorname{Br}(Y') = 0$ with Y' being a smooth projective rational surface, so W' and X are rational. Thus we may assume that $D' \neq \emptyset$ and π' is a standard conic bundle; see [Miy83, § 4.9 and Lemma 4.7] for the relevant material.

Let D be the one-dimensional part of the discriminant of π. Note that $\sigma_{y*}(D') = D$ because every reducible fibre over some $d \in D$ should be underneath only reducible fibres over some $d' \in D'$ and note that $\sigma_y: Y' \to Y$ is the blowup over the discriminant $D(W/Y)$; see the construction in [Miy83, Theorem 4.8]; note also that $(\pi')^* E$ is irreducible for every prime divisor $E \subset X'$ (and especially for those in D').

Our $h: Y \to Y$ satisfies $h^{-1}(D) \subseteq D$ since the reducibility of a fibre W_d over $d \in D$ implies that of $W_{d'}$ for $d' \in h^{-1}(d)$. So $D \supseteq h^{-1}(D) \supseteq h^{-2}(D) \supseteq \cdots$. Considering the number of components, we have $h^{-s}(D) = h^{-s-1}(D)$ for some $s > 1$. Since h is surjective and applying h^s and h^{s+1}, we have $h^{\pm}(D) = D$. Replacing f by its power, we may assume $h^{\pm}(D_i) = D_i$ for every irreducible

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component D_i of D. Therefore, $h^*D_i = qD_i$ by Lemma 2.5. Hence

$$K_Y + D = h^*(K_Y + D) + G$$

with G an effective Weil divisor. Noting that $h_*H_Y = (\deg(h)/q)H_Y = qH_Y$ and by the projection formula,

$$H_Y.(K_Y + D) = h_*H_Y.(K_Y + D) + H_Y.G, \quad (1-q)H_Y.(K_Y + D) = H_Y.G \geqslant 0.$$

This proves the second assertion below. For the first, see [KM98, Proposition 3.36] and [MP08, Theorem 1.2.7]. For the third, see [Miy83, Lemma 4.1 and Remark 4.2]. The fifth is due to Iskovskikh in his 1987 paper in the Duke Mathematical Journal (see, e.g., his survey [Isk97, Theorem 8]).

Claim 3.13.

(1) The surface Y is $\mathbb{Q}$-factorial with at worst Du Val singularities.

(2) If $K_Y + D$ is pseudo-effective, then $K_Y + D \equiv 0$ in $N^1(Y)$.

(3) The divisor D' is of normal crossing. Every smooth rational component of D' meets at least two points of other components.

(4) We have $\sigma_{y*}(D') = D$.

(5) If π' is a standard conic bundle, D' is connected and $D'.F \leqslant 3$ for a free pencil $|F|$ of rational curves, then W' and hence W and X are rational.

We factor $Y' \to Y$ as $Y' \to \widetilde{Y} \to Y$ with $\widetilde{Y} \to Y$ the minimal resolution. Let $\widetilde{D} \subset \widetilde{Y}$ be the image of D'. Since $D' \neq \emptyset$ and by Claim 3.13(3) and the Riemann–Roch theorem, we have $|K_{Y'} + D'| \neq \emptyset$; the latter implies $K_{\widetilde{Y}} + \widetilde{D} \sim E$ for some effective divisor. Hence $K_Y + D \sim \hat{E}$ with $\hat{E} \subset Y$ the image of E. By Claim 3.13(2), $\hat{E} = 0$ and $K_Y + D \sim 0$. Thus $\operatorname{Supp} E = \bigcup_i E_i$ is supported on the exceptional locus of $\widetilde{Y} \to Y$, so each E_i is a (-2)-curve. Now $h^0(\widetilde{Y}, K_{\widetilde{Y}} + \widetilde{D}) = 1$. Our $\widetilde{D}$ is connected and is either a smooth elliptic curve, or a nodal rational curve, or a simple loop of smooth rational curves; in fact, one may use Claim 3.13(3) and [CCZ05, the proof of Lemma 2.3].

