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# ON THE DIVISOR CLASS GROUPS OF A TWO-DIMENSIONAL LOCAL RING AND ITS FORM RING

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## Introduction

Let $A$ be a noetherian ring and let $I$ be an ideal of $A$ contained in the Jacobson radical of $A$: $\mathrm{Rad}(A)$. We assume that the form ring of $A$ with respect to the ideal $I$: $G = \mathrm{Gr}(A, I)$, is a normal integral domain. Hence $A$ is a normal integral domain and one can ask for the links between $\mathrm{Cl}(A)$ and $\mathrm{Cl}(G)$.

Let $R = \oplus_{n \in \boldsymbol{Z}} I^n$ be the Rees algebra of $A$ with respect to the ideal $I$ (see § 2). In a previous paper [20], the authors have proved that $\mathrm{Cl}(A) \simeq \mathrm{Cl}(R)$; moreover there exists a "canonical" map $j$: $\mathrm{Cl}(R) \to \mathrm{Cl}(G)$ deduced from the hypersurface section $R \to G = R/uR$ (§ 1). Following the ideas of Lipman's paper [18], in [20] an attempt was made to find out sufficient conditions for $\ker(j) = 0$, (resp.: for $\ker(j)$ to be a torsion group). But this sufficient conditions become almost tautological when $\dim(A) = 2$ and $\mathrm{ht}(I) = 2$ (i.e. when $A$ is a local ring and $I = \mathrm{Rad}(A)$; see §1). This paper deals with this last case.

The main result of the paper is Theorem 4; this theorem can be proved also by using the geometrical machinery of Grothendieck, Danilov, Boutot and Bădescu-Fiorentini [15, 8, 9, 6, 3] (see also the Remark 3 after the proof of Theorem 4).

Our proof mainly uses simple tools of Commutative Algebra and standard facts of Local Cohomology theory. A key point is the finiteness of a suitable local cohomology module which we derive from [15].

It is also interesting that the short exact sequences which appear in Theorem 1 of [18] are the same which appear in our proof. In a certain sense, this circumstance unifies the two techniques.

However the problem of the injectivity of $j$: $\mathrm{Cl}(R) \to \mathrm{Cl}(G)$ for a general hypersurface section $R \to G = R/uR$, $\dim(G) = 2$, is rather different

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from that considered in this paper as the example given in Section 3 shows.

## § 1.

In [10] Danilov has studied the links between the groups $\mathrm{Cl}(A[[T]])$ and $\mathrm{Cl}(A)$, where $A$ is a normal integral domain. To do this, he has defined a canonical map $j\colon \mathrm{Cl}(A[[T]]) \to \mathrm{Cl}(A)$. But in fact Danilov's definition works more generally to give a map $j\colon \mathrm{Cl}(R) \to \mathrm{Cl}(R/uR)$ for any normal integral domain $R$ and nonunit $u \in R$ such that $R/uR = G$ is also a normal integral domain. Let us recall the construction of $j$ from the viewpoint of this paper.

First of all, $\mathrm{Cl}(R)$ can be thought as the group of isomorphism classes of finitely generated, reflexive, rank one $R$-modules [18, 31]; a similar interpretation holds for $\mathrm{Cl}(G)$. Let $F$ be a finitely generated, reflexive, rank one $R$-module; we set: $[F]_R =$ (isomorphism class of $F$) $\in \mathrm{Cl}(R)$. Let $E = F \otimes_R G$ with $F$ as above; then $E^{**} = \mathrm{Hom}_G(\mathrm{Hom}_G(E, G), G)$ is a finitely generated, reflexive, rank one $G$-module. By this interpretation of the class group, we have, following [18]: $j([F]_R) = [E^{**}]_G$.

