ON THE DISTRIBUTION OF PRIMES IN SHORT INTERVALS

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One of the formulations of the prime number theorem is the statement that the number of primes in an interval \((n, n + h]\), averaged over \(n \leq N\), tends to the limit \(\lambda\), when \(N\) and \(h\) tend to infinity in such a way that \(h \sim \lambda \log N\), with \(\lambda\) a positive constant.

In this note we study the distribution of values of \(\pi(n + h) - \pi(n)\), for \(n \leq N\) and \(h \sim \lambda \log N\). We show that, assuming a certain uniform version of the (unproved) prime \(r\)-tuple conjecture of Hardy and Littlewood [3], the distribution tends to the Poisson distribution with parameter \(\lambda\) as \(N \to \infty\). Using a sieve upper bound for the \(r\)-tuple problem, we also get an unconditional exponential upper bound for the tail of the distribution.

Our method has many features in common with the argument by which Hooley [4] has studied the distribution of values of the differences between consecutive integers prime to \(n\), for \(n/\phi(n)\) large. An analogous result for primes has been announced by Hooley in [5].

Explicitly, the \(r\)-tuple conjecture is an asymptotic formula for the number \(\pi_d(N)\) of positive integers \(n \leq N\) for which \(n + d_1, \ldots, n + d_r\) are all prime. Here \(d_1, \ldots, d_r\) are distinct integers. The formula is

\[
\pi_d(N) \sim \mathcal{S}_d \frac{N}{\log^r N} \quad (N \to \infty),
\]

(1)

provided \(\mathcal{S}_d \neq 0\), where

\[
\mathcal{S}_d = \prod_p p^{r-1} \frac{p - v_d(p)}{(p - 1)^r},
\]

and where \(v_d(p)\) is the number of distinct residue classes mod \(p\) occupied by \(d_1, \ldots, d_r\).

Formula (1) is the prime number theorem, for \(r = 1\). For \(r \geq 2\), it has not been proved for any \(d\); the source of (1) in these cases is a heuristic application of the circle method, and a summation of the corresponding (multiple) singular series [3]. Lavrik [8] has proved that (1) holds in mean over cubes \(1 \leq d_1, \ldots, d_r \leq H\), in the range \(N/\log^2 N \leq H \leq N\); a similar mean result for the (small) cubes of side \(h\) would suffice for our purpose.

**Theorem 1.** Denote by \(P_k(h, N)\) the number of integers \(n \leq N\) for which the interval \((n, n + h]\) contains exactly \(k\) primes. Then

\[
P_k(h, N) \sim N \frac{e^{-\lambda} \lambda^k}{k!}
\]

(2)

for \(N \to \infty\), \(h \sim \lambda \log N\), provided, for each \(r\), (1) holds, uniformly for \(1 \leq d_1, \ldots, d_r \leq h\), with \(d_1, \ldots, d_r\) distinct and \(\mathcal{S}_d \neq 0\).
Our argument for (2) goes through a computation of the moments of 
\( \pi(n + h) - \pi(n) \), and depends on the fact that, for each \( r \), \( \mathcal{S}_d \) averages to 1 over cubes:

\[
\sum_{1 \leq d_1, \ldots, d_r \leq h} \mathcal{S}_d \sim h^r \quad (h \to \infty).
\] (3)

For \( r = 2 \), a smoothed variant of this was used by Hardy and Littlewood to refute earlier asymptotic Goldbach conjectures. A simple proof of (3) for \( r = 2 \), starting with the singular series representation for \( \mathcal{S}_d \), was given by Bombieri and Davenport in [1]. Our proof of (3) starts with the product definition of \( \mathcal{S}_d \), and is closer to an argument of Hooley in [5].

