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THE FUNCTIONAL EQUATION OF ZETA DISTRIBUTIONS ASSOCIATED WITH PREHOMOGENEOUS VECTOR SPACES $(\tilde{G}, \tilde{\rho}, M(n, C))$

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Introduction

Let (G, ρ, V) be a triple of a linear algebraic group G and a rational representation ρ on a finite dimensional vector space V, all defined over the complex number field C.

We call the triple (G, ρ, V) a prehomogeneous vector space if G has a Zariski-open orbit. Assume that the triple (G, ρ, V) is a prehomogeneous vector space. Then there exists a proper algebraic subset S of V such that $V-S$ is a single G-orbit. The algebraic set S is called the singular set of (G, ρ, V). For a rational character of G, a non-zero rational function P on V is called a relative invariant of (G, ρ, V) corresponding to χ if

$$P(\rho(g)x) = \chi(g)P(x) \qquad (g \in G,\ x \in V).$$

Let $P_1, \cdots, P_n$ be irreducible polynomials defining the components of S with codimension 1. It is known that $P_1, \cdots, P_n$ are relative invariants of (G, ρ, V) (cf. [1]). The set $\{P_1, \cdots, P_n\}$ is called a complete set of irreducible relative invariants of (G, ρ, V).

The purpose of this paper is to give an explicit expression for the Fourier transform of relative invariants on a certain class of prehomogeneous vector spaces.

NOTATION. We denote by Z, R and C the ring of integers, the rational number field and the complex number field, respectively. For $z \in C$, we set $e(z) = \exp(2\pi\sqrt{-1}z)$. We denote by $M(n, C)$ (resp. $M(n, R)$) the complex (resp. real) vector space consisting of all n by n matrices with entries in C (resp. R). For any matrix x, ${}^t x$ denotes the transposed matrix. For

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$x \in M(n, C)$, we set $x^* = {}^t x^{-1}$. For a C^∞-manifold X, $C_0^\infty(X)$ denotes the space of C^∞-functions with compact support on X. We denote by $\Gamma(z)$ the usual Gamma function. We denote by $B_n(C)$ (resp. $B_n(R)$) the subgroup of the general linear group $GL(n, C)$ (resp. $GL(n, R)$) consisting of all upper triangular matrices.

§1. Prehomogeneous vector space $(\tilde{G}, \tilde{\rho}, M(n, C))$

1.1. Let G be a linear algebraic group, ρ: $G \to GL(n, C)$ a rational representation of G both defined over C. We denote by $\tilde{G}$ the direct product group $G \times B_n(C)$. For any $x \in M(n, C)$ and $\tilde{g} = (g, a) \in \tilde{G}$, set $\tilde{\rho}(\tilde{g})x = \rho(g)xa^{-1}$. Then $\tilde{\rho}$ is a rational representation of $\tilde{G}$. We denote by $\tilde{\rho}^*$ the contragredient representation to $\tilde{\rho}$. It is known that the triple $(\tilde{G}, \tilde{\rho}, M(n, C))$ is a prehomogeneous vector space if and only if the triple $(G, \tilde{\rho}^*, M(n, C))$ is a prehomogeneous vector space. In what follows we assume that the triplet $(\tilde{G}, \tilde{\rho}, M(n, C))$ is a P.V. Let $\{P_0, \cdots, P_k\}$ be a complete set of irreducible relative invariants of $(\tilde{G}, \tilde{\rho}, M(n, C))$ and $\chi_0, \cdots, \chi_k$ characters of $P_0, \cdots, P_k$, respectively. Since $\det x$ is an irreducible relative invariant of $(\tilde{G}, \tilde{\rho}, M(n, C))$, we may set $P_0(x) = \det x$. Let $P(x)$ be any relative invariant polynomial of $(\tilde{G}, \tilde{\rho}, M(n, C))$. For any $x \in M(n, C)$, we denote by x^ℓ the ℓ-th column vector of x. Then it is known that $P(x)$ is homogeneous with respect to each column vector x^ℓ $(1 \leq \ell \leq n)$. Denoting by λ_ℓ the homogeneous degree of $P(x)$ with respect to x^ℓ, one can show that λ_ℓ's satisfy

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n\,, \qquad \text{(cf. [6])}\,.$$

Denoting by λ the n-tuple $(\lambda_1, \cdots, \lambda_n)$, we call λ the partition corresponding to the relative invariant polynomial $P(x)$. Let $\lambda(0), \cdots, \lambda(k)$ be partitions corresponding to $P_0, \cdots, P_k$, respectively. We set

$$\begin{aligned} &P_0^*(x) = P_0(x) = \det x\,, \qquad \chi_0^* = \chi_0\,, \\ &P_i^*(x) = P_i(x^*)P_0(x)^{\lambda(i)_1} \quad \text{and} \quad \chi_i^* = \chi_i^{-1}\chi_0^{\lambda(i)_1} \quad (1 \leq i \leq k)\,. \end{aligned}$$

Then one sees easily that $P_0^*, \cdots, P_k^*$ are relative invariants of $(\tilde{G}, \tilde{\rho}^*, M(n, C))$ and satisfy

$$P_i^*(\tilde{\rho}^*(\tilde{g})x) = \chi_i^*(\tilde{g})^{-1}P_i(x) \qquad (0 \leq i \leq k)\,.$$

Moreover the set $\{P_0^*, \cdots, P_k^*\}$ is a complete set of irreducible relative invariants of $(\tilde{G}, \tilde{\rho}^*, M(n, C))$ (cf. [6]).