We assert that $E = 0$. Indeed, since E is negative definite, we may assume that $E.E_1 < 0$. Then $0 > E_1.(K_{\widetilde{Y}} + \widetilde{D}) = E_1.\widetilde{D}$ and hence $E_1 \leqslant \widetilde{D}$. If $\widetilde{D}$ is irreducible then $E_1 = \widetilde{D}$ and $K_{\widetilde{Y}} \sim E - E_1 \geqslant 0$, contradicting the fact that $\widetilde{Y}$ is a smooth rational surface. Hence $\widetilde{D}$ is a simple loop of smooth rational curves and contains E_1. Thus $0 > E_1.E_1 + E_1.(\widetilde{D} - E_1) \geqslant -2 + 2$ by Claim 3.13(3). This is absurd. So our assertion is true and $K_{\widetilde{Y}} + \widetilde{D} \sim 0$.

If $\widetilde{Y}$ is ruled with a general fibre F then $\widetilde{D}.F = -K_{\widetilde{Y}}.F = 2$; if $\widetilde{Y} = \mathbb{P}^2$, then for a line F we have $F.\widetilde{D} = 3$. Denoting by the same F its total transform on Y', we have $F.D' \leqslant 3$. Thus W' and hence X are rational by Claim 3.13. This proves Theorem 3.3. □

3.14 Proof of Theorem 1.2

We apply Theorem 3.2. By MMP, we may assume that W has no extremal contraction of birational type. Since X is rationally connected, both K_X and K_W are non-nef, so there is a contraction $W \to Y$ of an extremal ray. We have $\dim Y \leqslant 2$. Now Theorem 1.2(1) follows from Theorems 2.7 and 3.2 and Lemma 3.11(4) and (2). Indeed, when $\dim Y = 2$, Y is rational with only Du Val singularities by [MP08, Theorem 1.2.7] and hence K_Y is not trivial in $N^1(Y)$.

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Theorem 1.2(3) follows from the following claim.

Claim 3.15. Replace f by its power so that $f^*|N^1(X) = q\,\mathrm{id}$. We have the following assertions.

(1) If $M \subset X$ is an irreducible divisor with $\kappa(X, M) = 0$ then $f^*M = qM$.

(2) There are only finitely many f^{-1}-periodic irreducible divisors M_i. Hence there is a $v > 0$ such that $(f^v)^*M_i = q^v M_i$ for all i. The ramification divisor R_{f^v} equals $(q^v - 1)\sum_i M_i + \Delta$, where Δ is an effective integral divisor containing no M_i.

(3) We have $-K_X \sim_{\mathbb{Q}} \sum_i M_i + \Delta/(q^v - 1) \geqslant 0$ and $\kappa(X, -K_X) = \kappa(X, \sum M_i - K_X) \geqslant 0$.

Proof. Since $q(X) = 0$, we have $f^*M \sim_{\mathbb{Q}} qM$ for every irreducible integral divisor M. Hence $f^{-1}(M) = M$ when $\kappa(X, M) = 0$. Thus assertion (1) follows.

Suppose that M_i $(1 \leqslant i \leqslant N)$ are f^{-1}-periodic, so a power $h_N = f^{s(N)}$ of f satisfies $h_N^{-1}(M_i) = M_i$ for all $1 \leqslant i \leqslant N$. Then $h_N^* M_i = q^{s(N)} M_i$ and $K_X + \sum M_i = h_N^*(K_X + \sum M_i) + \Delta_N \sim_{\mathbb{Q}} q^{s(N)}(K_X + \sum M_i) + \Delta_N$, where Δ_N is an effective integral divisor containing no any M_i. Thus $-K_X \sim_{\mathbb{Q}} \sum_{i=1}^N M_i + \Delta/(q^{s(N)} - 1) \geqslant 0$, which also implies (3). Multiplying the above equivalence by two copies of an ample divisor H, we see that N is bounded. This proves assertion (2). $\square$

We now prove Theorem 1.2(2). By Theorem 3.3, we may assume that the end product of MMP for X is of Picard number one, i.e., there is a composite $X = X_0 \dashrightarrow X_1 \cdots \dashrightarrow X_r$ of divisorial contractions and flips such that $\rho(X_r) = 1$, so $-K_{X_r}$ is ample because all X_i are rationally connected with only $\mathbb{Q}$-factorial terminal singularities by MMP. Let $g_i : X_i \dashrightarrow X_i$ be the dominant rational map induced from $f : X \to X$ (with $g_0 = f$).