Using Selberg's sieve, Klimov [7] obtained for each \( r \) the upper bound†

\[
\pi_d(N) \leq 2^r r! \mathcal{S}_d \frac{N}{\log^2 N}
\] (4)

for \( N \to \infty \), uniformly for \( d \) in small cubes. For this, see Halberstam and Richert [2], Theorem 5.7. Using (4) instead of (1), we get upper bounds for the \( k \)th moments of 
\( \pi(n + h) - \pi(n) \) for \( n \leq N \), as Bombieri and Davenport did for \( k = 2 \). For large \( k \), these bounds give

**THEOREM 2.** For positive constants \( \mu \geq \lambda \geq 1 \), the number of \( n \leq N \) for which 
\( \pi(n + \lambda \log N) - \pi(n) > \mu \) is \( \leq N e^{-C \mu / \lambda} \), where \( C \) is an absolute constant.

1. Reduction to (3). For each positive integer \( k \),

\[
\sum_{n \leq N} (\pi(n + h) - \pi(n))^k = \sum_{n \leq N} \sum_{\substack{d_1, \ldots, d_r \leq h \\text{distinct}}} 1
\]

\[
= \sum_{r=1}^{k} \sigma(k, r) \sum_{\pi_{d_1, \ldots, d_r}(N)},
\]

where the inner sum is over all \( r \)-tuples \( d_1, \ldots, d_r \) satisfying \( 1 \leq d_1 < \ldots < d_r \leq h \), and \( \sigma(k, r) \) is the number of maps from the set \( \{1, \ldots, k\} \) onto \( \{1, \ldots, r\} \). For the \( d \) with \( \mathcal{S}_d \neq 0 \), we use (1); for the others, \( d_1, \ldots, d_r \) occupy all residue classes modulo some prime, so \( \pi_d(N) \leq r \). Using (3), it follows that

\[
\sum_{\pi_{d_1, \ldots, d_r}(N)} \sim \frac{h^r}{r!} \frac{N}{\log^2 N},
\]

and hence

\[
\frac{1}{N} \sum_{n=1}^{N} (\pi(n + h) - \pi(n))^k \to m_k(\lambda),
\] (5)

with

\[
m_k(\lambda) = \sum_{r=1}^{k} \sigma(k, r) \frac{\lambda^r}{r!}.
\]

In §3, it is shown that \( m_k(\lambda) \) is the \( k \)th moment of the Poisson distribution with

† The notation \( F \leq G \) stands for \( \lim F/G \leq 1 \).
parameter \( \lambda \), and that the corresponding moment generating function is entire. The result (2) now follows from general theorems on moments [6, Chapter 4].

Putting \( h = \lambda \log N \), and using (4) instead of (1), we get

\[
\sum \pi_{d_1, \ldots, d_r}(N) \leq (2\lambda)^r N,
\]

from which it follows that

\[
\frac{1}{N} \sum_{n=1}^{N} (\pi(n+h) - \pi(n))^k \leq \sum_{r=1}^{k} \sigma(k, r)(2\lambda)^r \leq k(2\lambda)^k.
\]

Hence the proportion of \( n \leq N \) for which \( \pi(n+h) - \pi(n) \geq \mu \) is \( \leq k(2\lambda \mu)^k \) if \( \mu/\lambda \geq 4 \), we choose \( k = [\{\mu/\lambda}\}]. \) Then \( k \geq \{\mu/\lambda}\}, \) so the proportion is

\[
\leq k2^{-k} \leq e^{-C \mu/\lambda}.
\]

If \( \mu/\lambda < 4 \), the result is trivial.

2. Proof of (3). Let

\[
D_d = \prod_{i < j} (d_i - d_j).
\]

Then \( 1 \leq v_d(p) \leq r \), with equality at the right, unless \( p \mid D_d \). The \( p \)th factor in \( \mathscr{S}_d \) is

\[
1 + \frac{p^r - v_d(p)p^{r-1} - (p-1)^r}{(p-1)^r} = 1 + a(p, v_d(p)),
\]

where

\[
a(p, v) \ll r, \begin{cases} (p-1)^{-2}, & v = r; \\ (p-1)^{-1}, & v < r. \end{cases}
\]

It follows that the product for \( \mathscr{S}_d \) converges. Defining \( a_d(q) \) for squarefree \( q \) by

\[
a_d(q) = \prod_{p \mid q} a(p, v_d(p)),
\]

we get an absolutely convergent series expansion

\[
\mathscr{S}_d = \sum q a_d(q),
\]

where the sum is over squarefree \( q \).