From now on we assume that $R$ is a $Z$-graded ring, i.e. $R = \oplus_{n \in Z} R_n$, and $u \in R$ is a homogeneous element. Let $\xi \in \mathrm{Cl}(R)$; then $\xi = [\mathfrak{b}]_R$ for some homogeneous integral divisorial ideal $\mathfrak{b}$ of $R$. If $\mathfrak{b} = \mathfrak{b}' \cap u^n R$, $(n \geqslant 0)$, where $\mathfrak{b}'$ is a homogeneous divisorial ideal with $\mathfrak{b}' \not\subseteq uR$; then $[\mathfrak{b}]_R = [\mathfrak{b}']_R$ since $u$ is a prime element of $R$. Therefore there is no loss of generality in assuming that $\mathfrak{b} \not\subseteq uR$, or equivalently that $u$ is regular for $R/\mathfrak{b}$. Then we get: $\mathfrak{a} = \mathfrak{b} \otimes_R G \simeq \mathfrak{b}/u\mathfrak{b} \simeq \mathfrak{b} + uR/uR$ and $j([\mathfrak{b}]_R) = [((\mathfrak{a})^{-1})^{-1}]_G$ where $((\mathfrak{a})^{-1})^{-1}$ denotes, as usual, the divisorial ideal associated to $\mathfrak{a}$. In the sequel we will always refer to this simpler setup whenever the map $j$ is concerned.

The homomorphism $j$ ties together the groups $\mathrm{Cl}(R)$ and $\mathrm{Cl}(G)$. In particular one can ask the following questions for $j$: when is $j$ surjective? and: when is $j$ injective?

The following proposition, concerning the latter question, has been proved in [20] following the general ideas of Lipman's paper [18]:

*Let $R = \oplus_{n \in Z} R_n$ be a $Z$-graded normal integral domain and let $u \in h$-$\mathrm{Rad}(R)$ be a non-zero homogeneous element such that $G = R/uR$ is also a normal domain. Suppose that the canonical map $j_Q\colon \mathrm{Cl}(R_Q) \to \mathrm{Cl}(G_P)$ is injective (resp.: $\ker(j_Q)$ is a torsion group) for every homogeneous prime ideal $Q$ of $R$ such that: $u \in Q$ and $\mathrm{depth}(R_Q) \leqslant 3$ (of course $P = Q/uR$). Then also $j\colon \mathrm{Cl}(R) \to \mathrm{Cl}(G)$ is an injective map (resp.: $\ker(j)$ is a torsion* *group*).

Let us observe that the hypotheses of this proposition forces $\dim(G) \geqslant 2$. But if $G$ is a normal integral domain and $\dim(G) = 2$, then $G$ is a C.M. ring. Hence also $R$ is a C.M. ring ($u \in h$-Rad$(R)$; see [7], Proposition 2.2) and the above proposition becomes almost tautological. If $\dim(G) \leqslant 1$, $\mathrm{Cl}(R)$ is simple to compute.

Therefore only the case $\dim(G) = 2$ remains still open. After all, this is not so surprising; in fact the case $\dim(A) = 2$ was the hardest to solve also for the problem of Danilov-Samuel, i.e. for the hypersurface section $A[[T]] \to A$ (see [26, 25, 28, 9]). Essentially, there are two (non tautological) ways to handle the case of a general hypersurface section $R \to G$ when $\dim(G) = 2$. The most recent one is due to Flenner (see Lemma 3.4 of [12]) and is inspired to Theorem 1 of [18]. The other one is used in this paper and comes from Hilfsatz 3 of [28], or Remarque p. 164 of [27]; it is summarized in the following proposition:

PROPOSITION 1. *Let $R = \oplus_{n \in Z} R_n$ be a Z-graded, normal integral domain and let $u \in h - \mathrm{Rad}(R)$ be a non-zero, homogeneous element such that $G = R/uR$ is also a normal integral domain. Assume that G is a C.M. ring. Let $\xi \in \mathrm{Cl}(R)$ and let $\mathfrak{b} \subset R$ be a homogeneous, proper, divisorial ideal such that $\xi = [\mathfrak{b}]_R$, $\mathfrak{b} \not\subseteq uR$ and $\mathfrak{a}^{-1}$ is a h-free G-module, where $\mathfrak{a} = \mathfrak{b} \otimes_R G$. Then $\xi = 0$, i.e. $\mathfrak{b}$ is h-free, if and only if $R/\mathfrak{b}$ is a C.M. ring.*