Claim 3.16. Replacing f by its positive power, we have the following assertions.

(1) For all $0 \leqslant t \leqslant r$, our g_t is holomorphic with $g_t^*|N^1(X_t) = q\,\mathrm{id}$. Let $E_t' \subset X_t$ be zero (respectively the (irreducible) exceptional divisor) when $X_t \dashrightarrow X_{t+1}$ is a flip (respectively $X_t \to X_{t+1}$ is divisorial). Then the strict transform $E_t \subset X$ of E_t' satisfies $f^{-1}(E_t) = E_t$.

(2) The space $N^1(X)$ is spanned by K_X and those E_t in assertion (1). Let $E = \sum E_t$.

Proof. Assertion (1) can be proved by ascending induction on the index t of X_t. Suppose assertion (1) is true for t. Since g_t^* is scalar, we may assume that both $g_t^{\pm}$ preserve the extremal ray corresponding to the birational map $X_t \dashrightarrow X_{t+1}$, so g_t descends to the holomorphic g_{t+1} as in the proof of Theorem 3.2, and also the last part of assertion (1) is true. The scalarity of g_t^* implies that of g_{t+1}^* because $N^1(X_{t+1})$ is isomorphic to (respectively regarded as a subspace of) $N^1(X_t)$ via the pullback when $X_t \dashrightarrow X_{t+1}$ is a flip (respectively $X_t \to X_{t+1}$ is divisorial); see [KM98, the proof of Proposition 3.37].

Assertion (2) is true because $N^1(X_r)$ is generated by K_{X_r}, $N^1(X_t)$ is isomorphic to $N^1(X_{t+1})$ (respectively spanned by E_t' and the pullback of $N^1(X_{t+1})$) when $X_t \dashrightarrow X_{t+1}$ is a flip (respectively divisorial). $\square$

To conclude Theorem 1.2(2), take an ample divisor $H \subset X$. By Claim 3.16, we can write $H \sim_{\mathbb{Q}} \sum a_t E_t + b(-K_X)$. So $H \leqslant m(E - K_X)$ for some $m \geqslant 1$, since $\kappa(X, -K_X) \geqslant 0$. This and Claim 3.15(3) and Claim 3.16(1) imply $\kappa(X, -K_X) = \kappa(X, E - K_X) \geqslant \kappa(X, H) = \dim X$. Thus, $-K_X$ is big. Theorem 1.2(2) is proved. $\square$

3.17 Proof of Theorem 1.3

Since X is Fano, X is rationally connected (by Campana and Kollár–Miyaoka–Mori), and $\overline{\mathrm{NE}}(X)$ has only finitely many extremal rays all of which are K_X-negative (cf. [KM98, Theorem 3.7]). Let $X \to X_1$ be the smooth blowdown such that X_1 is a primitive (smooth) Fano threefold in the sense of [MM81]. If $\rho(X) \geqslant 2$, by [MM81, Theorem 5], X_1 has an extremal contraction of conic bundle type. Now Theorem 1.3 follows from Theorem 3.3. $\square$

3.18 Proof of Theorem 3.4

By Lemma 2.9, we may assume that $S(X)$ is a finite set. We may also assume $\rho(X) \geqslant 3$. Suppose that $R_i := \mathbb{R}_{\geqslant 0}[C_i]$ $(i \geqslant 1)$ are pairwise distinct K_X-negative extremal rays with $\pi_i : X \to Y_i$ the corresponding contraction each of which is either divisorial or of Fano type (i.e., $\dim Y_i \leqslant 2$). We can take the generator C_i to be an irreducible curve in the fibre of π_i. Since $3 \leqslant \rho(X) = \rho(Y_i) + 1$, we have $\rho(Y_i) \geqslant 2$ and hence $\dim Y_i \in \{2, 3\}$.

If π_i is divisorial, we let E_i be the exceptional divisor of π_i; then E_i is necessarily irreducible and is in the finite set $S(X)$. If π_i is of Fano type (and hence onto a surface Y_i), then Y_i is a rational surface with at worst Du Val singularities (cf. [MP08, Theorem 1.2.7]); for each $G \in S(Y_i)$, the divisor $\pi_i^* G$ is irreducible and in $S(X)$.