We need an estimate for the remainder in (8) which is uniform for \( d \) in the \( h \)-cube. It follows from the bounds on \( a(p, v) \) that

\[
\sum_{q \neq x} |a_d(q)| \leq \sum_{q \neq x} \frac{\mu^2(q)C^{(q)}}{\phi^2(q)} \phi((q, D)),
\]

where \( \omega(q) \) is the number of prime factors of \( q \), and \( C \) is a positive constant depending only on \( r \). Putting \( q = de \) with \( d \mid D \) and \( (e, D) = 1 \), this is

\[
\sum_{d \mid b} \frac{\mu^2(d)C^{(d)}}{\phi(d)} \sum_{e \neq z/d \atop (e, D) = 1} \frac{\mu^2(e)C^{(e)}}{\phi^2(e)} \leq \sum_{d \mid b} \frac{\mu^2(d)C^{(d)}}{\phi(d)} \frac{d}{x} \log^b x \leq (xh)^r/x,
\]
with a constant depending only on \( r \) and \( \varepsilon \). It follows that

\[
\mathcal{S}_d = \sum_{d_1, \ldots, d_r \leq h \text{ distinct}} a_d(q) + O\left(\mathcal{H}(xh)^{r}/x\right),
\]

(9)

with a constant depending only on \( r \) and \( \varepsilon \).

The inner sum in (9) is

\[
\sum \prod_{p \mid q} a(p, v(p))\left\{\sum' 1 + O(h^{-1})\right\},
\]

where \( \sum' 1 \) stands for the number of \( r \)-tuples of not necessarily distinct integers \( d_1, \ldots, d_r \) with \( 1 \leq d_1, \ldots, d_r \leq h \) which, for each prime \( p \mid q \), occupy exactly \( v(p) \) residue classes mod \( p \); the outer sum is over all “vectors” \( (\ldots, v(p), \ldots)_{p \mid q} \) with components satisfying \( 1 \leq v(p) \leq p \). A simple lattice point argument using the Chinese remainder theorem gives, for \( q \leq h \),

\[
\sum' 1 = \left(\left(\frac{h}{q}\right)^r + O\left(\frac{h}{q}\right)^{r-1}\right) \prod_{p \mid q} \left(\frac{p}{v(p)}\right)\sigma(r, v(p));
\]

the product representing the number of ways of choosing the residue classes of \( d_1, \ldots, d_r \) mod \( q \) subject to the congruence restrictions in \( \sum' \).

Thus the inner sum in (9) is

\[
\left(\frac{h}{q}\right)^r A(q) + O\left(\left(\frac{h}{q}\right)^{r-1} B(q)\right) + O(h^{-1} C(q)),
\]

(10)

with

\[
A(q) = \sum \prod_{p \mid q} a(p, v(p))\left(\frac{p}{v(p)}\right)\sigma(r, v(p)),
\]

\[
B(q) = \sum \prod_{p \mid q} |a(p, v(p))|\left(\frac{p}{v(p)}\right)\sigma(r, v(p)),
\]

\[
C(q) = \sum \prod_{p \mid q} |a(p, v(p))|.
\]

We have

\[
A(q) = \prod_{p \mid q} \left\{\sum_{v=1}^r a(p, v)\left(\frac{p}{v}\right)\sigma(r, v)\right\},
\]

\[
B(q) = \prod_{p \mid q} \left\{\sum_{v=1}^r |a(p, v)|\left(\frac{p}{v}\right)\sigma(r, v)\right\},
\]

\[
C(q) = \prod_{p \mid q} \left\{\sum_{v=1}^r |a(p, v)|\right\}.
\]