We denote by $X_\rho(\tilde{G})$ the group of all rational characters corresponding to the relative invariants of $(\tilde{G}, \tilde{\rho}, M(n, \boldsymbol{C}))$. It is known that the group $X_\rho(\tilde{G})$ is a free abelian group of rank $k+1$ generated by $\chi_0, \cdots, \chi_k$ and hence there exists $k+1$-tuples $(\delta(\chi_0), \cdots, \delta(\chi_k))$ and $(\delta^*(\chi_0), \cdots, \delta^*(\chi)_k) \in \boldsymbol{Z}^{k+1}$ such that

$$\chi = \prod_{i=0}^{k} \chi_i^{\delta(\chi)_i} = \prod_{i=0}^{k} \chi_i^{*\delta^*(\chi)_i}, \qquad \text{(cf. [6])}.$$

For any $\chi \in X_\rho(\tilde{G})$, we set

$$P_\chi = \prod_{i=0}^{k} P_i^{\delta(\chi)_i}, \qquad P_\chi^* = \prod_{i=0}^{k} P_i^{*\delta^*(\chi)_i},$$

$$\delta(\chi) = (\delta(\chi)_0, \cdots, \delta(\chi)_k) \quad \text{and} \quad \delta^*(\chi) = (\delta(\chi)_0, \cdots, \delta^*(\chi)_k).$$

Let $\lambda(0), \cdots, \lambda(k)$ be partitions of $P_0, \cdots, P_k$, respectively. For any $s = (s_0, \cdots, s_k) \in \boldsymbol{C}^{k+1}$, we write

$$\lambda_\ell(s) = \sum_{i=0}^{k} s_i \lambda(i)_\ell$$

and

$$\lambda_\ell^*(s) = s_0 + \sum_{i=1}^{k} s_i(\lambda(i)_1 - \lambda(i)_\ell) \qquad (1 \leq \ell \leq n).$$

Furthermore we set

$$\gamma(s) = \prod_{\ell=1}^{n} \Gamma(\lambda_\ell(s) + n - \ell + 1)$$

and

$$\gamma^*(s) = \prod_{\ell=1}^{n} \Gamma(\lambda_\ell^*(s) + n - \ell + 1).$$

Then the "b-function" of $(\tilde{G}, \tilde{\rho}, M(n, \boldsymbol{C}))$ (resp. $(\tilde{G}, \tilde{\rho}, M(n, \boldsymbol{C}))$ is given by

$$b_\chi(s) = \frac{\gamma(s)}{\gamma(s - \delta(\chi))} \quad \left(\text{resp. } b_\chi^*(s) = \frac{\delta^*(s)}{\gamma^*(s - \delta^*(\chi))}\right), \quad \text{(cf. [6])}.$$

For $s = (s_0, s_1, \cdots, s_k) \in \boldsymbol{C}^{k+1}$, we set

$$s^* = (s_0^*, s_1^*, \cdots, s_k^* \; ,$$

where $s_0^* = -\lambda_1(s)$ and $s_i^* = s_i$ $(1 \leq i \leq k)$.

LEMMA 1. *Notations being as above, one has*

(i) $\lambda_\ell^*(s^*) = -\lambda_\ell(s)$ $(1 \leq \ell \leq n)$,

(ii) $\delta(\chi)^* = -\delta^*(\chi)$.

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Proof. (i) $\lambda_\ell^*(s^*) = \sum_{i=1}^{k} s_i(\lambda(i)_1 - \lambda(i)_\ell) - \sum_{i=0}^{k} s_i\lambda(i)_1$

$$= -\sum_{i=0}^{k} s_i\lambda(i)_\ell$$

$$= -\lambda_\ell(s)$$

(ii) $\delta(\chi)^* = (-\lambda_1(\delta(\chi)),\ \delta(\chi)_1, \cdots, \delta(\chi)_k)$

$$= \left(-\sum_{i=0}^{k}\delta(\chi)_i\lambda(i)_1,\ \delta(\chi)_1 \cdots, \delta(\chi)_k\right).$$

Then one sees immediately

$$\delta^*(\chi) = \left(\sum_{i=0}^{k}\delta(\chi)_i\lambda(i)_1,\ -\delta(\chi)_1, \cdots, \delta(\chi)_k\right). \qquad \text{Q.E.D.}$$

If x is not contained in the singular set of $(\tilde{G}, \tilde{\rho}, M(n, \boldsymbol{C}))$, it follows from the definition of $P_\chi^*(x)$ that

(1) $$P_\chi^*(x^*) = P_\chi(x)^{-1}.$$

For $\chi \in X_\rho(\tilde{G})$, we set

$$d(\chi) = \sum_{i=0}^{k}\delta(\chi)_i \deg P_i \quad \text{and} \quad d^*(\chi) = \sum_{i=0}^{k}\delta^*(\chi)_i \deg P_i^*.$$

1.2. In the following, we assume that G is defined over $\boldsymbol{R}$. Denoting by $G_{\boldsymbol{R}}$ the set of $\boldsymbol{R}$-rational points of G, we set

$$\tilde{G}_{\boldsymbol{R}} = G \times B_n(\boldsymbol{R}),$$
$$S_{\boldsymbol{R}} = S \cap M(n, \boldsymbol{R}),$$
$$S_{\boldsymbol{R}}^* = S^* \cap M(n, \boldsymbol{R}),$$
$$\tilde{\rho}_{\boldsymbol{R}} = \tilde{\rho}|_{\tilde{G}_{\boldsymbol{R}}}.$$

Furthermore we always assume the following conditions:

(A.1) $G_{\boldsymbol{R}}$ is a connected subgroup of $GL(n, \boldsymbol{R})$.

(A.2) the singular set S of $(\tilde{G}, \tilde{\rho}, M(n, \boldsymbol{C}))$ is the union of irreducible hypersurfaces of the form

$$S_i = \{x \in M(n, \boldsymbol{R});\ P_i(x) = 0\} \qquad (0 \leq i \leq k),$$

where, for each i, $P_i(x)$ is a $\boldsymbol{C}$-irreducible polynomial with real coefficients.

(A.3) $M(n, \boldsymbol{R}) - S_{\boldsymbol{R}}$ is a single $\tilde{\rho}_{\boldsymbol{R}}(\tilde{G}_{\boldsymbol{R}})$-orbit.