The claim below follows from the fact that $\rho(X/Y_i) = 1$.

CLAIM 3.19. Suppose that either D is the exceptional divisor E_i for a divisorial contraction $\pi_i : X \to Y_i$, or $D = \pi_i^* G$ for a Fano contraction $\pi_i : X \to Y_i$ to a surface with $G \subset Y_i$ an irreducible curve. Then $N^1(X)|D$, as a subspace of $N_1(X)$, is of rank less than or equal to two and contains the extremal ray R_i of $\overline{\mathrm{NE}}(X)$.

Suppose, after replacing with an infinite subsequence, that each π_i is either divisorial and we let $D_i := E_i$, or is of Fano type with $S(Y_i) \neq \emptyset$ and we let $D_i = \pi_i^* G$ for some $G \in S(Y_i)$. Since $D_i \in S(X)$ and $S(X)$ is finite, we may assume that $D_1 = D_2 = \cdots$ after replacing with an infinite subsequence. If $N^1(X)|D_i \subset N_1(X)$ contains only one extremal ray, i.e., R_i, then $R_1 = R_2$, which contradicts the hypothesis. If $N^1(X)|D_i$ has two extremal rays R_i, R_i', then either $R_i = R_j$ for some $i \neq j$, which contradicts the hypothesis; or $R_2 = R_1' = R_3$, which again contradicts the hypothesis.

Thus, replacing $\{\pi_i\}$ with an infinite subsequence, we may assume that for every $i \geqslant 1$, π_i is of Fano type and $S(Y_i) = \emptyset$. Hence Y_i is relatively minimal, $\rho(Y_i) = 2$ and there is a $\mathbb{P}^1$-fibration $Y_i \to B_i \cong \mathbb{P}^1$ with every fibre irreducible, noting that K_{Y_i} is not pseudo-effective (cf. [Sa87, Theorem 3.2]). Take a general fibre X_{b_i} of the composite $X \to Y_i \to B_i$ which is a smooth ruled surface, noting that $\mathrm{Sing}\, X$ and hence its image in B_i are finite sets. Then $R_i . X_{b_i} = 0$.

Now $\rho(X) = \rho(Y_i) + 1 = 3$. Any three of C_i are linearly independent in $N_1(X)$ and hence form a basis; otherwise, $C_3 = a_1 C_1 + a_2 C_2$, say, with $a_1 > 0$, $a_2 \geqslant 0$ and hence $R_1 = R_3$, since R_3 is extremal. This is impossible.

Suppose that $R_1 . X_{b_i} = 0$, i.e., $\pi_1(X_{b_i}) \neq Y_1$, for $i = 2, 3, 4$. Then $X_{b_i} = \pi_1^* M_i$ for an irreducible curve $M_i \subset Y_1$ since $\rho(X/Y_1) = 1$. Since $\rho(Y_1) = 2$ and $q(Y_1) = 0$, we may assume that $M_4 \sim_{\mathbb{Q}} a_2 M_2 + a_3 M_3$ and hence $X_{b_4} \sim_{\mathbb{Q}} a_2 X_{b_2} + a_3 X_{b_3}$. Note that $0 = X_{b_4}^2 = 2 a_2 a_3 X_{b_2} X_{b_3}$. After relabeling, we may assume that X_{b_3} and X_{b_4} are parallel in $N^1(X)$. Then $X_{b_3} = \pi_1^* M_3$ is perpendicular to all of C_1, C_3, C_4, a basis of $N_1(X)$. Hence $X_{b_3} = 0$ in $N^1(X)$. This contradicts the hypothesis.

