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Transformations on tensor products and the torsionless property in abelian groups

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Suppose T is the natural transformation

$$\hom(A, B) \otimes C \to \hom(\hom(C, A), B)$$

where the variables are abelian groups. We find conditions on A, B, and C that guarantee that T is injective.

1. Introduction

If C is an abelian group and C^* denotes its dual, $\hom(C, Z)$, then the natural transformation $C \to C^{**}$ is injective exactly if C is isomorphic to a subgroup of a product of infinite cyclic groups. Such groups are called "torsionless".

This transformation can be thought of as a special case of the natural transformation

$$T : \hom(A, B) \otimes C \to \hom(\hom(C, A), B)$$

given by $T(u \otimes x)(v) = (u \circ v)(x)$. It is the purpose of this study to show that T remains injective under relaxed conditions on A, B, and C.

The transformation T has been studied previously by several authors. Morita [4] and Warfield [7] have studied the natural map $C \to \hom(\hom(C, A), A)$. This map is the restriction of T in the case A

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and C are torsion-free since C can be considered as a subgroup of $\hom(A, A) \otimes C$. Nunke also has studied T. In [6] he proves that T is surjective in the case that I is a non-measurable set, $A = B = C_i = Z$, and C is a subgroup of $\prod_{i \in I} C_i$.

The proof of our result depends upon the injectivity of two other transformations, namely,

$$f : B \otimes \varprojlim C_i \to \varprojlim (B \otimes C_i)$$

and

$$S : \hom(A, B) \otimes C \to \hom(A, B \otimes C) .$$

These results are interesting in their own right and are shown to hold over an arbitrary Dedekind domain.

Throughout, I will denote an index set, Z an infinite cyclic group, A, B, C, and C_i for $i \in I$ will be abelian groups (or, sometimes, modules over a Dedekind domain R) and, when we are talking about modules over the Dedekind domain R, groups of homomorphisms and tensor products will be over that domain. Often an obvious map will not be explicitly defined or even named.

This study was begun with Professor Kaplansky's observation of the relation between the transformations f and S.

2. The transformations f and S

The transformation

$$f : B \otimes \varprojlim C_i \to \varprojlim (B \otimes C_i)$$

that sends a generator $x \otimes (y_i)$ to $(x \otimes y_i)$ has been studied in some detail [8], [9]. The relevant fact for this investigation is the following.

THEOREM 1 [8]. *Suppose B and, for every $i \in I$, C_i are modules over a Dedekind domain, every C_i is torsion-free, and the C_i's form an inverse system. Then $f : B \otimes \varprojlim C_i \to \varprojlim (B \otimes C_i)$ is injective.*

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We mean by $S : \hom(A, B) \otimes C \to \hom(A, B \otimes C)$ the transformation given by $S(u \otimes x)(a) = u(a) \otimes x$.

THEOREM 2. *Suppose* I *is an index set,* A, B , *and, for every* $i \in I$, C_i *are modules over a Dedekind domain* R , C *is a submodule of* ΠC_i , *and every* $S_i : \hom(A, B) \otimes C_i \to \hom(A, B \otimes C_i)$ *is injective (for example* $C_i = R$ *). Then* S *is injective for* C *if* C *is pure in* $\varprojlim C_i$, *and, in case* $\hom(A, B)$ *is torsion-free (for example* B *is torsion-free),* S *is injective for (arbitrary)* C .

Proof. By Theorem 1,

$$f_1 : \hom(A, B) \otimes \varprojlim C_i \to \varprojlim \left(\hom(A, B) \otimes C_i\right)$$

is injective. By assumption every

$$S_i : \hom(A, B) \otimes C_i \to \hom\left(A, B \otimes C_i\right)$$

is also injective. Now consider the commutative diagram

$$\begin{array}{ccccc}
\hom(A, B) \otimes \varprojlim C_i & \xrightarrow{S'} & \hom\left(A, B \otimes \varprojlim C_i\right) & \xrightarrow{f_2} & \hom\left(A, \varprojlim \left(B \otimes C_i\right)\right) \\
{\scriptstyle f_1}\big\downarrow & & & & \big\downarrow{\scriptstyle f_3} \\
\varprojlim \left(\hom(A, B) \otimes C_i\right) & & \xrightarrow{\qquad \Pi S_i \qquad} & & \varprojlim \left(\hom\left(A, B \otimes C_i\right)\right) .
\end{array}$$

f_2 is induced by the inclusion $B \otimes \varprojlim C_i \to \varprojlim \left(B \otimes C_i\right)$ (Theorem 1) and f_3 is the natural equivalence. Since $\Pi S_i \circ f_1 = f_3 \circ f_2 \circ S'$ is injective, S' must also be injective. It is now easy to see that S is injective for C by just chasing the following commutative diagram.