*Proof.* Let $R/\mathfrak{b}$ be a C.M. ring; since $u$ is regular for $R/\mathfrak{b}$, the ideal $\mathfrak{b} + uR/\mathfrak{b}$ is an unmixed ideal of height one of $R/\mathfrak{b}$. Therefore $\mathfrak{b} + uR$ is an unmixed ideal of height two of $R$, hence $\mathfrak{a} = \mathfrak{b} + uR/uR$ is an unmixed ideal of height one of $G$. It follows that $\mathfrak{a} = ((\mathfrak{a})^{-1})^{-1}$, so $\mathfrak{a}$ is $h$-free and then $\mathfrak{b}$ is $h$-free (see [5], Ch. II, 3.2, Proposition 5; with suitable modifications to the homogeneous case). The converse is trivial because $R$ is a C.M. ring.

## § 2.

Let $A$ be a ring and $I \subseteq \mathrm{Rad}(A)$ an ideal of $A$. We fix the following notation: $R = R(A, I) = \oplus_{n \in Z} I^n$ ($I^n = A$ for $n \leqslant 0$) is the Rees algebra of $A$ with respect to $I$. If $T$ is an indeterminate over $A$, let $u = T^{-1}$. We have $R = A[a_1T, \cdots, a_rT, u] \subseteq A[T, u]$ where $I = (a_1, \cdots, a_r)$. $G = \mathrm{Gr}(A, I) = \oplus_{n \geq 0} I^n/I^{n+1}$ is the form ring of $A$ with respect to $I$; the irrelevant ideal of $G$ is $G_+ = \oplus_{n>0} I^n/I^{n+1}$; let $\mathfrak{a}^* = \mathfrak{a} \cdot A[T, u] \cap R$ where $\mathfrak{a}$ is an ideal of

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$A$; $\mathfrak{a}^*$ is a graded ideal of $R$; $\mathrm{In}_I(\mathfrak{a})$ is the graded ideal of $G$ generated by the initial forms $\mathrm{In}(x)$ for all $x \in \mathfrak{a}$.

We refer to [7, 20, 22, 24] for the general properties of these rings and ideals. However, for the sake of completeness, let us recall the following ones: first $G \simeq R/uR$; $u$ is a homogeneous element and $\deg(u) = -1$. Moreover $u \in h - \mathrm{Rad}(R)$ and, finally, $\mathrm{In}_I(\mathfrak{a}) = \mathfrak{a}^* + uR/uR$. If $G$ is a normal integral domain, then also $R$ and $A$ are normal integral domains. If $G$ is a normal integral domain we can consider the map $j\colon \mathrm{Cl}(R) \to \mathrm{Cl}(G)$ defined in Section 1. Moreover, since $u$ is a prime element of $R$ it is easy to see that $\mathrm{Cl}(A) \underset{\psi}{\xrightarrow{\sim}} \mathrm{Cl}(R)$ (see [20], Proposition 1); to be precise, the isomorphism $\psi$ between $\mathrm{Cl}(A)$ and $\mathrm{Cl}(R)$ is given by $\psi([\mathfrak{a}]_A) = [\mathfrak{a}^*]_R$, where $\mathfrak{a}$ is an integral, divisorial ideal of $A$. Therefore, by composition, we get a homomorphism $i\colon \mathrm{Cl}(A) \to \mathrm{Cl}(G)$ such that $i([\mathfrak{a}]_A) = [((\mathrm{In}_I(\mathfrak{a}))^{-1})^{-1}]_G$ where $\mathfrak{a}$ is as above.