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Therefore, we may assume that $\pi_1(X_{b_i}) = Y_1$ for all $i \geqslant 2$, after replacing with a subsequence. Since $S(Y_1) = \emptyset$ and $\rho(Y_1) = 2$, our $\overline{\mathrm{NE}}(Y_1)$ is generated by two extremal pseudo-effective divisors L_{1k} with $L_{1k}^2 = 0$. We may assume that L_{11} is a fibre of $Y_1 \to B_1$. Let $M_{ik} := \pi_1^* L_{1k} | X_{b_i}$. Then $M_{ik}^2 = 0$ and M_{ik}'s span the (only) two extremal rays of $(N^1(X) \mid X_{b_i} \cap \overline{\mathrm{NE}}(X_{b_i})$. We may assume that C_i is a fibre of $\pi_i | X_{b_i}$ and hence is extremal and parallel to M_{i1} or M_{i2}. If C_i is parallel to M_{i1} for $i = r, s, t$, then by Claim 3.19 applied to $N^1(X) | \pi_1^* L_{11}$, two of the (extremal) C_i are parallel to each other in $N_1(X)$, contradicting the fact that the R_i are all distinct. If C_i is parallel to M_{i2} for $i = u, v, w$, then $(\pi_1 | X_{b_i})_* C_i$ is parallel to L_{12} and we may assume that L_{12} is an irreducible curve. Applying Claim 3.19 to $N^1(X) | \pi_1^* L_{12}$, we get a similar contradiction. This proves Theorem 3.4. □

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Appendix. Termination of extremal rays of fibration type for the iteration of surjective endomorphisms

Yoshio Fujimoto and Noboru Nakayama

The purpose of this note is to prove the following.

Theorem A.1. *Let X be a normal projective variety defined over an algebraically closed field of characteristic zero such that X has only log-terminal singularities. Let $R \subset \overline{\mathrm{NE}}(X)$ be an extremal ray such that $K_X R < 0$ and the associated contraction morphism cont_R is a fibration to a lower-dimensional variety. Then, for any surjective endomorphism $f \colon X \to X$, there exists a positive integer k such that $(f^k)_*(R) = R$ for the automorphism $(f^k)_* \colon \mathsf{N}_1(X) \xrightarrow{\simeq} \mathsf{N}_1(X)$ induced from the iteration $f^k = f \circ \cdots \circ f$.*

A special case is proved in Theorem 2.13 of a recent paper [Zha08b] of D.-Q. Zhang. We extend and simplify the idea of Zhang. The authors express their gratitude to Professor De-Qi Zhang for informing them of his paper [Zha08b].

Notation A.2. For a normal projective variety X, let $\mathsf{N}^1(X)$ denote the vector space $\mathrm{NS}(X) \otimes \mathbb{R}$ for the Néron–Severi group $\mathrm{NS}(X)$. The dimension of $\mathsf{N}^1(X)$ is called the *Picard number* and is denoted by $\rho(X)$. The numerical equivalence class $\mathrm{cl}(D)$ of a Cartier divisor D on X is regarded as an element of $\mathsf{N}^1(X)$. The dual vector space of $\mathsf{N}^1(X)$ is denoted by $\mathsf{N}_1(X)$, i.e., $\mathsf{N}_1(X) = \mathrm{Hom}(\mathrm{NS}(X), \mathbb{R})$. An element $u \in \mathsf{N}^1(X)$ is regarded as a linear function on $\mathsf{N}_1(X)$. We denote by $u^\perp$ the kernel of $u \colon \mathsf{N}_1(X) \to \mathbb{R}$. The cone $\mathrm{NE}(X)$ of the numerical equivalence classes $\mathrm{cl}(Z)$ of the effective 1-cycles Z on X is defined in $\mathsf{N}_1(X)$, by the intersection pairing $D \mapsto DZ \in \mathbb{Z}$ for Cartier divisors D on X.

The closure of $\mathrm{NE}(X)$ in $\mathsf{N}_1(X)$ is denoted by $\overline{\mathrm{NE}}(X)$, which is a *strictly convex* cone, i.e., $\overline{\mathrm{NE}}(X) + \overline{\mathrm{NE}}(X) \subset \overline{\mathrm{NE}}(X)$ and $\overline{\mathrm{NE}}(X) \cap (-\overline{\mathrm{NE}}(X)) = \{0\}$. An *extremal ray* R of $\overline{\mathrm{NE}}(X)$ is by definition a one-dimensional face of the cone $\overline{\mathrm{NE}}(X)$, i.e., $R = \mathbb{R}_{\geqslant 0} v = u^{\perp} \cap \overline{\mathrm{NE}}(X)$ for some $0 \neq v \in \overline{\mathrm{NE}}(X)$ and for some $u \in \mathsf{N}^1(X)$ which is non-negative on $\overline{\mathrm{NE}}(X)$ as a function on $\mathsf{N}_1(X)$. For a Cartier divisor D on X, $DR > 0$ means that the functional $\mathrm{cl}(D)$ on $\mathsf{N}_1(X)$ is positive on $R \setminus \{0\}$. The meanings of $DR = 0$ and $DR < 0$ are similar.

