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A CROSS-CONSTRAINED VARIATIONAL PROBLEM FOR THE GENERALIZED DAVEY–STEWARTSON SYSTEM

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Abstract We study the sharp threshold for blow-up and global existence and the instability of standing wave $e^{i\omega t}u_\omega(x)$ for the Davey–Stewartson system

$$i\phi_t + \Delta\phi + a|\phi|^2\phi + E_1(|\phi|^2)\phi = 0 \tag{DS}$$

in $\mathbb{R}^3$, where u_ω is a ground state. By constructing a type of cross-constrained variational problem and establishing so-called cross-invariant manifolds of the evolution flow, we derive a sharp criterion for global existence and blow-up of the solutions to (DS), which can be used to show that there exist blow-up solutions of (DS) arbitrarily close to the standing waves.

Keywords: generalized Davey–Stewartson system; cross-constrained variational problem; sharp threshold; global existence; blow-up; instability

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Secondary 35Q55; 35B30

1. Introduction

Consider the generalized Davey–Stewartson system:

$$i\phi_t + \Delta\phi + a|\phi|^2\phi + E_1(|\phi|^2)\phi = 0, \quad t \geqslant 0,\ x \in \mathbb{R}^3, \tag{1.1}$$

where $\phi = \phi(t, x)$ is a complex-valued function of $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^3$, $a > 0$, E_1 is the singular integral operator with symbol $\sigma_1(\xi) = \xi_1^2/|\xi|^2$, $\xi \in \mathbb{R}^3$, and $E_1(\psi) = F^{-1}(\xi_1^2/|\xi|^2)F\psi$, F^{-1} and F are the Fourier inverse transform and Fourier transform on $\mathbb{R}^3$, respectively (see [**4**–**6**, **9**]). When $x \in \mathbb{R}^2$, system (1.1) describes the evolution of weakly nonlinear water waves that travel predominantly in one direction (see [**4**–**6**]). More precisely, (1.1) is the three-dimensional extension of the generalized Davey–Stewartson system in the elliptic–elliptic case when $p = 3$, namely

$$\left.\begin{aligned} i\phi_t + \lambda\phi_{xx} + \phi_{yy} + a|\phi|^{p-1}\phi + \phi\psi_x &= 0, \\ \psi_{xx} + \mu\psi_{yy} &= (|\phi|^2)_x, \end{aligned}\right\} \tag{1.2}$$

where $\lambda, \mu > 0$, ϕ is a complex-valued function of $(t; x, y) \in \mathbb{R}_+ \times \mathbb{R}^2$ and ψ is a real-valued function of $(t; x, y) \in \mathbb{R}_+ \times \mathbb{R}^2$ (see [**5**]). The Davey–Stewartson system (1.2) is the model equation in the theory of shallow-water waves, and the functions ϕ and ψ are related to the amplitude and the mean velocity potential of the water wave, respectively. A large amount of work (see [**6**, **10**, **13**, **15**]) has been devoted to the study of the generalized Davey–Stewartson system (1.2). In 1990, Ghidaglia and Saut [**6**] studied the Cauchy problem of (1.2) and, except for the case when $\lambda, \mu < 0$, proved the solvability in the Sobolev spaces $H = H^1(\mathbb{R}^2)$ when $p = 3$. In the elliptic-hyperbolic case, i.e. $\lambda > 0$ and $\mu < 0$, Tsutsumi [**15**] obtained the $L^p(\mathbb{R}^2)$-decay estimates of solutions of (1.2) $(2 < p < \infty)$. In the elliptic–elliptic case, i.e. $\lambda > 0$ and $\mu > 0$, Cipolatti [**3**] proved the existence of the ground state for the N-dimensional extension of (1.2) by reducing the extension to a single nonlinear equation of Schrödinger type. In [**13**] Ozawa presented the exact blow-up solutions of the Cauchy problem for (1.2). Ohta [**10**] discussed the existence of stable standing waves under certain conditions.

For system (1.1), Guo and Wang [**9**] established the local well-posedness of the Cauchy problem in energy class $H^1(\mathbb{R}^3)$. Moreover, when $x \in \mathbb{R}^2$, Ohta [**12**] proved that if $a(p-3) > 0$, there exists a blow-up solution of (1.1) arbitrarily close to the standing wave. When $x \in \mathbb{R}^N$ $(N = 2$ or $3)$, Ohta [**11**] proved that if $p \geqslant 1 + (4/N)$, the ground state u_ω is unstable for any $\omega \in (0, \infty)$.