We begin the study of the map $j\colon \mathrm{Cl}(R(A, I)) \to \mathrm{Cl}(G(A, I))$ with a statement concerning the surjectivity of $j$.

PROPOSITION 2. *Let $(A, m)$ be a local, henselian ring with $\dim(A) = 2$. Suppose that $G = \mathrm{Gr}(A, m)$ is a normal integral domain. Then the map $j\colon \mathrm{Cl}(R) \to \mathrm{Cl}(G)$ is surjective.*

*Proof.* Let $P$ be a homogeneous, height one, prime ideal of $G$. Pick $\bar{x} \in P - P^{(2)}$ with $\bar{x}$ homogeneous. Let $x \in A$ be an element such that $\mathrm{In}(x) = \bar{x}$. Expand $xA$ to a prime ideal $Q$, maximal among those disjointed from the multiplicatively closed set $\{y \in A \mid \mathrm{In}(y) \notin P\}$; clearly $\mathrm{ht}(Q) = 1$. From the isomorphism $G(A/Q, m/Q) \simeq G(A, m)/\mathrm{In}(Q)$ and the choice of $\bar{x}$ it follows that $((\mathrm{In}(Q))^{-1})^{-1} = P$ (see Lemma 6 of [1]). Therefore $j([Q^*]_R) = [P]_G$, where $Q^* = Q \cdot A[T, u] \cap R$. Since $\mathrm{Cl}(G)$ is generated by the classes of homogeneous, height one prime ideals of $G$, the thesis follows.

*Remark.* The following example shows that we cannot delete the requirement "$G$ is normal" in Proposition 2. Let $A = \boldsymbol{R}[[X, Y, Z]]/(x^2 + Y^2 + Z^3)$ ($\boldsymbol{R}$ is the field of real numbers); $A$ is a local complete factorial ring, and $\dim(A) = 2$ (see ex. (25, 4) of [17]). But $G(A, m) \simeq \boldsymbol{R}\,[X, Y, Z]/(X^2 + Y^2)$ is not even normal, hence it cannot be factorial.

The next proposition deals with the case $\mathrm{ht}(I) \leqslant 1$.

PROPOSITION 3. *Let $A$ be a ring with $\dim(A) = 2$, and let $I \subset \mathrm{Rad}(A)$ be an ideal of $A$ such that $\mathrm{ht}(I) \leqslant 1$ and $G = \mathrm{Gr}(A, I)$ is a normal integral* *domain.*

a) *If $I$ is invertible, then* $\mathrm{Cl}(A)$ *is embedded in* $\mathrm{Cl}(G)$;

b) *If $G$ is an almost factorial ring* (*in particular if $G$ is a factorial ring*) *then $I$ is invertible.*

*Proof.* If $\mathrm{ht}(I) = 0$ there is nothing to prove, so we assume that $\mathrm{ht}(I) = 1$. We have that $G_0 = A/I$ is a Krull domain. Since $\dim(A/I) = 1$, $A/I$ is a Dedekind domain; in particular it satisfies the property $(R_1)$ of Serre and moreover $\mathrm{Cl}(A/I) \simeq \mathrm{Pic}(A/I)$. But $I$ is invertible and $I \subseteq \mathrm{Rad}(A)$; by localization at the maximal ideals, we have that $A$ is an $(R_2)$ ring, hence $A$ is locally factorial and $\mathrm{Cl}(A) \simeq \mathrm{Pic}(A)$. Since $I \subseteq \mathrm{Rad}(A)$, the canonical map $\mathrm{Pic}(A) \to \mathrm{Pic}(A/I)$ is injective (see [2], Proposition 1.4). From the hypothesis "I is invertible" it follows that $G$ is a flat $G_0$-module (Lemma 2.1 of [23]). So the extension $G_0 \to G$ satisfies condition (PDE) and the induced homomorphism $\mathrm{Cl}(G_0) \to \mathrm{Cl}(G)$ is injective ([13], Proposition 10.7). This completes the proof of *a*).