We denote by $\tilde{G}_{\boldsymbol{R}}^0$ the connected components of the identity and consider the $\tilde{\rho}_{\boldsymbol{R}}(\tilde{G}_{\boldsymbol{R}}^0)$-orbital decomposition of $M(n, \boldsymbol{R}) - S_{\boldsymbol{R}}$

$$M(n, \boldsymbol{R}) - S_{\boldsymbol{R}} = V_1 \cup \cdots \cup V_\nu.$$

For $\tilde{\rho}_{R}(G_{R}^{0})$-orbit V_i, we set

$$V_i^* = \{x \in M(n, \boldsymbol{R});\ x^* \in V_i\}.$$

Then one sees that the set $M(n, \boldsymbol{R}) - S_{\boldsymbol{R}}^*$ is decomposed into the disjoint union of $\tilde{\rho}_{\boldsymbol{R}}^*(G_{\boldsymbol{R}}^0)$-orbits

$$M(n, \boldsymbol{R}) - S_{\boldsymbol{R}}^* = V_1^* \cup \cdots \cup V_\nu^*.$$

For $s = (s_0, \cdots, s_k) \in \boldsymbol{C}^{k+1}$, we set

$$|P(x)|^s = \prod_{i=0}^{k} |P_i(x)|^{s_i}, \qquad |P^*(x)|^s = \prod_{i=0}^{k} |P_1^*(x)|^{s_i},$$

$$|\chi(g)|^s = \prod_{i=0}^{k} |\chi_i(g)|^{s_i}, \qquad |\chi^*(g)|^s = \prod_{i=0}^{k} |\chi_i^*(g)|^{s_i}$$

§ 2. Fourier transforms of relative invariants

2.1. We denote by $S(M(n, \boldsymbol{R}))$ the Schwartz space of the vector space $M(n, \boldsymbol{R})$. We consider the following integrals:

$$\Phi_i(f, s) = \int_{V_i} f(x)|P(x)|^s dx \tag{2}$$

and

$$\Phi_i^*(f, s) = \int_{V_i^*} f(x)|P^*(x)|^s dx \qquad (1 \leq i \leq \nu) \tag{3}$$

where dx is the Euclidean measure on V_i. If $\operatorname{Re}(s)_0 > 0, \cdots, \operatorname{Re}(s)_k > 0$, the above integrals $\Phi_i(f, s)$, $\Phi_i^*(f, s)$ are absolutely convergent.

For $\chi \in X_\rho(\tilde{G})$, we set

$$\varepsilon_i(\chi) = \operatorname{sgn} P_\chi|_{V_i} \quad \text{and} \quad \varepsilon_i^*(\chi) = \operatorname{sgn} P_\chi^*|_{V_i^*} \quad (1 \leq i \leq \nu).$$

By (1), one has $\varepsilon_i(\chi) = \varepsilon_i^*(\chi)$, $(1 \leq i \leq \nu)$. We also set, for $s = (s_0, \cdots, s_k) \in \boldsymbol{C}^{k+1}$,

$$d(s) = \sum_{\ell=0}^{k} s_\ell \deg P_\ell, \qquad d^*(s) = \sum_{\ell=0}^{k} s_\ell \deg P_\ell^*$$

$$\varepsilon_i(s) = e\Big(\frac{1}{4}\sum_{\ell=0}^{k} s_\ell(1 - \varepsilon_i(\chi_\ell))\Big), \qquad \varepsilon_i^*(s) = e\Big(\frac{1}{4}\sum s_\ell(1 - \varepsilon_i^*(\chi_\ell^*))\Big).$$

Then, one can easily check:

$$d(\chi) = d(\delta(\chi)), \qquad d^*(\chi) = d^*(\delta^*(\chi)),$$
$$\varepsilon_i(\chi) = \varepsilon_i(\delta(\chi)), \qquad \varepsilon_i^*(\chi) = \varepsilon_i^*(\delta^*(\chi)),$$
$$d(s) = -d^*(s^*) \quad \text{and} \quad d(\chi) = d^*(\chi).$$

We set

$$F_i(f, s) = \frac{1}{\gamma(s)} \Phi_i(f, s) \quad \text{and} \quad F_i^*(f, s) = \frac{1}{\gamma^*(s)} \Phi_i^*(f, s) .$$

Denoting by $\hat{f}$ the Fourier transform of f, one can easily prove the following

LEMMA 2. *If* $\mathrm{Re}(s_0), \cdots, \mathrm{Re}(s_k)$ *are sufficiently large, one has*
(i) *for any* $\chi \in X_\rho(\tilde{G})$, *such that* $\delta^*(\chi)_0, \cdots, \delta^*(\chi)_k \geq 0$,

$$F_i(\widehat{p_\chi^* f}, s) = (-2\pi\sqrt{-1})^{-d^*(\chi)} \varepsilon_i(\chi) F_i(\hat{f}, s - \delta(\chi))$$

and for any $\chi \in X_\rho(\tilde{G})$ *such that* $\delta(\chi)_0, \cdots, \delta(\chi)_k \geq 0$,

$$F_i^*(\widehat{P_\chi \cdot f}, s) = (2\pi\sqrt{-1})^{-d(\chi)} \varepsilon_i(\chi) F_i^*(\hat{f}, s - \delta^*(\chi))$$

(ii) *for any* $\chi \in X_\rho(\tilde{G})$ *such that* $\delta(\chi)_0, \cdots, \delta(\chi)_k \geq 0$,

$$F_i(\widehat{P_\chi(\mathrm{grad}) \cdot f}, s) = (-2\pi\sqrt{-1})^{d(\chi)} \varepsilon_i(\chi) b_\chi(s + \delta(\chi)) F_i(\hat{f}, s + \delta(\chi))$$

and for any $\chi \in X_\rho(\tilde{G})$ *such that* $\delta^*(\chi)_0, \cdots, \delta^*(\chi)_k \geq 0$,

$$F_i^*(\widehat{P_\chi^*(\mathrm{grad}) f}, s) = (2\pi\sqrt{-1})^{d^*(\chi)} \varepsilon_i^*(\chi) b_\chi^*(s + \delta^*(\chi)) F_i^*(\hat{f}, s + \delta^*(\chi)) .$$