In this paper, we construct a type of cross-constrained variational problem and establish its property, then apply it to the generalized Davey–Stewartson system (1.1). Through studying the corresponding cross-invariant manifolds under the flow generated by the system (1.1), we establish the sharp threshold for global existence and blow-up of the solutions. By this threshold and the property of the cross-constrained variational problem, we also show the strong instability of ground states in § 5. Berestycki and Cazenave [**1**] and Weinstein [**16**] have studied the similar problems of nonlinear Schrödinger equations. However, in [**1**, **16**], the related variational problems must be solved and the Schwarz symmetrization and complicated variational computations must be conducted. But in our new variational argument, we can refrain from inducing the Schwarz symmetrization and complicated variational computations as well as from solving the attached variational problem, and can establish directly the sharp criterion for global existence and blow-up of system (1.1). Furthermore, by using our sharp threshold for blow-up, the strong instability of the standing waves of system (1.1) is also shown. Moreover, the argument proposed here may be developed to treat system (1.1) with $3 \leqslant p < \infty$ when $x \in \mathbb{R}^2$ and $3 < p < 5$ when $x \in \mathbb{R}^3$.

Note that the result about the instability of ground states in the present paper is not new and the same result was proved by Cipolatti in [**4**].

For simplicity we denote $\int_{\mathbb{R}^3} \cdot \, \mathrm{d}x$ by $\int \cdot \, \mathrm{d}x$ throughout the present paper.

2. Preliminaries

We impose the initial data of (1.1) as follows:

$$\phi(0, x) = \phi_0(x), \quad x \in \mathbb{R}^3. \tag{2.1}$$

From [**9**] (see also [**6**]), we have the following local well-posedness for the Cauchy problem (1.1)–(2.1).

Proposition 2.1. *Let $\phi_0 \in H^1(\mathbb{R}^3)$. There then exists a unique solution $\phi(t,x)$ of the Cauchy problem (1.1)–(2.1) in $C([0,T);H^1(\mathbb{R}^3))$ for some $T \in (0,\infty)$ (maximal existence time), either $T=\infty$ or else $T<\infty$ and*

$$\lim_{t\to T^-} \|\phi(t)\|_{H^1(\mathbb{R}^3)} = \infty.$$

Furthermore, for all $t \in [0,T)$, we have conservation of momentum,

$$\int |\phi(t,x)|^2 \,\mathrm{d}x = \int |\phi_0(x)|^2 \,\mathrm{d}x, \tag{2.2}$$

and conservation of energy,

$$E(\phi) = \frac{1}{2}\int |\nabla\phi|^2 \,\mathrm{d}x - \frac{a}{4}\int |\phi|^4 \,\mathrm{d}x - \frac{1}{4}\int |\phi|^2 E_1(|\phi|^2)\,\mathrm{d}x = E(\phi_0). \tag{2.3}$$

Remark 2.2. From [**3**, Lemma 2.1] we have $E_1(\phi(\lambda\,\cdot))(x) = E_1(\phi)(\lambda x)$, $\lambda > 0$. In addition, from the definition of E_1 and the Parseval identity,

$$\int f\cdot\bar{g}\,\mathrm{d}x = \int F[f]\overline{F[g]}\,\mathrm{d}\xi, \quad \mathrm{d}\xi = (2\pi)^{-3}\,\mathrm{d}x,$$

we have

$$\int |\phi|^2 E_1(|\phi|^2)\,\mathrm{d}x = \int |\phi|^2 F^{-1}\sigma_1(\xi)F(|\phi|^2)\,\mathrm{d}x = \int \sigma_1(\xi)|F(|\phi|^2)|^2\,\mathrm{d}\xi > 0.$$

Moreover, by a direct calculation (see [**11**, **12**]), we have the following.

