Additional Note on Triangle Transformations.

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PART I.

The chief object of this Note is to develope some simple and rather interesting properties of the operators α, β, γ, referred to in my Note read at last meeting. I take for brevity the symbol μ to denote the compound operator

$$\alpha\beta\gamma \quad \text{or its equivalents} \quad \gamma\alpha\beta, \quad \beta\alpha\gamma, \quad \text{etc.}$$

1. If we apply μ to any function F, of a, b, c, s, s_1, s_2, s_3, r, r_1, r_2, r_3, h_1, h_2, h_3, Δ, R; then, as I pointed out before, $\mu F = F$, if F be an *even* function of certain letters; but if F be an *odd* function, then $\mu F = -F$; and lastly, if $F \equiv F_1 + F_2$, where F_1 is odd and F_2 even, then $\mu F = -F_1 + F_2$.

2. The order of the operations α, β, γ successively applied, is immaterial.

If F be any function of the letters above mentioned, we have

$$\alpha^2 F = F, \text{ or } \alpha^2 = 1 \therefore \alpha = \frac{1}{\alpha}.$$

Again $\qquad \beta\gamma\gamma\beta = \beta\gamma^2\beta = \beta\beta = 1 \qquad \therefore \alpha\beta\gamma\gamma\beta = \alpha$
$$\therefore \mu\gamma\beta = \alpha.$$

Similarly $\qquad \mu\beta\gamma = \alpha \qquad \therefore \gamma\beta = \beta\gamma.$

Thus any two successive operators may be transposed without altering the result. Hence the order of any number of successive operations is immaterial.

3. Next, any succession of operations is reducible to one of eight operations. For by the last paragraph, any such succession is equivalent to $\alpha^m\beta^n\gamma^p$, where m, n, p are positive integers.

But $a^m = 1$ or a according as m is even or odd. Hence $a^m\beta^n\gamma^p$ reduces to one of the following eight operations :

$$1, \ a, \ \beta, \ \gamma, \ \beta\gamma, \ \gamma a, \ a\beta, \ a\beta\gamma \ ;$$

which may be written

$$1, \ a, \ \beta, \ \gamma, \ \mu a, \ \mu\beta, \ \mu\gamma, \ \mu.$$

4. These form a "group" of operations, *i.e.*, any combination of them is equivalent to one or other of the eight.

And using the nomenclature explained by F. Klein in his "Lectures on the Icosahedron ;" the *periodicity* of each operation (excepting the first) is 2 ; the most extended "*sub-groups*" are of the type $1, \ a, \ \mu a, \ \mu$; which again contains two *sub-groups* $1, a$ and $1, \mu$.

5 When the operation is performed on an *even* function, then $\mu = 1$, and the group reduces to $1, a, \beta, \gamma$; with $1, a$, etc., as sub-groups : and if on an *odd* function, then $\mu = -1$, and the group is $\pm 1, \ \pm a, \ \pm \beta, \ \pm \gamma$.

6. As pointed out at the end of the preceding Note, the angles ABC may occur in any functions which unaltered in absolute magnitude when $-A$ is changed into $-A + 2\pi$, etc., without modifying the above results ; noting that A, B, C are reckoned along with r_1, r_1, etc., in counting the dimensions as *even* or *odd*.

PART II. (*Abstract*).

7. Reference was made to the following table for transforming a function of the general spherical triangle into the corresponding function of the *colunar* triangle opposite to A :

a, b, c, A, B, C, $s, s-a, s-b, s-c$, E, A – E,
      B – E, C – E, L, N, n, R, R_a, R_b, R_c, r, r_a, r_b, r_c

become respectively

$a, \pi - b, \pi - c$, A, π – B, π – C, $\pi - s + a, \pi - s, s - c, s - b$, A – E, E,
      C – E, B – E, L, N, n, R_a, R, R_c, R_b, r_a, r, r_c, r_y.

The notation used is that of Casey, who gives some of the transformations in his *Spherical Trigonometry*.

This, though similar to the transformation a in a plane triangle, is not strictly analogous.

8. Referring to the demonstration of the principle of the *transformation continue* as given by Mr Lemoine in the Report of the French Association for the Advancement of Science for 1891, Vol. II., p. 118, and ascribed by him to M. Laisant, it was pointed out that a somewhat simpler point of view is possible : from which it appears that many other kinds of transformations are equally valid. As interesting cases, the following were mentioned :

(1) ABC changed to $\dfrac{\pi}{2} - \dfrac{A}{2}$, $\dfrac{\pi}{2} - \dfrac{B}{2}$, $\dfrac{\pi}{2} - \dfrac{C}{2}$ and a into $a \operatorname{cosec} \dfrac{A}{2}$, etc., which corresponds to changing from the given triangle to that whose vertices are the excentres of ABC.

(2) s_1, s_2, s_3 changed to r_1, r_2, r_3.

Some properties of both were mentioned.

9. The suggestion made at the close of the preceding Note was verified by the example

$$\sin \frac{A}{2} = \sqrt{\frac{s_2 s_3}{bc}}$$

in which, after the transformation a, the square root must be taken *negatively*.

Added Note.—Dr Mackay having pointed out that a, β, γ have already been appropriated to denote certain quantities connected with the triangle, the author suggests the symbols t_1, t_2, t_3 to replace a, β and γ, in cases where these symbols have other meanings already ascribed to them.