(iii) *for any* $\chi \in X_\rho(\tilde{G})$ *such that* $\delta^*(\chi)_0, \cdots, \delta^*(\chi)_k \geq 0$,

$$F_i(P_\chi^*(\mathrm{grad}) f, s) = \varepsilon_i(\chi)(-1)^{d^*(\chi)} F_i(f, s - \delta(\chi))$$

and for any $\chi \in X_\rho(\tilde{G})$ *such that* $\delta(\chi)_0, \cdots, \delta(\chi)_k \geq 0$,

$$F_i^*(P_\chi(\mathrm{grad}) f, s) = \varepsilon_i^*(\chi)(-1)^{d(\chi)} F_i^*(f, s - \delta^*(\chi)) .$$

Let D be the domain in $\boldsymbol{C}^{k+1}$ defined by

$$D = \{(s_0, \cdots, s_k) \in \boldsymbol{C}^{k+1};\ \mathrm{Re}(s_0) > 0, \cdots, \mathrm{Re}(s_k) > 0\} .$$

Then one sees that s^* is contained in D when s is contained in D. By Lemma 2 (iii), one has, for any i $(1 \leq i \leq \nu)$,

$$F_i(P_0^m(\mathrm{grad}) f, s) = \varepsilon_i(\chi_0)^m (-1)^{nm} F_i(f, s - m\delta(\chi_0))$$

and

$$F_i^*(P_0^m(\mathrm{grad}) f, s) = \varepsilon_i(\chi_0)^m (-1)^{nm} F_i^*(f, s - m\delta^*(\chi_0))$$

if $\mathrm{Re}(s_0), \cdots, \mathrm{Re}(s_k)$ are sufficiently large. Hence we can continue analytically $F_i(f, s)$ and $F_i^*(f, s)$ to holomorphic functions on D. Again, by Lemma 2 (iii), one can easily show that the mapping $f \to F_i(f, s)$ (resp.

$F_i^*(f, s))$ defines a tempered distribution on the vector space $M(n, \boldsymbol{R})$ when s is contained in D (cf. Proposition 1.3 in [3]). We call this tempered distribution a zeta distribution associated with the prehomogeneous vector space $(\tilde{G}, \tilde{\rho}, M(n, \boldsymbol{C}))$.

Putting

$$\Phi(f, s) = {}^t(\Phi_1(f, s), \cdots, \Phi_\nu(f, s))$$

and

$$\Phi^*(f, s) = {}^t(\Phi_1^*(f, s), \cdots, \Phi_\nu(f, s)),$$

one has the following proposition.

PROPOSITION 1. *The vector valued functions $\Phi(f, s)$ and $\Phi^*(f, s)$ satisfy a functional equation of the following form*

$$\Phi(\hat{f}, s - n\delta(\chi_0)) = \gamma(s - \delta(\chi_0))C(s)\Phi^*(f, s^*),$$

where s varies in the domain D and $C(s)$ is a $\nu \times \nu$ matrix whose entries $C_{ij}(s)$ are holomorphic in D.

This proposition can be proved by the similar argument to Theorem 1.1 in [3]. For the sake of completement, we shall give a proof.

Proof of Proposition 1. For $f \in S(M(n, \boldsymbol{R}))$, set $g \cdot f(x) = f(\tilde{\rho}^*(g)^{-1} \cdot x)$ $(g \in \tilde{G}_{\boldsymbol{R}}^0)$. Then one has $\widehat{g \cdot f}(x) = \chi_0^{-n}(g)\hat{f}(g^{-1}x)$ and hence it follow that

$$F_i(\widehat{g \cdot f}, s - n\chi_0) = |\chi_0(g)|^{-n}|\chi(g)|^s F_i(\hat{f}, s - n\chi_0),$$
$$F_i(g \cdot f, s^*) = |\chi_0(g)|^{-n}|\chi^*(g)|^{-s^*} F_i(f, s^*).$$

On the other hand, one can easily check that $|\chi(g)|^s = |\chi^*(g)|^{-s^*}$. Then by a theorem of Bruhat (Theorem 3.1 in [7]), there exist holomorphic functions $C_{ij}(s)$ $(1 \leq i, j \leq \nu)$ such that

$$F_i(\hat{f}, s - n\chi_0) = \gamma^*(s^*) \sum_{j=1}^{\nu} C_{ij}(s) F_j^*(f, s^*)$$

for all $f \in C_0^\infty(M(n, \boldsymbol{R}) - S_{\boldsymbol{R}}^*)$. We denote by T_s a tempered distribution on $M(n, \boldsymbol{R})$ defined as

$$T_s(f) = F_i(\hat{f}, s - n\chi_0) - \gamma^*(s^*) \sum_{j=1}^{\nu} C_{ij}(s) F_j^*(f, s^*) \qquad (s \in D).$$

One can find a non-negative integer M such that the order of the tempered distribution T_s does not exceed M for all s contained in the set $D_0 = \{s \in D;\ -1 \leq \operatorname{Re} s_0 \leq 0\}$. If $\delta^*(\chi)_0, \cdots, \delta^*(\chi)_k \geq M$, it follows from Lemma

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1.3 in [3] that $T_s(P_\chi^* f) = 0$ ($s \in D$, $-1 \leq s_0 \leq 0$ and $f \in C_0^\infty(M(n, \boldsymbol{R}))$. Take a $\chi \in X_\rho(\tilde{G})$ such that $\delta^*(\chi)_0, \cdots, \delta^*(\chi)_k \geq 0$. From Lemma 2, it follows that, for every $f \in C_0^\infty(M(n, \boldsymbol{R}) - S_{\boldsymbol{R}})$,

$$F_i(\widehat{P_x^* \cdot f}, s - n\chi_0)$$
$$= (-2\pi\sqrt{-1})^{-d^*(\chi)} \varepsilon_i(\chi) \sum_{j=1}^{\nu} C_{ij}(s - \delta(\chi)) \gamma^*((s - \delta(\chi))^*) F_j^*(f^*, (s - \delta(\chi))^*)$$

and

$$F_i(\widehat{P_\chi^* \cdot f}, s - n\chi_0) = \sum C_{ij}(s) \varepsilon_j^*(\chi) \gamma^*(s^* + \delta^*(\chi)) F_j^*(f^*, s^* + \delta^*(\chi)) .$$