Proposition 2.3. *Let $\phi_0(x) \in H^1(\mathbb{R}^3)$ and let $\phi(t,x)$ be a solution of the Cauchy problem (1.1)–(2.1) on $[0,T)$. Set*

$$J(t) = \int |x|^2|\phi(t,x)|^2\,\mathrm{d}x. \tag{2.4}$$

Then

$$J''(t) = 8\int |\nabla\phi|^2\,\mathrm{d}x - 6a\int |\phi|^4\,\mathrm{d}x - 6\int |\phi|^2 E_1(|\phi|^2)\,\mathrm{d}x. \tag{2.5}$$

3. The cross-constrained variational problem

For $u \in H^1(\mathbb{R}^3)$, we define the following functionals:

$$I(u) := \frac{1}{2}\int |\nabla u|^2\,\mathrm{d}x + \frac{\omega}{2}\int |u|^2\,\mathrm{d}x - \frac{a}{4}\int |u|^4\,\mathrm{d}x - \frac{1}{4}\int |u|^2 E_1(|u|^2)\,\mathrm{d}x, \tag{3.1}$$

$$S(u) := \int |\nabla u|^2\,\mathrm{d}x + \omega\int |u|^2\,\mathrm{d}x - a\int |u|^4\,\mathrm{d}x - \int |u|^2 E_1(|u|^2)\,\mathrm{d}x, \tag{3.2}$$

$$Q(u) := \int |\nabla u|^2\,\mathrm{d}x - \frac{3a}{4}\int |u|^4\,\mathrm{d}x - \frac{3}{4}\int |u|^2 E_1(|u|^2)\,\mathrm{d}x. \tag{3.3}$$

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From Sobolev's embedding theorem, and

$$\int |u|^2 E_1(|u|^2)\,\mathrm{d}x \leqslant \int |u|^4\,\mathrm{d}x$$

(see also [**11**,**12**]), we see that the above functionals are well defined. Moreover, we define a manifold N as

$$N := \{u \in H^1(\mathbb{R}^3) \setminus \{0\},\ S(u) = 0\}, \tag{3.4}$$

and a cross-manifold M as

$$M := \{u \in H^1(\mathbb{R}^3),\ Q(u) = 0,\ S(u) < 0\}. \tag{3.5}$$

Then the following results are true.

Lemma 3.1. *There exists $u \in H^1(\mathbb{R}^3) \setminus \{0\}$ such that $S(u) = 0$ and $Q(u) = 0$.*

Proof. From [**11**, **12**], it follows that there exists $u \in H^1(\mathbb{R}^3) \setminus \{0\}$ such that u is a solution of the following Euler–Lagrangian equation

$$-\Delta u + \omega u - a|u|^2 u - E_1(|u|^2)u = 0. \tag{3.6}$$

Thus, $S(u) = 0$. Moreover, from (3.6) we have the Pohozaev identity

$$\frac{1}{3}\int |\nabla u|^2\,\mathrm{d}x + \omega \int |u|^2\,\mathrm{d}x - \frac{a}{2}\int |u|^4\,\mathrm{d}x - \frac{1}{2}\int |u|^2 E_1(|u|^2)\,\mathrm{d}x = 0, \tag{3.7}$$

which is obtained by multiplying (3.6) by $x \cdot \nabla u$, then integrating. Note that $S(u) = 0$. Thus, $Q(u) = 0$. □

Lemma 3.2. *M is not empty.*

Proof. From Lemma 3.1, there exists $u \in H^1(\mathbb{R}^3) \setminus \{0\}$ such that both $S(u) = 0$ and $Q(u) = 0$. Now we let $u_\lambda = \lambda u(\lambda x)$ for $\lambda > 0$, we get

$$S(u_\lambda) = \lambda\biggl(\int |\nabla u|^2\,\mathrm{d}x - a\int |u|^4\,\mathrm{d}x - \int |u|^2 E_1(|u|^2)\,\mathrm{d}x\biggr) + \lambda^{-1}\omega \int |u|^2\,\mathrm{d}x, \tag{3.8}$$

$$Q(u_\lambda) = \lambda\biggl(\int |\nabla u|^2\,\mathrm{d}x - \frac{3a}{4}\int |u|^4\,\mathrm{d}x - \frac{3}{4}\int |u|^2 E_1(|u|^2)\,\mathrm{d}x\biggr). \tag{3.9}$$

Thus, $S(u) = 0$ implies that there exists $\lambda^* > 1$ such that $S(u_{\lambda^*}) < 0$.

On the other hand, from $\lambda^* > 1$, we still have $Q(u_{\lambda^*}) = 0$. So $u_{\lambda^*} \in M$. This proves that M is not empty. □

Now we consider the constrained variational problem

$$d_N = \inf_{u \in N} I(u), \tag{3.10}$$

and the cross-constrained minimization problem

$$d_M = \inf_{u \in M} I(u). \tag{3.11}$$

First we have the following two lemmas.