FACT A.3 [Kaw84]. Let X be a normal projective variety with only log-terminal singularities, i.e., $(X, 0)$ has only log-terminal singularities in the sense of [Kaw84]. For an extremal ray R of $\overline{\mathrm{NE}}(X)$ with $K_X R < 0$, there exist a proper surjective morphism $\mathrm{cont}_R \colon X \to Y$ onto a normal projective variety Y satisfying the following two conditions.

(1) Every fiber of cont_R is connected.

(2) For an irreducible closed curve C on X, $\mathrm{cont}_R(C)$ is a point if and only if $\mathrm{cl}(C) \in R$.

The morphism cont_R is uniquely determined by the conditions (1) and (2), and is called the *contraction morphism* associated with R. The following property holds by [Kaw84, Corollary 4.4].

(3) If D is a Cartier divisor on X with $DR = 0$, then $D \sim \mathrm{cont}_R^*(E)$ for a Cartier divisor E on Y.

Remark A.4. Let $f \colon X \to Y$ be a surjective morphism between normal projective varieties. Then, we have the pullback homomorphism $f^* \colon \mathsf{N}^1(Y) \to \mathsf{N}^1(X)$ which is well-defined by $f^*(\mathrm{cl}(D)) := \mathrm{cl}(f^*(D))$ for Cartier divisors D on Y. We have also the push-forward homomorphism $f_* \colon \mathsf{N}_1(X) \to \mathsf{N}_1(Y)$ as the dual of f^*. Here, for any irreducible closed curve C on X, we have $f_*(\mathrm{cl}(C)) = \mathrm{cl}(f_*(C))$ for the 1-cycle

$$f_*(C) = \begin{cases} \deg(C/f(C))C & \text{if } f(C) \text{ is not a point,} \\ 0 & \text{otherwise.} \end{cases}$$

Since f is surjective, $f^* \colon \mathsf{N}^1(Y) \to \mathsf{N}^1(X)$ is injective and $f_* \colon \mathsf{N}_1(X) \to \mathsf{N}_1(Y)$ is surjective. Assume that $\rho(X) = \rho(Y)$. Then f^* and f_* above are both isomorphisms, since $\mathsf{N}^1(X)$ and $\mathsf{N}^1(Y)$ have the same dimension. In particular, we have $f_*(\overline{\mathrm{NE}}(X)) = \overline{\mathrm{NE}}(Y)$ from the obvious equality $f_*(\mathrm{NE}(X)) = \mathrm{NE}(Y)$. Moreover, f is a finite morphism; in fact, $f(C)$ is not a point for any irreducible closed curve C on X by $f_*(\mathrm{cl}(C)) \neq 0$.

LEMMA A.5. *In the situation of Theorem A.1, $f_*(R)$ is also an extremal ray of $\overline{\mathrm{NE}}(X)$ such that $K_X f_*(R) < 0$.*

Proof. The push-forward map $f_* \colon \mathsf{N}_1(X) \to \mathsf{N}_1(X)$ is an automorphism preserving the cone $\overline{\mathrm{NE}}(X)$. Thus, $f_*(R)$ is extremal. Let E_f be the ramification divisor of $f \colon X \to X$, i.e., $K_X = f^*(K_X) + E_f$. Since E_f is effective, the restriction of E_f to a general fiber of cont_R is also effective. Hence, $E_f \gamma \geqslant 0$ for a general curve γ contracted to a point by cont_R. Thus $0 > K_X \gamma \geqslant (f^* K_X)\gamma = K_X(f_* \gamma)$. Therefore, $K_X f_*(R) < 0$. □