Hence, by using the relation $(s - \delta(\chi))^* = s^* + \delta^*(\chi)$, one obtains

$$C_{ij}(s) = (-2\pi\sqrt{-1})^{-d^*(\chi)} \varepsilon_i(\chi) \varepsilon_j^*(\chi) C_{ij}(s - \delta(\chi)) .$$

Therefore, for any $f \in C_0^\infty(M(n, \boldsymbol{R}))$ and $\delta^*(\chi)_0, \cdots, \delta^*(\chi)_k \geq M$, one has

$$T_s(P_\chi^* f) = (-2\pi\sqrt{-1})^{-d(\chi)} \varepsilon_i(\chi) T_{s-\delta(\chi)}(f) .$$

This implies that, for any $s \in D_0$,

$$T_{s-\delta(\chi)}(f) = 0 \qquad (f \in C_0^\infty(M),\ \delta^*(\chi)_0, \cdots, \delta^*(\chi)_k \geq M) .$$

Since T_s is a tempered distribution and $T_s(f)$ is a holomorphic function of s in D, we can conclude $T_s(f) = 0$ ($s \in D, f \in S(M(n, \boldsymbol{R}))$ which proves our proposition.

Remark. It is known that the integrals $\Phi_1(f, s), \cdots, \Phi_\nu(f, s), \Phi_1^*(f, s), \cdots, \Phi_\nu^*(f, s)$ have analytic continuation to meromorphic functions of s in $\boldsymbol{C}^{k+1}$ (cf. [3], [5]) and hence Proposition 1 holds in $\boldsymbol{C}^{k+1}$.

We denote by $\varepsilon(s)$ the ν by ν matrix whose entries are given by

$$\varepsilon_{ij}(s) = (2\pi)^{-d(s)} e\left(\frac{d(s)}{4}\right) \varepsilon_i(s) \varepsilon_j^*(s^*) , \quad 1 \leq i, \quad j \leq \nu .$$

Then it is easy to verify the following relation, for any $\chi \in X_\rho(\tilde{G})_{\boldsymbol{R}}$,

$$\varepsilon_{ij}(s) = (-2\pi\sqrt{-1})^{-d(\chi)} \varepsilon_i(\chi) \varepsilon_j^*(\chi) \varepsilon_{ij}(s - \delta(\chi)) .$$

We set $t_{ij}(s) = C_{ij}(s) \varepsilon_{ij}(s)^{-1}$ $(1 \leq i, j \leq \nu)$. Then one sees

$$t_{ij}(s) = t_{ij}(s - \delta(\chi)) \qquad (\chi \in X_\rho(\tilde{G}_{\boldsymbol{R}})) .$$

2.2. In this paragraph, we shall give an explicit expression for the functions $t_{ij}(s)$. We denote by D the group consisting of n by n diagonal matrices whose diagonal entries are 1 or -1. For two subsets A and B

of $GL(n, \boldsymbol{R})$, we write $A \underset{D}{\sim} B$ if there exists a matrix g in D such that $A = B\cdot g$. We set, for any integer i $(1 \leq i \leq \nu)$,

$$K_i = \{k \in SO(n, \boldsymbol{R});\ ^{\exists}a \in B_n^0(\boldsymbol{R}) \text{ such that } k\cdot a \in V_i e_i\}$$

and

$$K_i^* = \{k \in SO(n, \boldsymbol{R});\ ^{\exists}a \in B_n^0(\boldsymbol{R}) \text{ such that } k\cdot a^* \in V_i^* e_i\}\,,$$

where

$$e_i = \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1, & \\ & & & a_i \end{bmatrix} \qquad (a_i = \varepsilon_i(\chi_0))\,.$$

From (A.3), it follows that $V_i \underset{D}{\sim} V_j$ $(1 \leq i, j \leq \nu)$. On the other hand, the Iwasawa decomposition for the group $GL(n, \boldsymbol{R})^0$ shows that, for any i $(1 \leq i \leq \nu)$,

$$V_i = K_i \cdot B_n(\boldsymbol{R})^0 \cdot e_i\,, \tag{4}$$

$$V_i^* = K_i^* \cdot B_n(\boldsymbol{R})_-^0 \cdot e_i\,, \tag{5}$$

where $B_n(\boldsymbol{R})_-$ stands for the subgroup of $GL(n, \boldsymbol{R})$ consisting of lower triangular matrices. Since the mapping $x \to x^*$ gives a one-to-one correspondence between V_i and V_i^*, one has $K_i = K_i^*$ $(1 \leq i \leq \nu)$.

Using the Iwasawa decomposition

$$(x_{ij}) = g\cdot(t_{ij}), \quad ((x_{ij}) \in GL(n, \boldsymbol{R})^0,\ (t_{ij}) \in B_n(\boldsymbol{R})^0,\ g \in SO(n, \boldsymbol{R}))\,,$$

we normalize a Haar measure dg on $SO(n, \boldsymbol{R})$ by setting

$$\prod_{1\leq i, j\leq n} dx_{ij} = \prod_{i=1}^{n} t_{ii}^{n-i} \prod_{i\leq j} dt_{ij}\cdot dg\,.$$

Then, one has

$$\int_{K_i} |P(g)|^s dg = \int_{K_i^*} |P^*(g)|^s dg\,, \qquad (1 \leq i, j \leq \nu)\,.$$

Let $f(s)$ be a function on $\boldsymbol{C}^{k+1}$. For a character $\chi \in X_\rho(\tilde{G})$, we set:

$$\sigma_i(\chi) f(s) = \varepsilon_i(\chi) f(s + \delta(\chi))\,, \qquad (1 \leq i \leq \nu)\,. \tag{6}$$

We also set

$$E_i(s) = (-2)^n (2\pi)^{n(n-1)/2} e\Big(-\frac{d(s)}{4}\Big)\varepsilon_i(-s) \prod_{\ell=1}^{n} \sin\frac{\pi}{2}(\lambda_\ell(s) - \ell)\,.$$

We set

$$\varepsilon_j^* = (\varepsilon_j(\chi_0^*), \cdots, \varepsilon_j(\chi_r^*)), \qquad 1 \leq j \leq n.$$