Lemma 3.3. $d_N > 0$.

Proof. From (3.1), (3.2), (3.4) and (3.10) on N one has

$$I(u) = \frac{1}{4}\int |\nabla u|^2 \,\mathrm{d}x + \frac{\omega}{4}\int |u|^2 \,\mathrm{d}x = \frac{a}{4}\int |u|^4 \,\mathrm{d}x + \frac{1}{4}\int |u|^2 E_1(|u|^2)\,\mathrm{d}x > 0. \tag{3.12}$$

It follows from $\int |u|^2 E_1(|u|^2)\,\mathrm{d}x \leqslant \int |u|^4 \,\mathrm{d}x$ and Sobolev's inequality that

$$\begin{aligned}\frac{1}{4}\int |\nabla u|^2 \,\mathrm{d}x + \frac{\omega}{4}\int |u|^2 \,\mathrm{d}x &= \frac{a}{4}\int |u|^4 \,\mathrm{d}x + \frac{1}{4}\int |u|^2 E_1(|u|^2)\,\mathrm{d}x \\ &\leqslant \frac{a}{4}\int |u|^4 \,\mathrm{d}x + \frac{1}{4}\int |u|^4 \,\mathrm{d}x \\ &\leqslant c\bigg(\frac{1}{4}\int |\nabla u|^2 \,\mathrm{d}x + \frac{\omega}{4}\int |u|^2 \,\mathrm{d}x\bigg)^2,\end{aligned}$$

where c is a positive constant. Thus, we have

$$\frac{1}{4}\int |\nabla u|^2 \,\mathrm{d}x + \frac{\omega}{4}\int |u|^2 \,\mathrm{d}x \geqslant \frac{1}{c},$$

so $I(u) \geqslant 1/c > 0$. That is, $d_N > 0$. □

Lemma 3.4. $d_M \geqslant d_N$.

Proof. Let

$$u \in M \quad \text{and} \quad u_\lambda = \lambda u(\lambda x). \tag{3.13}$$

Then

$$S(u_\lambda) = \lambda\bigg(\int |\nabla u|^2 \,\mathrm{d}x - a\int |u|^4 \,\mathrm{d}x - \int |u|^2 E_1(|u|^2)\,\mathrm{d}x\bigg) + \lambda^{-1}\omega\int |u|^2 \,\mathrm{d}x, \tag{3.14}$$

$$Q(u_\lambda) = \lambda\bigg(\int |\nabla u|^2 \,\mathrm{d}x - \frac{3a}{4}\int |u|^4 \,\mathrm{d}x - \frac{3}{4}\int |u|^2 E_1(|u|^2)\,\mathrm{d}x\bigg). \tag{3.15}$$

Thus, $S(u) < 0$ implies that there exists a unique $0 < \lambda^* < 1$ such that $S(u_{\lambda^*}) = 0$, $S(u_\lambda) > 0$ for $\lambda \in (0, \lambda^*)$ and $S(u_\lambda) < 0$ for $\lambda \in (\lambda^*, 1)$. It is clear that $u \neq 0$ and $u_{\lambda^*} \neq 0$. By (3.10) it follows that

$$I(u_{\lambda^*}) \geqslant d_N. \tag{3.16}$$

At the same time, $Q(u) = 0$ implies that, for any $\lambda > 0$, $Q(u_\lambda) = 0$. It follows that

$$I(u_\lambda) = \frac{1}{8}\int (|u_\lambda|^4 + |u_\lambda|^2 E_1(|u_\lambda|^2))\,\mathrm{d}x + \frac{\omega}{2}\int |u_\lambda|^2 \,\mathrm{d}x, \tag{3.17}$$

$$S(u_\lambda) = -\frac{1}{4}\int (|u_\lambda|^4 + |u_\lambda|^2 E_1(|u_\lambda|^2))\,\mathrm{d}x + \omega\int |u_\lambda|^2 \,\mathrm{d}x. \tag{3.18}$$

By (3.17) we have

$$\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} I(u_\lambda) = \frac{\lambda}{8} \int (|u|^4 + |u|^2 E_1(|u|^2))\,\mathrm{d}x - \frac{\lambda^{-1}\omega}{2} \int |u|^2\,\mathrm{d}x. \tag{3.19}$$

Then (3.18) and (3.19) imply that

$$\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} I(u_\lambda) = -\tfrac{1}{2} S(u_\lambda). \tag{3.20}$$

