ALMOST CONVERGENCE AND WELL-DISTRIBUTED SEQUENCES

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1. Introduction. A sequence (x_n) of real numbers is said to be well-distributed modulo 1 (abbreviated w.d.) if for each subinterval $I = [a, b]$ of $[0, 1]$ we have that

$$\lim_{n\to\infty} \left(\frac{1}{n}\right) \sum_{k+1}^{k+n} \chi_I(x_m) = b - a \quad \text{uniformly in } k = 0, 1, 2, \ldots,$$

where χ_I is the characteristic function of I modulo 1. A sequence (r_n) of positive numbers is lacunary if

$$\liminf_{n\to\infty} (r_{n+1}/r_n) > 1.$$

It is a consequence of a general theorem due to Koksma (**1**) that if (r_n) is a lacunary sequence, then for almost all x (in the sense of Lebesgue measure) the sequence $(r_n x)$ is uniformly distributed modulo 1. In contrast to this result, it is shown in (**3**) and (**4**) that if (r_n) is lacunary, then for almost all x the sequence $(r_n x)$ is *not* w.d. It is easy to extend this result to sequences which contain lacunary subsequences of positive density (in a certain sense). My aim in this note is to show that this result holds under more general conditions, thereby answering a question raised in (**3**).

2. Main results. Let (n_i) be a strictly increasing sequence of positive integers. We say that (n_i) has lower density 0 if for each $\epsilon > 0$ there exists an integer N such that for all $n \geqq N$ and for $k = 0, 1, 2, \ldots,$

$$\left(\frac{1}{n}\right) \sum_{k+1}^{k+n} \phi(m) < \epsilon,$$

where $\phi(m) = 1$ if $m \in \{n_i\}$ and $\phi(m) = 0$ otherwise, that is, the sequence (n_i) has lower density 0 if the sequence $(\phi(m))$ is almost convergent to 0 in the sense of Petersen (**2**).

If there are a $\delta > 0$ and strictly increasing sequences of positive integers (k_i) and (p_i) such that

$$\left(\frac{1}{p_i}\right) \sum_{k_i+1}^{k_i+p_i} \phi(m) \geqq \delta$$

(i.e., if (n_i) does not have lower density 0), then we shall write $\text{dens}(n_i) \geqq \delta$.

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Then we have the following.

THEOREM 1. *Suppose that* (r_n) *is a seqenuce of positive numbers such that there exist* $r > 1$, $\delta > 0$, *and a subsequence* $(r_{n(i)})$ *for which*

(i) $r_{n(i+1)}/r_{n(i)} \geqq r$ *for all* i *and*

(ii) $\text{dens}(n(i)) \geqq \delta$.

Then for almost all x, *the sequence* $(r_n x)$ *is not w.d. modulo* 1.

For the proof of this theorem we shall need two lemmas, the first of which was proved in (**4**).

LEMMA 1. *Suppose that* (r_n) *is a sequence of positive numbers such that* $r_{n+1}/r_n \geqq r \geqq 2$ *for all* n. *Let* I *be a closed subinterval of* $[0, 1]$ *of length* $||I||$ *such that* $r||I|| \geqq 2$. *Then for almost all* x *and any positive integer* k *there exists an integer* $m = m(x)$ *such that each of the terms* $r_m x, r_{m+1}x, \ldots, r_{m+n}x$ *lies in* I *modulo* 1.

LEMMA 2. *Let* m, *a positive integer, and* $\delta > 0$ *be given. Then for* n *sufficiently large (depending only upon* m *and* δ*) the following is true: if* $A \subset \{1, 2, \ldots, n\}$ *with* $|A| \geqq \delta n$ *(*$|A|$ *is the number of elements in* A*) and*

$$B = \{x \in A \colon |A \cap \{x, x+1, \ldots, x+m-1\}| \geqq \delta m/2\},$$

then $|B| \geqq \delta n/2$.

Proof. Let

$$C_i = (A - B) \cap \{im+1, im+2, \ldots, (i+1)m\} \quad (i = 0, 1, 2, \ldots).$$

It is clear that $|C_i| < \delta m/2$ so that, in fact, $C_i \leqq \langle \delta m/2 \rangle$ (where by $\langle x \rangle$ we mean the largest integer *less than* x). Hence

$$|A - B| \leqq [n/m]\langle \delta m/2 \rangle + m \leqq \delta n/2$$

for n sufficiently large, depending only upon m and δ.