$$\begin{array}{ccc}
\hom(A, B) \otimes C & \xrightarrow{S} & \hom(A, B \otimes C) \\
\big\downarrow & & \big\downarrow \\
\hom(A, B) \otimes \varprojlim C_i & \xrightarrow{S'} & \hom\left(A, B \otimes \varprojlim C_i\right) .
\end{array}$$

THEOREM 3. *If* B *is torsion-free module over a Dedekind domain then* S *is injective.*

Proof. Denote by tA the torsion submodule of A . Then

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$\hom(A/tA, B) \approx \hom(A, B)$ and the natural map

$$\hom(A/tA,\ B\otimes C) \to \hom(A,\ B\otimes C)$$

is injective.

Consider the commutative diagram

$$\begin{array}{ccc} \hom(A,\ B)\otimes C & \to & \hom(A,\ B\otimes C) \\ \downarrow & & \downarrow \\ \hom(A/tA,\ B)\otimes C & \to & \hom(A/tA,\ B\otimes C)\ . \end{array}$$

The injectivity of S, in the case B is torsion-free, reduces to the situation where A is also torsion-free.

Following [3] we may write $A \approx \lim F_i$ where F_i is a free R-module of rank $n_i < \infty$. We have the following, where the natural equivalences are the standard ones;

$$\begin{array}{ccc} \hom(A,\ B)\otimes C & \to & \hom(A,\ B\otimes C) \\ \wr\wr & & \wr\wr \\ \hom\left(\varinjlim F_j,\ B\right)\otimes C & & \hom\left(\varinjlim F_j,\ B\otimes C\right) \\ \wr\wr & & \wr\wr \\ \varprojlim \hom\left(F_j,\ B\right)\otimes C & & \varprojlim \hom\left(F_j,\ B\otimes C\right) \\ \wr\wr & & \wr\wr \\ \left(\varprojlim B^{n_j}\right)\otimes C & \to & \varprojlim \left(B^{n_j}\otimes C\right) \end{array}$$

That the bottom line is injective is Theorem 1. The theorem is proved.

3. Main theorem

Henceforth all objects will be abelian groups. Preparatory to proving our main theorem we recall some definitions.

A torsion-free group A is said to be "slender" if every homomorphism from $\prod^{\infty} Z$ (countable product) into A is zero on all but finitely many components. Examples of slender groups are direct sums of reduced rank 1 torsion-free groups. Nunke [5] has characterized slender groups as those reduced torsion-free groups not containing either a copy of $\prod^{\infty} Z$ or the

p-adic integers (for any prime p).

The set I is called "measurable" if there is a countably additive measure m on I whose range is the two point set $\{0, 1\}$, which is 0 on points of I , and such that $m(I) = 1$. Apparently it is not known if such sets exist although, assuming that strongly inaccessible cardinals do exist, many of them would be measurable. However, our interest is in sets that are *not* measurable. (We can at least say countable sets are not.)

The reader is referred to [*1*, *2*] for a discussion of slender groups and the role of non-measurable sets in their study. At this point we isolate the theorem, due to Łoś, concerning these two ideas that will interest us.

PROPOSITION [*1*, *2*]. *If* A *is a slender group,* I *is a non-measurable set,* C_i *is torsion free for all* $i \in I$ *, then there is a natural isomorphism*

$$\hom(\Pi C_i, A) \approx \oplus \hom(C_i, A) .$$

THEOREM 4. *Suppose* A *is slender,* B *is torsion-free,* I *is not measurable,* C_i *is torsion-free for all* $i \in I$ *, and for all* $0 \neq a \in A$ *and all finite subsets* $C_i' \subset C_i$ *there exists* $r \in \hom(C_i, A)$ *such that* $r(C_i') = \{a\}$. *Then* T *is injective for every subgroup* C *of* ΠC_i .