So $I(u_\lambda)$ makes the minimal value at $\lambda = \lambda^*$ since $S(u_{\lambda^*}) = 0$, $S(u_\lambda) > 0$ for $\lambda \in (0, \lambda^*)$ and $S(u_\lambda) < 0$ for $\lambda \in (\lambda^*, 1)$. Thus, for $\lambda = 1 > \lambda^*$, we have $I(u) = I(u_\lambda) \geqslant I(u_{\lambda^*})$. Recalling (3.16), we obtain $I(u) \geqslant d_N$. Therefore, $d_M \geqslant d_N$. □

Now we define a cross-invariant manifold

$$K := \{\phi \in H^1(\mathbb{R}^3),\ I(\phi) < d_N,\ Q(\phi) < 0,\ S(\phi) < 0\}.$$

Then we have the following.

Proposition 3.5. *K is an invariant manifold of (1.1). More precisely, from $\phi_0 \in K$ it follows that the solution $\phi(t, x)$ of the Cauchy problem (1.1)–(2.1) satisfies $\phi(t, \cdot) \in K$ for any $t \in [0, T)$.*

Proof. Let $\phi_0 \in K$. By Proposition 2.1, there exists a unique

$$\phi(t, \cdot) \in C([0, T); H^1(\mathbb{R}^3))$$

with $T \leqslant \infty$ such that $\phi(t, x)$ is a solution of the Cauchy problem (1.1)–(2.1). From (2.2) and (2.3) we have

$$I(\phi(t, \cdot)) = I(\phi_0(x)), \quad t \in [0, T). \tag{3.21}$$

Thus, $I(\phi_0) < d_N$ implies that $I(\phi(t, \cdot)) < d_N$ for any $t \in [0, T)$.

Now we show $S(\phi(t, \cdot)) < 0$ for $t \in [0, T)$. Otherwise, from the continuity, there would be a $t_0 \in [0, T)$ such that $S(\phi(t_0, \cdot)) = 0$. By (3.21), $\phi(t_0, \cdot) \neq 0$. From (3.10), it follows that $I(\phi(t_0, \cdot)) \geqslant d_N$. This contradicts $I(\phi(t, \cdot)) < d_N$ for any $t \in [0, T)$. Therefore, $S(\phi(t, \cdot)) < 0$ for all $t \in [0, T)$.

Finally, we show that $Q(\phi(t, \cdot)) < 0$ for $t \in [0, T)$. Otherwise, from the continuity, there would be a $t_1 \in [0, T)$ such that $Q(\phi(t_1, \cdot)) = 0$. Because we have shown that $S(\phi(t_1, \cdot)) < 0$, it follows that $\phi(t_1, \cdot) \in M$. Thus, (3.11) and Lemma 3.4 imply that $I(\phi(t_1, \cdot)) \geqslant d_M \geqslant d_N$. This contradicts $I(\phi(t, \cdot)) < d_N$ for $t \in [0, T)$. Therefore, $Q(\phi(t, \cdot)) < 0$ for all $t \in [0, T)$.

By the above we have proved that $\phi(t, \cdot) \in K$ for any $t \in [0, T)$.

By the argument in Proposition 3.5, we get the following result. □

Proposition 3.6. *Define*

$$K_+ := \{\phi \in H^1(\mathbb{R}^3),\ I(\phi) < d_N,\ Q(\phi) > 0,\ S(\phi) < 0\},$$
$$R_- := \{\phi \in H^1(\mathbb{R}^3),\ I(\phi) < d_N,\ S(\phi) < 0\},$$
$$R_+ := \{\phi \in H^1(\mathbb{R}^3),\ I(\phi) < d_N,\ S(\phi) > 0\}.$$

Then K_+, R_- and R_+ are all invariant manifolds of (1.1).

By the definition of K, R_+ and R_-, as well as (3.10) and Lemma 3.4, we can easily obtain the following result.

Proposition 3.7. $\{\phi \in H^1(\mathbb{R}^3) \setminus \{0\},\ I(\phi) < d_N\} = \mathbb{R}_+ \cup K_+ \cup K$.