PROPOSITION 2. *Assume that, for all p, q ($p \neq q$), $\varepsilon_p^* \neq \varepsilon_q^*$. Then $t_{ij}(s)$ is given by*

$$t_{ij}(s) = \left(\frac{1}{2}\right)^{k+1} \varepsilon_j^*(-s^*) \prod_{\ell=0}^{k} (1 + \sigma_j(\chi_\ell) E_i(s)).$$

Proof. Let f be a function on the vector space $M(n, \boldsymbol{R})$ defined by $f(x) = \exp(-\pi(x, x))$. Then $f = \hat{f}$. We make change of variables (4) and (5). Then, using an well known formula:

$$\frac{1}{2}\pi^{-(1+z)/2}\Gamma\left(\frac{1+z}{2}\right) = \int_0^\infty t^z e^{-\pi t^2} dt,$$

one has

$$\begin{aligned}\Phi_i(f, s) &= \int_{K_i} |P(g)|^s dg \cdot \int_{0<t_\ell<\infty} \exp\left(-\pi \sum_{\ell=1}^n t_\ell^2\right) \prod_{\ell=1}^n t_\ell^{\lambda_\ell(s)+n-\ell} \prod_{\ell=1}^n dt_\ell \\ &= \int_{K_i} |P(g)|^s dg \cdot \left(\frac{1}{2}\right)^n \pi^{-\frac{1}{4}n(n+1)} \pi^{-\frac{1}{2}d(s)} \sum_{\ell=1}^n \Gamma\left(\frac{\lambda_\ell(s) + n - \ell + 1}{2}\right)\end{aligned}$$

and

$$\begin{aligned}\Phi_i^*(f, s^*) &= \int_{K_i^*} |P^*(g)|^{s^*} \cdot \int_{0<t_\ell<\infty} \exp\left(-\pi \sum_{\ell=1}^n t_\ell^2\right) \prod_{\ell=1}^n t_\ell^{\lambda_\ell^*(s^*)+\ell-1} \prod_{\ell=1}^n dt_\ell \\ &= \int_{K_i^*} |P(g)|^s dg \cdot \left(\frac{1}{2}\right)^n \pi^{-\frac{1}{4}n(n+1)} \pi^{\frac{1}{2}d(s)} \prod_{\ell=1}^n \Gamma\left(\frac{-\lambda_\ell(s) + \ell}{2}\right).\end{aligned}$$

Thus, from Proposition 1, one obtains

$$\begin{aligned}&\frac{1}{\gamma(s - n\chi_0)} \prod_{\ell=1}^n \Gamma\left(\frac{\gamma_\ell(s) - \ell + 1}{2}\right) \\ &\quad = \sum_{j=1}^{\nu} \varepsilon_{ij}(s) t_{ij}(s) \prod_{\ell=1}^n \left(\frac{-\lambda_\ell(s) + \ell}{2}\right), \qquad (1 \leq i \leq \nu).\end{aligned} \tag{7}$$

Using well known formulas of Γ-function:

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

and

$$\Gamma(2z) = \frac{2^{z-1}}{\sqrt{\pi}} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right),$$

we can rewrite (7) as

$$(8)\qquad \sum_{j=1}^{\nu} \varepsilon_{ij}(s)t_{ij}(s) = (-2)^n(2\pi)^{-d(s)+\frac{1}{2}n(n-1)} \prod_{\ell=1}^{n} \sin \pi\left(\frac{\lambda_\ell(s)-\ell}{2}\right), \quad (1 \le i \le \nu).$$

Hence one has

$$E_i(s) = \sum_{\ell=1}^{\nu} \varepsilon_\ell^*(s^*)t_{i\ell}(s).$$

From (6), it follows that

$$(9)\qquad \begin{aligned}\sigma_j(\chi)E_i(s) &= \varepsilon_j^*(\chi)\sum_{\ell=1}^{\nu} \varepsilon_\ell^*((s+\delta(\chi))^*)t_{i\ell}(s)\\ &= \sum_{\ell=1}^{\nu} \varepsilon_j^*(\chi)\varepsilon_\ell^*(\chi)\varepsilon_\ell^*(s^*)t_{i\ell}(s).\end{aligned}$$

Putting $L_j(\chi) = \{\ell;\ \varepsilon_j^*(\chi) = \varepsilon_\ell^*(\chi).\}$, one can rewrite (9) as

$$\frac{1}{2}(1+\sigma_j(\chi))E_i(s) = \sum_{\ell \in L_j(\chi)} \varepsilon_\ell^*(s^*)t_{i\ell}(s).$$

Then, by our assumption of Proposition 2, one obtains

$$\left(\frac{1}{2}\right)^{k+1} \sum_{\ell=0}^{k} (1+\sigma_j(\chi_\ell))E_i(s) = \varepsilon_j^*(s^*)t_{ij}(s),$$

which proves our assertion.

By Proposition 1 and Proposition 2, we have the following theorem.

THEOREM 1. *Assume that, for all p, q* $(p \ne q,\ 1 \le p,\ q \le \nu)$, $\varepsilon_p^* \ne \varepsilon_q^*$. *Then the tempered distributions* $\Phi_1(f, s), \cdots, \Phi_\nu(f, s)$, $\Phi_1^*(f, s), \cdots, \Phi_\nu^*(f, s)$ *defined by* (2) *and* (3) *satisfy a system of functional equations of the following form.*

$$\Phi(\hat{f}, s - n\delta(\chi_0)) = \gamma(s - n\delta(\chi_0))C(s)\Phi^*(f, s^*)),$$

where $C(s)$ *is the* ν *by* ν *matrix whose entries are given by*

$$(10)\qquad \begin{aligned} C_{ij}(s) &= (-2)^n\left(\frac{1}{2}\right)^{k+1}(2\pi)^{n(n-1)/2-d(s)}e\left(\frac{d(s)}{4}\right)\varepsilon_i(s)\\ &\times \prod_{r=0}^{k}(1+\sigma_j(\chi_r))e\left(-\frac{d(s)}{4}\right)\varepsilon_i(-s)\prod_{\ell=1}^{n}\sin\frac{\pi}{2}(\lambda_\ell(s)-\ell),\\ &\qquad (1 \le i,\ j \le n).\end{aligned}$$

§ 3. Examples

Let G be a connected semi-simple linear algebraic group and ρ an n-dimensional irreducible representation both defined over $\boldsymbol{C}$. Assume that the triple $(\tilde{G}, \tilde{\rho}, M(n, \boldsymbol{C}))$ is a prehomogeneous vector space. Then the group G must be one of the following subgroups of $SL(n, \boldsymbol{C})$ and ρ the identity representation of G;

$$G = SL(n, \boldsymbol{C}), \quad SO(n, \boldsymbol{C}) \text{ or } Sp(m, \boldsymbol{C}) \text{ witn } n = 2m, \quad (\text{cf. [6]}).$$

Case 1. (cf. [2], [6]) $G = SL(n, \boldsymbol{C})$.

