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LETTERS TO THE EDITOR

A NON-RENEWAL PROCESS WITH RENEWAL COUNTING DISTRIBUTIONS

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Abstract

A simple example is given of a point process with dependent inter-event intervals for which the distributions of the counting random variables N_t are given by the same formulas as for a renewal process. This example shows the care needed in establishing that a given point process has the renewal property, but also raises interesting questions about the class of point processes with given counting distributions.

In the proof of Theorem 6.1 in [1], the authors verify that a given point process $[S_n, n \geqq 0]$, on the positive reals, with $S_0 = 0$, has the property that for all $t \geqq 0$, the marginal probability densities $\{P[N_t = n]\}$ of the counting variables N_t, are given by

$$(1) \qquad P[N_t = n] = P[S_n \leqq t < S_{n+1}] = F^{(n)}(t) - F^{(n+1)}(t), \text{ for } n \geqq 0 \quad \text{and} \quad t \geqq 0,$$

where $F^{(n)}(.)$ is the n-fold convolution of a probability distribution $F(.)$ on $[0, \infty)$. They assert that this establishes that $\{S_n\}$ is a *renewal* process. We construct a counter-example to that assertion by exhibiting a point process for which formula (1) holds, yet for which, conditional on the value of S_1, the sequence $\{S_n\}$ is *deterministic*. The following lemma is readily verified.

Lemma. Let $F(x)$ and $G(x)$ be continuous probability distributions on $[0, \infty)$, such that $F(x) < G(x)$, for $x > 0$, then the random variables $X = F^{-1}(U)$, and $Y = G^{-1}(U)$, where U is a uniformly distributed random variable on $[0, 1]$, have the marginal distributions F and G. X and Y are related by $X = F^{-1}[G(Y)]$, and $X > Y$ a.s.

Let now $\{F_n(.),\ n \geqq 1\}$ be a sequence of stochastically increasing continuous probability distributions on $[0, \infty)$, that is, for all $x \geqq 0$ and $n \geqq 1$, $F_{n+1}(x) < F_n(x)$.

The example. Let U be a uniform $[0, 1]$ random variable and consider the point process $\{S_n, n \geqq 0\}$ for which $S_0 = 0$, and S_n is defined by $S_n = F_n^{-1}(U)$, for $n \geqq 1$. The probability distributions F_n, $n \geqq 1$ are clearly the marginal distributions of the random variables S_n, for $n \geqq 1$. Moreover

$$(2) \qquad P[N_t = n] = F_n(t) - F_{n+1}(t), \text{ for } t \geqq 0.$$

Proof. Clearly for $n \geqq 0$ and $t \geqq 0$,

$$P[N_t = n] = P[S_n \leqq t < S_{n+1}] = P[F_n(t) \leqq U < F_{n+1}(t)] = F_n(t) - F_{n+1}(t).$$

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The counterexample to the assertion in [1] is now obtained by letting the probability distributions $F_n(.)$ be the n-fold convolutions $F^{(n)}$ of a continuous distribution $F(.)$ on $[0, \infty)$.

Remarks. A renewal process satisfying (1) and the point process constructed in the counterexample are, in a sense, *extreme* Markovian sequences for which (1) holds. The first has *independent increments*; the second is, conditional on S_1, completely deterministic. It is an interesting open problem to characterize the family of point processes, and in particular, of Markovian sequences for which property (1) holds. Explicit constructions of less extreme cases appear to be quite difficult. We refer to Whitt [2] for a comprehensive review of the literature.

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References

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[2] Whitt, W. (1987) Stochastic ordering. In *Encyclopedia of Statistical Sciences,* Vol. 7, ed. S. Kotz and N. L. Johnson, Wiley, New York.