We show first that \( A(q) = 0 \) for \( q > 1 \). Using (6), the \( p \)-th factor in \( A(q) \) is

\[
(p - 1)^{-r} \left\{(p^r - (p - 1)^r) \sum_{v=1}^r \left(\frac{p}{v}\right)\sigma(r, v) - p^{r-1} \sum_{v=1}^r v \left(\frac{p}{v}\right)\sigma(r, v)\right\}.
\]
By formulae (i) and (ii) of §3, the two sums here are \( p^r \) and \( p^{r+1} - (p - 1)^r p \) respectively, and the factor vanishes.

Using the bounds (7) for \( a(p, v) \), we may estimate \( B(q) \) and \( C(q) \). By (i) of §3, the \( p \)th factor in \( B(q) \) is \( \ll p^r/(p-1) \), so

\[
B(q) \ll C_{\text{opt}}(q) \frac {q^r} {\phi(q)}.
\]

More simply, the \( p \)th factor in \( C(q) \) is \( \ll p/(p-1) \), so

\[
C(q) \ll C_{\text{opt}}(q) \frac {q} {\phi(q)}.
\]

Returning to (9) and (10), it follows that (9) is \( h^x \) plus a remainder term which is

\[
\ll h^{-1} \sum_{q \leq x} C_{\text{opt}}(q) \frac{q}{\phi(q)} + h^x (xh)^y/x
\]

\[
\ll h^{-1} x^{1+\varepsilon} + h^x (hx)^y/x
\]

\[
\ll h^{-1+\varepsilon},
\]

choosing \( x = h^{\varepsilon} \). Since \( x \ll h \), the conditions \( q \ll h \), assumed earlier, are satisfied.

3. Combinatorial identities. We prove here the standard identities for the "Stirling numbers of the second kind" \( \sigma(k, r)/r! \) which have been used above. These are

(i)

\[
\sum_{v=1}^{p} \binom{p}{v} \sigma(r, v) = p^r,
\]

(ii)

\[
\sum_{v=1}^{p} \binom{p}{v} \sigma(r, v) = p^{r+1} - (p - 1)^r p,
\]

(iii)

\[
\sum_{v=1}^{r} \sigma(r, v) \frac {\lambda^v} {v!} = \sum_{p=0}^{\infty} p^r e^{-\lambda} \frac {e^{\lambda p}} {p!},
\]

(iv)

\[
\sum_{r=0}^{\infty} \frac {m_r(\lambda) x^r} {r!} = e^{-\lambda} e^{\lambda x},
\]

the last two identities show that \( m_r(\lambda) \), the left side of (iii), is the \( r \)th moment of the Poisson distribution with parameter \( \lambda \), and that the corresponding moment generating function (iv) is entire.

To prove (i), classify the maps from \( \{1, \ldots, r\} \) to \( \{1, \ldots, p\} \) by the size of the image. There are \( (\_)^r \) subsets of size \( v \) in \( \{1, \ldots, p\} \); for each such subset, the number of maps with this image is \( \sigma(r, v) \). To prove (ii), write

\[
\binom{p}{v} = p \binom{p - 1}{v - 1} = p \binom{p}{v} - p \binom{p - 1}{v}.
\]
and use (i). To prove (iii), multiply (i) by $\lambda^p/p!$ and sum over $p$:

$$\sum_{v=1}^{r} \sigma(r, v) \sum_{p=0}^{\infty} \binom{p}{v} \frac{\lambda^p}{p!} = \sum_{p=0}^{\infty} \frac{p^r \lambda^p}{p!}.$$  

From this and

$$\sum_{p=0}^{\infty} \binom{p}{v} \frac{\lambda^p}{p!} = \frac{\lambda^v}{v!} e^\lambda,$$

the identity (iii) follows. To prove (iv), multiply (iii) by $z^r/r!$ and sum over $r$.

References


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