Proof of Theorem 1. Let $E = \{n(i)\}$. Then dens $E \geqq \delta$. Let R be any positive integer and

$$E(R) = \left\{ n(i) \colon \left(\frac{1}{R}\right) \sum_{n(i)}^{n(i)+R-1} \phi(m) \geqq \delta/2 \right\}$$

(where ϕ is, again, the characteristic function of E). By Lemma 2, dens $E(R) \geqq \delta/2$. In particular, $E(R)$ is infinite. Now let

$$F(R) = \{n(i) \colon \text{there exists } n(p) \in E(R) \text{ with } 0 \leqq i - p \leqq R\delta/4\}.$$

Since $F(R) \supset E(R)$, we have that $F(R)$ is infinite. Label the elements of $F(R)$ so that $F(R) = \{\bar{m}(i)\}$.

Let $\mathscr{A}$ be any fixed arithmetic progression of positive integers of difference a, where a is large enough so that

$$r^a(\delta/(8a)) \geqq 2.$$

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Let $G(R) = \{\bar{m}(i): i \in \mathscr{A}\}$. Relabel so that $G(R) = \{m(i)\}$. Then, by Lemma 1, for fixed R, almost all x and arbitrary n, there exists an integer $k = k(x)$ such that

$$(*) \qquad r_j x \in I = [0, \delta/(8a)] \pmod 1$$

for $j = m(k), m(k+1), \ldots, m(k+n)$. Hence, the same conclusion holds for all R and n and almost all x. We shall now show that for x for which this is true for all R and n, the sequence $(r_n x)$ is not w.d., thereby proving the theorem.

Accordingly, let x be as above, take R to be large compared to a and let n be large compared to R. Find k such that $r_j x \in I \pmod 1$ for all

$$j \in S = \{m(k), m(k+1), \ldots, m(k+n)\}.$$

Then S contains a term $m(s)$, with $m(s) + R - 1 \leqq m(k+n)$, such that for some integers q and p, with $\bar{m}(q) = m(s)$ and $\bar{m}(p) \in E(R)$ we have that $|p - q| \leqq a$, so that from among the terms

$$m(s), m(s)+1, m(s)+2, \ldots, m(s)+R-1,$$

at least $\delta R/4 - a$ belong to $F(R)$, and hence at least $\delta R/(4a) - 2$ belong to $G(R)$. It thus follows that from among the terms

$$r_{m(s)}x, r_{m(s)+1}x, \ldots, r_{m(s)+R-1}x,$$

at least $\delta R/(4a) - 2$ belong to $I = [0, \delta/(8a)] \pmod 1$, so that

$$\left(\frac{1}{R}\right) \sum_{m(s)}^{m(s)+R-1} \chi_I(r_p x) \geqq \delta/(4a) - 2/R > \delta/(5a)$$

for R sufficiently large. But, if $(r_n x)$ were w.d., this fraction would have to uniformly approach $\delta/(8a)$, as R approaches infinity, a contradiction.

3. Further remarks. In a slightly different direction we can prove the following.

THEOREM 2. *Let (r_n) be a lacunary sequence and let (s_n) be a re-arrangement of (r_n). Then for almost all x, the sequence $(s_n x)$ is not well-distributed.*

Of course, this result is not true if we only assume that (r_n) contains lacunary subsequences. Again, Koksma's result in (**1**) shows that for almost all x, the sequence $(s_n x)$ is uniformly distributed.

Proof. The proof follows the lines of the proof for the lacunary case given in (**4**) therefore we shall only indicate the differences. We first show that the conclusion of Lemma 1 of this paper is valid for the sequence (s_n) under the assumptions of Lemma 1 on (r_n). In fact, the following more general result is valid.

LEMMA 3. *Let $r \geqq 2$ and $I \subset [0, 1]$ be such that $r\|I\| \geqq 2$. Let k be any positive integer and for each positive integer n let S_n be a set of positive numbers, $|S_n| = k$, such that*

(i) $\liminf_{n\to\infty}\{x: x \in S_n\} = \infty$,

(ii) *$x, y \in S_n$, $x < y$ imply that $y/x \geqq r$.*

Then for almost all real γ there exists an integer $m = m(\gamma)$ such that

$$x\gamma \in I \pmod 1 \quad \textit{for all } x \in S_m.$$

The proof of this lemma is similar to the proof of Lemma 1 given in (**4**) and will be omitted.

We can then use this lemma to complete the proof of Theorem 2 by passing to an appropriate subsequence of (s_n) much as was done in (**4**), although here we cannot use an arithmetic progression to define the sequence. Notice, at any rate, that if $\liminf(r_{n+1}/r_n) > 2$, then the theorem follows almost immediately from the lemma.

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