4. Sharp threshold for global existence and blow-up

Theorem 4.1. *If $\phi_0 \in K_+ \cup R_+$, then the solution $\phi(t,x)$ of the Cauchy problem (1.1)–(2.1) exists globally in $t \in (0,\infty)$.*

Proof. Firstly we let $\phi_0 \in K_+$. Thus, Proposition 3.6 implies that the solution $\phi(t,x)$ of the Cauchy problem (1.1)–(2.1) satisfies the condition that $\phi(t,\cdot) \in K_+$ for $t \in [0,T)$. For fixed $t \in [0,T)$, define $\phi(t,\cdot) = \phi$. Thus, we have $I(\phi) < d_N$, $Q(\phi) > 0$. If follows from (3.1) and (3.3) that

$$\frac{1}{6}\int |\nabla\phi|^2\,\mathrm{d}x + \frac{\omega}{2}\int |\phi|^2\,\mathrm{d}x < d_N. \tag{4.1}$$

From (2.2) and (4.1) we always get

$$\int |\nabla\phi|^2\,\mathrm{d}x < c. \tag{4.2}$$

Therefore, Proposition 2.1 implies that $\phi(t,x)$ exists globally in $t \in [0,T)$. Thus, for $\phi_0 \in K_+$, we have proved that the solution $\phi(t,x)$ of the Cauchy problem (1.1)–(2.1) exists globally in $t \in (0,\infty)$.

Now we see that $\phi_0 \in R_+$. By $\phi_0 \in R_+$, Proposition 3.6 implies that the solution $\phi(t,x)$ of the Cauchy problem (1.1)–(2.1) satisfies the condition that $\phi(t,\cdot) \in R_+$ for $t \in [0,T)$. We then have $I(\phi) < d_N$, $S(\phi) > 0$. It follows that

$$\frac{1}{4}\int (|\nabla\phi|^2 + \omega|\phi|^2)\,\mathrm{d}x < d_N. \tag{4.3}$$

Therefore, Proposition 2.1 implies that $\phi(t,x)$ exists globally in $t \in [0,\infty)$. □

Theorem 4.2. *If $\phi_0 \in K$ and satisfies $|x|\phi_0(x) \in L^2(\mathbb{R}^3)$, then the solution $\phi(t,x)$ of the Cauchy problem (1.1)–(2.1) blows up in a finite time.*

Proof. According to Ginibre and Velo [**7**, **8**], from $|x|\phi_0(x) \in L^2(\mathbb{R}^3)$, one has $|x|\phi \in L^2(\mathbb{R}^3)$. From $\phi_0 \in K$, Proposition 3.5 implies that $\phi(t,\cdot) \in K$ with $t \in [0,T)$. Now we set

$$J(t) = \int |x|^2|\phi(t,x)|^2\,\mathrm{d}x; \tag{4.4}$$

then (2.4) and (2.5) imply that

$$J''(t) = 8Q(\phi(t,\cdot)). \tag{4.5}$$

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Fix $t \in [0, T)$ and define $\phi(t, \cdot) = \phi$. $\phi(t, x)$ satisfies the condition that $Q(\phi) < 0$ and $S(\phi) < 0$. For $\lambda > 0$, we let $\phi_\lambda = \lambda^{3/2}\phi(\lambda x)$. Thus,

$$S(\phi_\lambda) = \lambda^2 \int |\nabla\phi|^2 \, dx + \omega \int |\phi|^2 \, dx - a\lambda^3 \int |\phi|^4 \, dx - \lambda^3 \int |\phi|^2 E_1(|\phi|^2) \, dx, \quad (4.6)$$

$$Q(\phi_\lambda) = \lambda^2 \int |\nabla\phi|^2 \, dx - \tfrac{3}{4}a\lambda^3 \int |\phi|^4 \, dx - \tfrac{3}{4}\lambda^3 \int |\phi|^2 E_1(|\phi|^2) \, dx. \quad (4.7)$$

Since $S(\phi) < 0$, this yields that there exists $0 < \lambda^* < 1$ such that $S(\phi_{\lambda^*}) = 0$ and, when $\lambda \in [\lambda^*, 1]$, $S(\phi_\lambda) \leqslant 0$. For $\lambda \in [\lambda^*, 1]$, $Q(\phi_\lambda)$ has the following three possibilities:

(i) $Q(\phi_\lambda) < 0$ for $\lambda \in [\lambda^*, 1]$;

(ii) $Q(\phi_{\lambda^*}) = 0$;

(iii) there exist $\mu \in (\lambda^*, 1)$ such that $Q(\phi_\mu) = 0$.