In this case, $\{\det x\}$ is a complete set of irreducible relative invariants and the b-function is given by

$$\gamma(s) = \Gamma(s+n) \cdots \Gamma(s+1), \qquad s \in \boldsymbol{C}.$$

Since the singular set is given by

$$S = S^* = \{x \in M(n, \boldsymbol{C}), \ \det x = 0\}.$$

We have the orbit decomposition

$$M(n, \boldsymbol{R}) - S_{\boldsymbol{R}} = V_1 \cup V_2,$$

where

$$V_1 = \{x \in M(n, \boldsymbol{R}), \ \det x > 0\} \quad \text{and} \quad V_2 = \{x \in M(n, \boldsymbol{R}), \ \det x < 0\}.$$

Then one has

$$\begin{aligned} C_{ij}(s) &= 2^{n-1}(2\pi)^{n(n-1)/2-ns} \\ &\cdot \left\{\prod_{\ell=1}^{n} \cos\frac{\pi}{2}(s-\ell+1) + (\sqrt{-1})^n(-1)^{i+j}\prod_{\ell=1}^{n} \sin\frac{\pi}{2}(s-\ell+1)\right\}, \\ &(1 \leq i, j \leq 2). \end{aligned} \tag{11}$$

By Theorem 1, one has a system of functional equations.

PROPOTISON 3. *The zeta distributions for* $G = SL(n, \boldsymbol{C})$ *have the following system of functional equations*:

$$\Phi_i(\hat{f}, s-n) = \Gamma(s)\Gamma(s-1) \cdots \Gamma(s-n+1) \sum_{j=1}^{2} C_{ij}(s)\Phi_j(f, -s),$$

where $C_{ij}(s)$ *is given by* (11), $(i = 1, 2)$.

Case 2. $G = SO(n, \boldsymbol{C})$.

For $x \in M(n, \boldsymbol{R})$, we denote by x^i the i-th column vector of x. Put

$$P_0(x) = \det x$$

and

$$P_i(x) = \det \begin{bmatrix} (x^1, x^1), & \cdots, & (x^1, x^i) \\ \vdots & & \vdots \\ (x^i, x^1), & \cdots, & (x^i, x^i) \end{bmatrix} \qquad (1 \leq i \leq n-1),$$

where (x^j, x^k) denotes the usual inner product,

$$\left(\text{i.e. } (x^j, x^k) = \sum_{\alpha=1}^{n} x_\alpha^j x_\alpha^k\right).$$

Then $\{P_0, \cdots, P_{n-1}\}$ is a complete set of irreducible relative invariants of this prehomogeneous vector space, and the singular set S is given by

$$S = \cup S_i,$$
$$S_i = \{x \in M(n, \boldsymbol{R});\ P_i(x) = 0\} \qquad (0 \leq i \leq n-1).$$

The orbit decomposition of $M(n, \boldsymbol{R}) - S_{\boldsymbol{R}}$ is given by

$$M(n, \boldsymbol{R}) - S_{\boldsymbol{R}} = V_1 \cup V_2,$$

where

$$V_1 = \{x \in M(n, \boldsymbol{R}) - S_{\boldsymbol{R}};\ \det x > 0\}$$

and

$$V_2 = \{x \in M(n, \boldsymbol{R}) - S_{\boldsymbol{R}};\ \det x < 0\}.$$

For $s = (s_0, s_1, \cdots, s_{n-1}) \in \boldsymbol{C}^n$, one sees

$$d(s) = ns_0 + \sum_{\ell=1}^{n-1} 2\ell s_\ell$$

$$\lambda_\ell(s) = s_0 + \sum_{m=\ell}^{n} 2ms_m, \qquad (1 \leq \ell \leq n),$$

and

$$\varepsilon_i(s) = e\left(\frac{1-(-1)^{i-1}}{4} s_0\right), \qquad (i = 1, 2).$$

Thus one has:

$$C_{ij}(s) = 2^{n-1}(2\pi)^{n(n-1)/2}\left(\prod_{\ell=1}^{n} \cos\frac{\pi}{2}(\lambda_\ell(s) - \ell + 1) + (-1)^{i+j}(\sqrt{-1})^n \prod_{\ell=1}^{n} \sin\frac{\pi}{2}(\lambda_\ell(s) - \ell + 1)\right), \qquad (1 \leq i, j \leq 2). \tag{12}$$

From Theorem 1, we obtain the following proposition.

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PROPOSITION 4. *The zeta distributions for $G = SO(n, \boldsymbol{C})$ have the following system of functional equation: for $i = 1, 2$,*

$$\Phi_i(\hat{f}, s - n\delta(\chi_0)) = \prod_{\ell=1}^{n} \Gamma\left(s_0 + \sum_{m=\ell}^{n} 2ms_m - \ell + 1\right) \sum_{j=1}^{2} C_{ij}(s)\Phi_j(f, s^*) ,$$

where $C_{ij}(s)$ is given by (12) *and*

$$s = (s_0, s_1, \cdots, s_{n-1}) , \qquad s^* = \left(-s_0 - \sum_{m=1}^{n} 2ms_m, s_2, \cdots, s_{n-1}\right) .$$

Case 3. $G = Sp(m, \boldsymbol{C})$, $(n = 2m)$.