The irrelevant ideal $G_+$ of $G$ is a prime ideal. We easily get $G_+^{(p)} = \oplus_{n \geqslant p} G_n = G_+^p$ for all $p > 0$. Therefore, since $G_+ = \mathrm{In}(I)$ and $\mathrm{ht}(\mathrm{In}(I)) = \mathrm{ht}(I) = 1$, we have that $G_+$ is a projective $G$-module, since $G$ is an almost factorial ring. But $I/I^2 \simeq G_+/G_+^2 \simeq G_+ \otimes_G G_0$, so $I/I^2$ is a projective $A/I$-module. Then $I$ is locally principal since $A_I$ is a DVR (see Lemma 2.1 of [23]).

*Remark.* In general the embedding $f\colon \mathrm{Cl}(A) \to \mathrm{Cl}(G)$ constructed in the proof is different from the map obtained by the composition of the isomorphism $\psi\colon \mathrm{Cl}(A) \xrightarrow{\sim} \mathrm{Cl}(R)$ and $j\colon \mathrm{Cl}(R) \to \mathrm{Cl}(G)$, i.e. from the map $i$.

If $\mathrm{ht}(I) = 2$, from $I \subseteq \mathrm{Rad}(A)$ and $\dim(A) = 2$ it follows that $(A, I)$ is a local ring, i.e. $I = \mathrm{Rad}(A)$. In the next theorem a sufficient condition is given for the map $j\colon \mathrm{Cl}(R) \to \mathrm{Cl}(G)$ to be injective when $\mathrm{ht}(I) = 2$.

Local cohomology is the key tool in the proof of Theorem 4, so let us make some general remarks on it. Let $S$ be a graded ring, $J \subset S$ a graded ideal and $M$ a graded $S$-module. Since $H_J^i(M) = \varinjlim_n \mathrm{Ext}_S^i(S/J^n, M)$ for all $i \geqslant 0$, the local cohomology modules are graded modules in this case. Moreover, let

$$0 \longrightarrow M' \underset{(d)}{\overset{\rho}{\longrightarrow}} M \underset{(t)}{\overset{\pi}{\longrightarrow}} M'' \longrightarrow 0$$

be a short exact sequence of graded $S$-modules. Suppose $\rho$ and $\pi$ graded, respectively of degree $d$ and $t$. Then the corresponding long exact cohomology sequence is graded as follows:

$$\cdots \longrightarrow H_J^i(M') \xrightarrow[(d)]{H_J^i(\rho)} H_J^i(M) \xrightarrow[(t)]{H_J^i(\pi)} H_J^i(M'') \xrightarrow[(-d-t)]{\theta_i} H_J^{i+1}(M') \longrightarrow \cdots$$

THEOREM 4. *Let* $(A, Q)$ *be a local ring with* $\dim(A) = 2$. *Assume that*:

(1) $$(H^2_{G+}(G))_n = 0 \text{ for all } n > 0$$

*Then the map* $j\colon \mathrm{Cl}(R) \longrightarrow \mathrm{Cl}(G)$ *is injective.*

*Proof.* We shall give the proof in several steps.