For cases (i) and (ii) we have $S(\phi_{\lambda^*}) = 0$ and $Q(\phi_{\lambda^*}) \leqslant 0$. It follows that $I(\phi_{\lambda^*}) \geqslant d_N$. Moreover, by

$$I(\phi_\lambda) = \frac{\lambda^2}{2} \int |\nabla\phi|^2 \, dx + \frac{\omega}{2} \int |\phi|^2 \, dx - \frac{a}{4}\lambda^3 \int |\phi|^4 \, dx - \frac{\lambda^3}{4} \int |\phi|^2 E_1(|\phi|^2) \, dx,$$

we have

$$I(\phi) - I(\phi_{\lambda^*}) = \tfrac{1}{2}(1 - \lambda^{*2}) \int |\nabla\phi|^2 \, dx - \tfrac{1}{4}(1 - \lambda^{*3}) \int (a|\phi|^4 + |\phi|^2 E_1(|\phi|^2)) \, dx, \quad (4.8)$$

$$Q(\phi) - Q(\phi_{\lambda^*}) = (1 - \lambda^{*2}) \int |\nabla\phi|^2 \, dx - \tfrac{3}{4}(1 - \lambda^{*3}) \int (a|\phi|^4 + |\phi|^2 E_1(|\phi|^2)) \, dx \quad (4.9)$$

and $0 < \lambda^* < 1$ implies that

$$I(\phi) - I(\phi_{\lambda^*}) \geqslant \tfrac{1}{2}Q(\phi) - \tfrac{1}{2}Q(\phi_{\lambda^*}) \geqslant \tfrac{1}{2}Q(\phi). \quad (4.10)$$

For case (iii), we have $Q(\phi_\mu) = 0$ and $S(\phi_\mu) < 0$. Thus, Lemma 3.4 implies that $I(\phi_\mu) \geqslant d_M \geqslant d_N$ and

$$I(\phi) - I(\phi_\mu) \geqslant \tfrac{1}{2}Q(\phi) - \tfrac{1}{2}Q(\phi_\mu) \geqslant \tfrac{1}{2}Q(\phi). \quad (4.11)$$

Since $I(\phi_{\lambda^*}) \geqslant d_N$, $I(\phi_\mu) \geqslant d_N$, from (4.10) and (4.11), we get

$$Q(\phi) \leqslant 2[I(\phi) - d_N]. \quad (4.12)$$

From (2.2), (2.3) and (3.1), $I(\phi) = I(\phi_0)$. Thus, by $\phi_0 \in K$ and (4.5), we have

$$J''(t) = 8Q(\phi) < 2[I(\phi_0) - d_N] < 0. \quad (4.13)$$

Obviously, $J(t)$ cannot verify (4.13) for all time. Therefore, from Proposition 2.1 it must be the case that $T < \infty$, which implies that

$$\lim_{t \in T} \|\phi(t, \cdot)\|_{H^1(\mathbb{R}^3)} = \infty.$$

From Proposition 3.7 and Theorem 4.2, it follows that if $|x|\phi_0(x) \in L^2(\mathbb{R}^3)$, then Theorem 4.1 is sharp. □

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Corollary 4.3. *Let $\phi_0 \in H^1(\mathbb{R}^3)$ and let ϕ_0 satisfy $\int |\nabla\phi_0|^2\,\mathrm{d}x + \omega\int|\phi_0|^2\,\mathrm{d}x < 2d_N$. Then the solution $\phi(t,x)$ of the Cauchy problem (1.1)–(2.1) exists globally in $t \in [0,\infty)$.*

Proof. From $\int |\nabla\phi_0|^2\,\mathrm{d}x + \omega\int|\phi_0|^2\,\mathrm{d}x < 2d_N$, we have $I(\phi_0) < d_N$. Moreover, we claim that $S(\phi_0) > 0$. Otherwise, there would be a $0 < \lambda \leqslant 1$ such that $S(\lambda\phi_0) = 0$. Thus, $I(\lambda\phi_0) \geqslant d_N$. On the other hand,

$$\lambda^2\bigg(\int|\nabla\phi_0|^2\,\mathrm{d}x + \omega\int|\phi_0|^2\,\mathrm{d}x\bigg) \leqslant 2d_N.$$

It follows that $I(\lambda\phi_0) < d_N$. This is a contradiction. Therefore, we have $\phi_0 \in R_+$. Thus, Theorem 4.1 implies this corollary. □