Proof. Our argument concerns the following diagram:

$$\begin{array}{ccccc}
\hom(A, B) \otimes \Pi C_i & \xrightarrow{S} & \hom(A, B \otimes \Pi C_i) & \xrightarrow{f_1} & \hom(A, \Pi(B \otimes C_i)) \\
\downarrow T & & & & \downarrow f_2 \\
\hom(\hom(\Pi C_i, A), B) & & & & \Pi \hom(A, B \otimes C_i) \\
\downarrow g_1 & & & & \downarrow f_3 \\
\hom(\oplus \hom(C_i, A), B) & & \xrightarrow{\qquad g_2 \qquad} & & \Pi \hom(\hom(C_i, A), B) .
\end{array}$$

f_1 is the injective map induced by the inclusion $B \otimes \Pi C_i \to \Pi(B \otimes C_i)$ (Theorem 1); f_2 is the natural equivalence; g_1 is the isomorphism of the proposition (this is the point requiring the conditions on A and I); and g_2 is the natural equivalence.

Let D_i be the image of $\hom(A, B) \otimes \Pi C_i$ in $\hom(A, B \otimes C_i)$; that is, $D_i = \operatorname{im}(\Pi_i \circ f_2 \circ f_1 \circ S)$ where Π_i is the ith projection. We intend to show that if $h_i : D_i \to \hom(\hom(C_i, A), B)$ is defined as it must to make the diagram commute, then h_i is an injective homomorphism.

We define $[(h_i \circ \Pi_i \circ f_2 \circ f_1 \circ S)(u \otimes c)](r) = (u \circ r)(c)$, where $u \in \hom(A, B)$, $c \in C_i$, and $r \in \hom(C_i, A)$. To show h_i is a homomorphism we suppose $u_1, \ldots, u_n \in \hom(A, B)$, $c_1, \ldots, c_n \in C_i$ and $\sum_{j=1}^{n} u_j(a) \otimes c_j = 0$ for all $a \in A$. We must show $\sum_{j=1}^{n} (u_j \circ r)(c_j) = 0$ for all $r \in \hom(C_i, A)$.

Let D be the subgroup of C_i generated by $c_1, \ldots, c_n$. Then $c_j = \sum n_{jk} e_k$ for integers $\{n_{jk}\}_{\substack{1 \le j \le n \\ 1 \le k \le m}}$ where $e_1, \ldots, e_m$ generate the infinite cyclic components of D .

Since B is torsion-free, $\sum_{j=1}^{n} u_j(a) \otimes c_j = 0$ in $B \otimes C_i$ for all $a \in A$ means $\sum_{j=1}^{n} u_j(a) \otimes c_j = 0$ in $B \otimes D$ for all $a \in A$. Hence

$$0 = \sum_{j=1}^{n} u_j(a) \otimes c_j = \sum_{j=1}^{n} u_j(a) \otimes \left(\sum_{k=1}^{n} n_{jk} e_k\right) = \sum_{\substack{1 \le j \le n \\ 1 \le k \le m}} n_{jk} u_j(a) \otimes e_k$$

for all $a \in A$. This implies

$$\sum_{\substack{1 \le j \le n \\ 1 \le k \le m}} n_{jk} u_j(a) = 0$$

in B for all $a \in A$.

But, for $r \in \hom(C_i, A)$,

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$$\sum_{j=1}^{n} (u_j \circ r)(c_j) = \sum_{j=1}^{n} (u_j \circ r)\left(\sum_{k=1}^{n} n_{jk}e_k\right) = \sum_{\substack{1 \le j \le n \\ 1 \le k \le m}} n_{jk}u_j(r(e_k)) .$$

By our preceding paragraph, this sum is always zero for all $r \in \hom(C_i, A)$. This proves h_i is a homomorphism.

To show h_i is injective we suppose $\sum_{j=1}^{n} (u_j \circ r)(c_j) = 0$; that is,

$$\sum_{\substack{1 \le j \le n \\ 1 \le k \le m}} n_{jk}u_j(r(e_i)) = 0 ,$$

for all $r \in \hom(C_i, A)$. Assuming hypothesis (3) holds, then for an arbitrary $a \in A$ there exists $r \in \hom(C_i, A)$ such that $r(e_k) = a$ for all $1 \le k \le m$. Hence

$$0 = \sum_{\substack{1 \le j \le n \\ 1 \le k \le m}} n_{jk}u_j(r(e_k)) = \sum_{\substack{1 \le j \le n \\ 1 \le k \le m}} n_{jk}u_j(a) ;$$

that is, $\sum_{j=1}^{n} u_j(a) \otimes c_j = 0$. Therefore, h_i is injective.

To complete the proof, set $f_3 = \Pi h_i$. Theorem 3 says S is injective. Therefore $f_3 \circ f_2 \circ f_1 \circ S = g_2 \circ g_1 \circ T$ is injective. This constrains T to be injective. That T is injective for $C \subseteq \Pi C_i$ follows by an argument similar to the conclusion of Theorem 2.

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