*Step* 1. Let $\mathfrak{b}$ be a homogeneous integral (proper) divisorial ideal of $R$ such that $\mathfrak{b} \not\subset uR$; suppose that $\mathfrak{a}^{-1}$ is a $h$-free $G$-module (where $\mathfrak{a}$ denotes, as usual, $\mathfrak{b} \otimes_R G \simeq \mathfrak{b} + uR/uR$). By Proposition 1 we have only to show that $R/\mathfrak{b}$ is a C.M. ring. $R$ is a $h$-local ring; indeed $\mathfrak{m} = (Q^*, u)$ is a maximal ideal of $R$ and $\mathfrak{m} = h - \mathrm{Rad}(R)$. Then also $R/\mathfrak{b}$ is a $h$-local ring and $\mathfrak{n} = \mathfrak{m}/\mathfrak{b} = h - \mathrm{Rad}(R/\mathfrak{b})$. $R/\mathfrak{b}$ is a C.M. ring if and only if $(R/\mathfrak{b})_\mathfrak{n}$ is a C.M. ring (see [19], Theorem 1.1). But $(R/\mathfrak{b})_\mathfrak{n}$ is a C.M. ring if and only if $H^0_{\bar{\mathfrak{n}}}((R/\mathfrak{b})_\mathfrak{n}) = H^1_{\bar{\mathfrak{n}}}((R/\mathfrak{b})_\mathfrak{n}) = 0$ (where $\bar{\mathfrak{n}} = \mathfrak{n}\cdot(R/\mathfrak{b})_\mathfrak{n}$). Now $H^0_{\bar{\mathfrak{n}}}((R/\mathfrak{b})_\mathfrak{n}) = 0$ since $u$ is a regular element for $R/\mathfrak{b}$. $R$ is a C.M. ring and $\mathrm{depth}(R_\mathfrak{m}) = 3$; then from the long exact sequence for the local cohomology and from Theorem 4.3 of [29] we get: $H^1_\mathfrak{n}(R/\mathfrak{b}) \simeq H^1_\mathfrak{m}(R/\mathfrak{b}) \simeq H^2_\mathfrak{m}(\mathfrak{b})$. As $H^1_\mathfrak{n}(R/\mathfrak{b}) \otimes_{R/\mathfrak{b}} (R/\mathfrak{b})_\mathfrak{n} \simeq H^1_{\bar{\mathfrak{n}}}((R/\mathfrak{b})_\mathfrak{n})$ (see [29], Theorem 5.1), it will be sufficient to show that $H^2_\mathfrak{m}(\mathfrak{b}) = 0$.

*Step* 2. $H^2_\mathfrak{m}(\mathfrak{b})$ is a finitely generated $R$-module. To see this it is sufficient to prove that $H^2_{\hat{\mathfrak{m}}}(\hat{\mathfrak{b}})$ is a finitely generated $\hat{R}$-module, where $\hat{R} = \widehat{(R, \mathfrak{m})} \simeq \widehat{(R_\mathfrak{m}, \mathfrak{m}R_\mathfrak{m})}$. In fact we have (see [30], Theorem 4.5): $H^2_{\hat{\mathfrak{m}}}(\hat{\mathfrak{b}}) \simeq H^2_{\mathfrak{m}R_\mathfrak{m}}(\mathfrak{b}R_\mathfrak{m}) \otimes_{R_\mathfrak{m}} \hat{R}$; therefore $H^2_{\hat{\mathfrak{m}}}(\hat{\mathfrak{b}})$ is finitely generated over $\hat{R}$ if and only if $H^2_{\mathfrak{m}R_\mathfrak{m}}(\mathfrak{b}R_\mathfrak{m})$ is a finitely generated $R_\mathfrak{m}$-module (see [5], Proposition 11, Ch. I. 3.6.) and this last condition is equivalent to "$H^2_\mathfrak{m}(\mathfrak{b})$ is finitely generated over $R$" since $H^2_{\mathfrak{m}R_\mathfrak{m}}(\mathfrak{b}R_\mathfrak{m}) \simeq H^2_\mathfrak{m}(\mathfrak{b}) \otimes_R R_\mathfrak{m}$ (see [21], Proposition 11). Since $\hat{R}$ is a C.M. ring and since $\hat{\mathfrak{b}}$ is an unmixed ideal of height one of $\hat{R}$ (see [14], 9.3 and 13.8), for every prime ideal $P$ of $\hat{R}$ such that $\mathrm{ht}(P) = 2$, $\mathrm{depth}(\hat{\mathfrak{b}}_P) = 2$. The finite generation of $H^2_{\hat{\mathfrak{m}}}(\hat{\mathfrak{b}})$ over $\hat{R}$ then follows from [15], Expose VIII, Corollaire 2.3.

