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NOTE ON INTEGERS REPRESENTABLE BY BINARY QUADRATIC FORMS

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Let B be the set of positive integers prime to d which are representable by some primitive, positive, integral binary quadratic form of discriminant d. It is the purpose of this note to show that the following asymptotic estimate for the number of integers in B less than or equal to x can be proved using only elementary arguments:

(1) $$B(x) = \sum_{m\le x,\, m\in B} 1 = c_1 \frac{x}{(\log x)^{1/2}} \{1+\mathcal{O}((\log\log x)^{-1}\} \qquad (x\to\infty),$$

where c_1 is the positive constant given in (17) below. (Using the deeper methods of complex analysis James [2] has proved this result with the error term $\mathcal{O}((\log x)^{-1/2})$ replacing $\mathcal{O}((\log\log x)^{-1})$. Heupel [1] using a transcendental method as in James [2] improved this to $\mathcal{O}((\log x)^{-1})$.)

We follow closely the ideas of Rieger [6] and set $M=\{n:p \mid n \Rightarrow (d/p)=1\}$ and $N=\{n:p \mid n\Rightarrow(d/p)=-1\}$. From the work of Selberg [7] we have

(2) $$\sum_{p\le x,\,(d/p)=1} \frac{\log p}{p} = \tfrac{1}{2}\log x+\mathcal{O}(1).$$

Appealing to (2) and a result of Rieger [5] we obtain

(3) $$m(x) = \sum_{m\le x, m\in M} \frac{1}{m} = \frac{e^{-c/2}}{\Gamma(\frac{3}{2})} \prod_{\substack{p\le x\\ (d/p)=1}} \left(1-\frac{1}{p}\right)^{-1} \{1+\mathcal{O}((\log\log x)^{-1})\},$$

where

$$c = -\int_0^\infty e^{-t}\log t\, dt.$$

Next we recall Merten's theorem ([3], p. 139)

(4) $$\prod_{p\le x}\left(1-\frac{1}{p}\right) = e^{-c}(\log x)^{-1}+\mathcal{O}((\log x)^{-2}),$$

and a result of Landau ([3], §109)

(5) $$\prod_{p\le x}\left(1-\frac{(d/p)}{p}\right) = \frac{1}{L(1)}+\mathcal{O}((\log x)^{-1}),$$

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where

$$L(1) = \sum_{n=1}^{\infty} \left(\frac{d}{n}\right) n^{-1} > 0.$$

Using (4) and (5) and an argument of Uchiyama [8] we obtain

$$(6) \qquad \prod_{p\le x,\,(d/p)=1} \left(1-\frac{1}{p}\right) = e^{-c/2} \prod_{p\mid d} \left(1-\frac{1}{p}\right)^{-1/2} \prod_{(d/p)=-1} \left(1-\frac{1}{p^2}\right)^{-1/2} L(1)^{-1/2}(\log x)^{-1/2} + \mathcal{O}((\log x)^{-3/2}).$$

Putting (6) into (3) we obtain

$$(7) \qquad m(x) = c_2(\log x)^{1/2}(1+\mathcal{O}((\log\log x)^{-1})),$$

where

$$(8) \qquad c_2 = \frac{2}{\sqrt{\pi}} \prod_{p\mid d} \left(1-\frac{1}{p}\right)^{1/2} \prod_{(d/p)=-1} \left(1-\frac{1}{p^2}\right)^{1/2} L(1)^{1/2}.$$

Selberg [7] (see also Wirsing [9]) has shown that if $(l, k)=1$ the following form of the prime number theorem for arithmetic progressions can be proved by elementary means

$$(9) \qquad \sum_{\substack{p\le x \\ p\equiv l \pmod k}} \log p = \frac{x}{\phi(k)} + \mathcal{O}\left(\frac{x}{(\log x)^{\alpha}}\right),$$

where α is a positive constant. Since $p \in M$ if and only if $(d/p)=1$ and this latter condition means that p lies in one of $(\frac{1}{2})\phi(4\prod_{p\mid d} p)$ residue classes mod $4\prod_{p\mid d} p$ which are prime to $4\prod_{p\mid d} p$, we have using (9)

$$(10) \qquad \sum_{p\le x,\, p\in M} \log p = \frac{x}{2}(1+\mathcal{O}((\log\log x)^{-1})).$$

Now using (7) and (10) in an argument of Rieger [6] (p. 199) we obtain

$$(11) \qquad M(x) = \sum_{m\le x,\, m\in M} = c_2 \frac{x}{2(\log x)^{1/2}}(1+\mathcal{O}((\log\log x)^{-1})).$$

Finally, noting that if k is prime to d, then k is represented by some primitive positive integral binary quadratic form of discriminant d, if and only if $k=mn^2$, where $m \in M$, $n \in N$, we have

$$(12) \qquad B(x) = \sum_{\substack{mn^2\le x \\ m\in M,\, n\in N}} 1 = \sum_{n\le\sqrt{x},\, n\in N} M(xn^{-2}),$$

and, since $M(t)\le t$, (12) gives

$$(13) \qquad B(x) = \sum_{n\le \log x,\, n\in N} M(xn^{-2}) + \mathcal{O}(x(\log x)^{-1}).$$

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From (11) we have

$$(14)\quad M(xn^{-2}) = c_2 \frac{x}{2n^2(\log x)^{/12}}(1+\mathcal{O}((\log\log x)^{-1})) \qquad (1 \le n < \log x),$$

and as

$$(15)\qquad \sum_{n\le y,\, n\in N} n^{-2} = \prod_{(d/p)=-1} (1-p^{-2})^{-1} + \mathcal{O}(y^{-1}) \qquad (y \to \infty),$$

from (13), (14) and (15) we obtain (1) with

$$(16)\qquad c_1 = \tfrac{1}{2}c_2 \prod_{(d/p)=-1} (1-p^{-2})^{-1}.$$

(16) together with (8) gives

$$(17)\qquad c_1 = \left(\frac{L(1)}{\pi}\right)^{1/2} \prod_{(d/p)=-1} (1-p^{-2})^{-1/2} \prod_{p\mid d} (1-p^{-1})^{1/2}.$$

(1) can be extended to all positive integers k by following Pall's argument in [4].

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