Denoting by $[x, y]$ the skew symmetric bilinear form on $\boldsymbol{C}^n \times \boldsymbol{C}^n$ defined as

$$[x, y] = \sum_{i=1}^{m} (x_i y'_i - x'_i y_i)$$

with $x = {}^t(x_1, x'_1, \cdots, x_m, x'_m)$ and $y = {}^t(y_1, y'_1, \cdots, y_m, y'_m)$, we set

$$P_0(x) = \det x$$

and, for $i = 1, 2, \cdots, m-1$,

$$P_i(x) = \mathrm{Pff}\begin{bmatrix} [x^1, x^1], & \cdots, & [x^1, x^{2i}] \\ \vdots & & \vdots \\ [x^{2i}, x^1], & \cdots, & [x^{2i}, x^{2i}] \end{bmatrix},$$

where Pff denotes the Pfaffian.

Then $\{P_0, \cdots, P_{m-1}\}$ is a complete set of the irreducible relative invariants of this prehomogeneous vector space, and the orbit decompositions are given as follows:

$$M(n, \boldsymbol{R}) - S_{\boldsymbol{R}} = \bigcup_{i \in I} V_i \text{ and } M(n, \boldsymbol{R}) - S^*_{\boldsymbol{R}} = \bigcup_{i \in I} V^*_i ,$$

where I denotes a set consisting of all m-tuples $(i_0, \cdots, i_{m-1})$ with each i_j is equal to 1 or -1, and V_i is described as

$$V_i = \{x \in M(n, \boldsymbol{R}) - S_{\boldsymbol{R}};\ \mathrm{sgn}\, P_\ell = i_\ell\} , \qquad (0 \leq \ell \leq m) .$$

In this case, one has:

$$d(s) = 2ms_0 + \sum_{\ell=1}^{m-1} 2\ell s_\ell ,$$

$$\lambda_\ell(s) = s_0 + \sum_{2i \geq \ell} s_i ,$$

$$\gamma(s) = \prod_{\ell=1}^{n} \Gamma(s_0 + \sum_{2i \geq \ell} s_i + n - \ell + 1)\,,$$
$$\varepsilon_i(s) = e\Big(\frac{1}{4}\sum_{\ell=0}^{m-1} s_\ell(1 - i_\ell)\Big).$$

Thus one has

$$C_{ij}(s) = 2^m (2\pi)^{m(2m-1)-d(s)} e\Big(\frac{d(s)}{4}\Big)\varepsilon_i(s) \prod_{r=0}^{k} (1 + \sigma_j(\chi_r))$$
$$\times\, e\Big(-\frac{d(s)}{4}\Big)\varepsilon_i(-s) \prod_{\ell=1}^{n} \sin\frac{\pi}{2}(s_0 + \sum_{2i \geq \ell} s_i - \ell)\,.$$

From Theorem 1, we obtain the following proposition.

PROPOSITION 5. *The zeta distributions for $G = Sp(m, \boldsymbol{C})$ have the following system of functional equations,*

$$\Phi_i(\hat{f}, s - n\delta(\chi_0)) = \prod_{\ell=1}^{n} (s_0 + \sum_{2i \geq \ell} s_i + n - \ell + 1) \cdot \sum_{j \in I} C_{ij}(s) \Phi_i(f, s^*)\,.$$

Now, we shall give an example such that G is not reductive. Let G be a subgroup of $SL(n, \boldsymbol{C})$ consisting of all lower triangular matrices whose diagonal entries are all equal to 1 and ρ a representation of G defined by

$$\rho(g)x = g \cdot x\,, \qquad x \in M(n, \boldsymbol{C})\,.$$

Then the triplet $(\tilde{G}, \tilde{\rho}, M(n, \boldsymbol{C}))$ is a prehomogeneous vector space. For $x = (x_{\alpha\beta}) \in M(n, \boldsymbol{C})$, we set

$$P_0(x) = \det x$$

and

$$P_i(x) = \det \begin{bmatrix} x_{11}, \cdots, x_{1i} \\ \vdots \qquad \vdots \\ x_{i1}, \cdots, x_{ii} \end{bmatrix}, \qquad (1 \leq i \leq n-1)\,.$$

One sees that $\{P_0, \cdots, P_{n-1}\}$ is a complete set of irreducible relative invariants of this space and the orbit decomposition is given by

$$M(n, \boldsymbol{R}) - S_{\boldsymbol{R}} = \bigcup_{i \in I} V_i$$

where I denotes the set of all n-tuples $(i_0, \cdots, i_{n-1})$ with $i_\ell = 1$ or -1, and

$$V_i = \{x \in M(n, \boldsymbol{R}) - S_{\boldsymbol{R}};\ \operatorname{sgn} P_\ell = i_\ell,\ 0 \leq \ell \leq n-1\}\,.$$

In this case, one has

$$d(s) = ns_0 + \sum_{\ell=1}^{n-1} \ell s_\ell$$

$$\lambda_\ell(s) = s_0 + \sum_{i \geq \ell} s_i\,, \qquad (1 \leq \ell \leq n)\,,$$

and

$$\varepsilon_i(s) = e\left(\frac{1}{4}\sum_{\ell=0}^{n-1} s_\ell(1-i_\ell)\right), \qquad (i \in I)\,.$$

Thus, one obtains

$$C_{ij}(s) = (-1)^n(2\pi)^{n(n-1)/2-d(s)}e\left(\frac{d(s)}{4}\right)\varepsilon_i(s) \tag{13}$$

$$\times \prod_{r=0}^{k}(1+\sigma_j(\chi_r))e\left(-\frac{d(s)}{4}\right)\varepsilon_i(-s)\prod_{\ell=1}^{n}\sin\frac{\pi}{2}(s_0+\sum_{i\geq\ell}s_i-\ell)\,.$$

By Theorem 1, we have the following proposition

PROPOSITION 6. *The zeta distributions for this group have the following system of functional equations, for any* $i \in I$

$$\Phi_i(\hat{f}, s - n\delta(\chi_0)) = \prod_{\ell=1}^{n}\Gamma(s_0 + \sum_{i\geq\ell}s_i + n - \ell + 1)\cdot\sum_{j\in I}C_{ij}(s)\phi_j(f, s^*)\,,$$

where $C_{ij}(s)$ *is given by* (13).

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