5. Instability of the standing waves

Using the methods in [**1, 14**], one can easily find that the variational problem (3.10) is attained. Let u be a solution of (3.10), that is we have

$$d_N = \min_{u\in N} I(u). \tag{5.1}$$

Then, by a standard variational computation, we have that u is a solution of the following nonlinear Euclidean scalar equation

$$-\Delta u + \omega u - a|u|^2u - E_1(|u|^2)u = 0. \tag{5.2}$$

Thus, $\phi(t,x) = \mathrm{e}^{\mathrm{i}\omega t}u(x)$ is a standing wave solution of (1.1). Since u is a minimizer of (5.1), we call $u(x)$ a ground state solution of (5.2). Using the method in [**1**] as well as [**2**], we can prove the strong instability of the standing wave, but the proof has to rely on the solvability of the following variational problem:

$$d_Q := \inf_{\{u\in H^1(\mathbb{R}^3)\setminus\{0\},\ Q(u)=0\}} I(u). \tag{5.3}$$

But here, by Lemma 3.4 and Theorem 4.2, we can refrain from solving problem (5.3), and show the instability of the standing waves directly. First we give two lemmas.

Lemma 5.1. *Let $\phi \in H^1(\mathbb{R}^3)\setminus\{0\}$. There then exists a unique $\mu > 0$ such that $S(\mu\phi) = 0$ and $I(\mu\phi) > I(\lambda\phi)$ for any $\lambda > 0$ and $\lambda \neq \mu$.*

Proof. For $\lambda > 0$, we have

$$S(\lambda\phi) = \lambda^2\int|\nabla\phi|^2\,\mathrm{d}x + \lambda^2\omega\int|\phi|^2\,\mathrm{d}x - a\lambda^4\int|\phi|^4\,\mathrm{d}x - \lambda^4\int|\phi|^2E_1(|\phi|^2)\,\mathrm{d}x, \tag{5.4}$$

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}I(\lambda\phi) = \lambda^{-1}S(\lambda\phi). \tag{5.5}$$

From (5.4) and (5.5), Lemma 5.1 is obtained. □

Lemma 5.2. *Let u be a minimizer of (5.1). Then $Q(u) = 0$.*

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Proof. Since u is a minimizer of (5.1), u is also a solution of (5.2). Thus, we have the Pohozaev identity,

$$\frac{\omega}{2}\int |u|^2\,\mathrm{d}x + \frac{1}{6}\int |\nabla u|^2\,\mathrm{d}x - \frac{a}{4}\int |u|^4\,\mathrm{d}x - \frac{1}{4}\int |u|^2 E_1(|u|^2)\,\mathrm{d}x = 0, \tag{5.6}$$

which is obtained from multiplying (5.2) by $x \cdot \nabla u$, then integrating. Note that $S(u) = 0$. Thus, $Q(u) = 0$. □

Now we give the following blow-up theorem.

Theorem 5.3. *Let u be a minimizer of (5.1). Then, for any $\varepsilon > 0$, there exists $\phi_0 \in H^1(\mathbb{R}^3)$ with $\|\phi_0 - u\|_{H^1(\mathbb{R}^3)} < \varepsilon$ such that the solution $\phi(t,x)$ of the Cauchy problem (1.1)–(2.1) blows up in a finite time.*

Proof. By Lemma 5.2, $Q(u) = 0$. Thus, we have $S(u) = 0$ and $Q(u) = 0$. It follows that, for any $\lambda > 1$, we have

$$S(\lambda u) < 0, \quad Q(\lambda u) < 0, \quad \lambda > 1. \tag{5.7}$$

On the other hand, from Lemma 5.1, $S(u) = 0$ implies that $I(\lambda u) < I(u)$ for any $\lambda > 1$. Note that $I(u) = d_N$. Thus, for any $\lambda > 1$ we have $\lambda u \in K$. Furthermore, it is clear that $|x|u(\cdot) \in L^2(\mathbb{R}^3)$, and thus $\lambda|x|u(\cdot) \in L^2(\mathbb{R}^3)$. Now we take $\lambda > 1$, and λ is sufficiently close to 1 such that

$$\|\lambda u - u\|_{H^1(\mathbb{R}^3)} = (\lambda - 1)\|u\|_{H^1(\mathbb{R}^3)} < \varepsilon. \tag{5.8}$$

Then we take $\phi_0 = \lambda u(x)$. From Theorem 4.2, the solution $\phi(t,x)$ of the Cauchy problem (1.1)–(2.1) blows up in a finite time. □

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