Notation A.6. For the extremal ray R in Theorem A.1, let R_k be the extremal ray $f_*^k(R)$ for $k \geqslant 0$. By Fact A.3 and Lemma A.5, we have the associated contraction morphism cont_{R_k}, which is denoted by $\pi_k \colon X \to Y_k$. Then, $\pi_{k+1} \circ f = h_k \circ \pi_k$ for a finite surjective morphism $h_k \colon Y_k \to Y_{k+1}$

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by the condition (2) in Fact A.3; in particular, we have the following commutative diagram.

$$
\begin{tikzcd}
X \arrow[r, equal] \arrow[d, "\pi"'] & X \arrow[r, "f"] \arrow[d, "\pi_0"'] & X \arrow[r, "f"] \arrow[d, "\pi_1"'] & \cdots \arrow[r, "f"] & X \arrow[r, "f"] \arrow[d, "\pi_k"'] & X \arrow[r, "f"] \arrow[d, "\pi_{k+1}"'] & \cdots \\
Y \arrow[r, equal] & Y_0 \arrow[r, "h_0"] & Y_1 \arrow[r, "h_1"] & \cdots \arrow[r] & Y_k \arrow[r, "h_k"] & Y_{k+1} \arrow[r, "h_{k+1}"] & \cdots
\end{tikzcd}
$$

Here, we simply write $\pi = \pi_0$ and $Y = Y_0$. We define $m := \dim Y$ and $\rho := \rho(X) - 1 \geqslant 0$. Then $m = \dim Y_k$, $\rho = \rho(Y_k)$, and $h_k^* \colon \mathsf{N}^1(Y_{k+1}) \to \mathsf{N}^1(Y_k)$ is an isomorphism for any $k \geqslant 0$.

LEMMA A.7. *Theorem A.1 is true if $\rho \leqslant 1$.*

Proof. Assume that $\rho = \rho(X) - 1 = 0$. Then $\mathsf{N}_1(X)$ is one-dimensional and $\overline{\mathrm{NE}}(X)$ is just a single ray. Thus $R_k = R$ for any k. Assume next that $\rho = \rho(X) - 1 = 1$. Then $\overline{\mathrm{NE}}(X)$ has exactly two extremal rays. Hence, f_*^2 preserves each extremal ray. Therefore, $R = R_{2k}$ for any k. $\square$

LEMMA A.8. *Let D be a Cartier divisor on Y such that $\pi^*(D)R_k = 0$ for some $k \geqslant 1$. If the self-intersection number $D^m \neq 0$, then $R = R_k$.*

Proof. By the property (3) in Fact A.3 of the contraction morphism of an extremal ray, we have a Cartier divisor D_k on Y_k such that $\pi^*(D) \sim \pi_k^*(D_k)$. Let A be an ample divisor on X. Then the product $\pi^*(D)^m A^{n-m-1}$ in the Chow ring of X is numerically equivalent to δZ for a non-zero effective 1-cycle Z and for $\delta := D^m \neq 0$. Thus,

$$\pi^*(L)Z = \delta^{-1}\pi^*(LD^m)A^{n-m-1} = 0 \quad \text{and} \quad \pi_k^*(L_k)Z = \delta^{-1}\pi_k^*(L_k D_k^m)A^{n-m-1} = 0$$

for any Cartier divisor L on Y and any Cartier divisor L_k on Y_k. In particular, the numerical equivalence class $\mathrm{cl}(Z)$ is contained in $R \cap R_k$. Therefore, $R = R_k$. $\square$

Proof of Theorem A.1. We shall derive a contradiction from the converse assumption that $R \neq R_k$ for any $k \geqslant 1$. Then, $R_k \neq R_j$ for any $j \neq k$, since $f_* \colon \mathsf{N}_1(X) \to \mathsf{N}_1(X)$ is an automorphism by Remark A.4. We have $\rho \geqslant 2$ by Lemma A.7. In particular, $\dim Y = m \geqslant 2$. Let $\{H_1, \ldots, H_\rho\}$ be a set of ample divisors of Y such that $\{\mathrm{cl}(H_1), \ldots, \mathrm{cl}(H_\rho)\}$ is a basis of $\mathsf{N}^1(X)$. We have $(\pi^* H_i)R_k > 0$ for any $1 \leqslant i \leqslant \rho$ and $k \geqslant 1$ by the property (3) in Fact A.3, since $R \neq R_k$. Hence, we can define a positive rational number $a_k^{(j)}$ for $2 \leqslant j \leqslant \rho$ and $k \geqslant 1$ by the equation:

$$\pi^*(H_j - a_k^{(j)} H_1) \cdot R_k = 0. \tag{A1}$$

Then $(H_j - a_k^{(j)} H_1)^m = 0$ for any j and k by Lemma A.8. On the other hand, for each $2 \leqslant j \leqslant \rho$, there exist at most m solutions for $x \in \mathbb{C}$ of the equation: $(H_j - xH_1)^m = 0$. Then, there exist rational numbers $\alpha_2, \ldots, \alpha_\rho$ such that, for infinitely many integers k, the equalities $\alpha_j = a_k^{(j)}$ hold for any $2 \leqslant j \leqslant \rho$. In fact, we can find a rational number α_2 such that the set S_2 of positive integers k with $\alpha_2 = a_k^{(2)}$ is infinite. Next, we can find a rational number α_3 such that the set S_3 of integers $k \in S_2$ with $\alpha_3 = a_k^{(3)}$ is infinite. If the rational numbers α_j with the sets S_j up to $l < \rho$ are selected, then we can find a rational number α_{l+1} such that the set S_{l+1} of integers $k \in S_l$ with $\alpha_{l+1} = a_k^{(l+1)}$ is infinite. In this way, we can find $\alpha_2, \alpha_3, \ldots, \alpha_\rho$ satisfying the required property.

The real vector subspace

$$F := \pi^*(\mathrm{cl}(H_2 - \alpha_2 H_1))^\perp \cap \cdots \cap \pi^*(\mathrm{cl}(H_\rho - \alpha_\rho H_1))^\perp \subset \mathsf{N}_1(X)$$

is two-dimensional, since $\pi^*(\mathrm{cl}(H_2 - \alpha_2 H_1)), \ldots, \pi^*(\mathrm{cl}(H_\rho - \alpha_\rho H_1))$ are linearly independent. We have $R_k \subset F$ for infinitely many k by the choice of $\alpha_2, \ldots, \alpha_\rho$ and by (A1). This is a contradiction, since there exist at most two extremal rays of $\overline{\mathrm{NE}}(X)$ contained in the two-dimensional vector subspace F. Thus, we are done. □

Remark A.9. In Theorem A.1, we can not allow the case where cont_R is a birational morphism. In fact, there exist a smooth projective surface X with an automorphism f and a (-1)-curve γ on X such that $\{f^k(\gamma) \mid k \geqslant 0\}$ is infinite. Here, $R = \mathbb{R}_{\geqslant 0}\,\mathrm{cl}(\gamma)$ is an extremal ray with $K_X R < 0$ and $f_*^k(R) = \mathbb{R}_{\geqslant 0}\,\mathrm{cl}(f^k(\gamma))$ for the (-1)-curve $f^k(\gamma)$. Thus $f_*^k(R) \neq R$ for any k. One of such a surface X is given as a blown-up surface of $\mathbb{P}^2$ whose center is the intersection of two sufficiently general cubic curves. In fact, X is a rational elliptic surface and any exceptional curve of the blowing up is a section of the elliptic fibration. Let Γ_0 and Γ_1 be two exceptional curves. Let X_K be the generic fiber of the elliptic fibration and P_i the point $\Gamma_i|_{X_K}$ defined over the function field K of the base curve. We give a group structure of the elliptic curve X_K such that P_0 is the zero element. Then, P_1 is not torsion by the choice of cubic curves. The translation mapping $X_K \to X_K$ by P_1 gives rise to a birational automorphism $f\colon X \to X$, which is in fact regular, since the elliptic surface X is relatively minimal over the base curve. Therefore, f is an automorphism of infinite order and $f^k(\Gamma_1) \neq \Gamma_1$ for any k. Thus, the conclusion of Theorem A.1 does not hold for X, f, and $R = \mathbb{R}_{\geqslant 0}\,\mathrm{cl}(\Gamma_1)$.

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