*Step* 3. The hypothesis "$\mathfrak{a}^{-1}$ is $h$-free" implies $((\mathfrak{a})^{-1})^{-1} = xG$, where $x$ is a homogeneous element of $G$ of degree $d > 0$. Then $\mathfrak{a} = \mathfrak{b} + uR/uR = xG \cap I$, where $I$ is an eventual embedded primary component; since $\mathfrak{a}$ is homogeneous and $\dim(G) = 2$, $I$ is irrelevent, i.e. $\sqrt{I} = G_+$. Now we have $H^1_{G+}(G) = H^0_{G+}(G) = 0$ since $G$ is a C.M. ring. But $((\mathfrak{a})^{-1})^{-1} = xG \simeq G(-d)$, hence $H^i_{G+}(((\mathfrak{a})^{-1})^{-1}) \simeq H^i_{G+}(G)(-d)$. From the short exact sequence:

$$(2) \qquad 0 \longrightarrow \mathfrak{a} \longrightarrow ((\mathfrak{a})^{-1})^{-1} \longrightarrow C \longrightarrow 0$$

it follows that $H^1_{G+}(\mathfrak{a}) \simeq H^0_{G+}(C) = C$ where the isomorphism is of degree zero. Then from $(((\mathfrak{a})^{-1})^{-1})_n = 0$ for all $n < d$ we get: $(H^1_{G+}(\mathfrak{a}))_n = 0$ for all $n < d$. Since $\operatorname{Supp}(C) \subseteq \{G_+\}$ we have $H^i_{G+}(C) = 0$ for all $i > 0$. Therefore from the long exact cohomology sequence associated to (2) we get: $H^2_{G+}(\mathfrak{a}) \simeq H^2_{G+}(((\mathfrak{a})^{-1})^{-1}) \simeq H^2_{G+}(G)(-d)$ where both isomorphisms are of degree zero. Now the hypothesis (1) comes into play to get: $(H^2_{G+}(\mathfrak{a}))_n = 0$ for all $n > d$.

Finally, from the canonical isomorphisms (of degree zero) $H^1_{G+}(\mathfrak{a}) \simeq H^1_{\mathfrak{m}}(\mathfrak{a})$ and $H^2_{G+}(\mathfrak{a}) \simeq H^2_{\mathfrak{m}}(\mathfrak{a})$ we get:

$$(3) \qquad (H^1_{\mathfrak{m}}(\mathfrak{a}))_n = 0 \qquad \text{for all } n < d\,,$$

$$(4) \qquad (H^2_{\mathfrak{m}}(\mathfrak{a}))_n = 0 \qquad \text{for all } n > d\,.$$

*Step* 4. Let

$$(5) \qquad \cdots \longrightarrow (H^1_{\mathfrak{m}}(\mathfrak{a}))_n \longrightarrow (H^2_{\mathfrak{m}}(\mathfrak{b}))_{n+1} \xrightarrow{\cdot u} (H^2_{\mathfrak{m}}(\mathfrak{b}))_n \longrightarrow (H^2_{\mathfrak{m}}(\mathfrak{a}))_n \longrightarrow \cdots$$

be the long exact cohomology sequence corresponding to the short exact sequence:

$$0 \longrightarrow \mathfrak{b} \underset{(-1)}{\xrightarrow{\cdot u}} \mathfrak{b} \underset{(0)}{\longrightarrow} \mathfrak{a} \longrightarrow 0$$

From (5) and (3) it follows that $u$ is a regular element for all homogeneous elements of $H^2_{\mathfrak{m}}(\mathfrak{b})$ of degree $\leqslant d$. Let $x \in (H^2_{\mathfrak{m}}(\mathfrak{b}))_n$ with $n \leqslant d$; by definition of local cohomology there exists a positive integer $t$ such that $\mathfrak{m}^t \cdot x = 0$; but $u \in \mathfrak{m}$, hence $u^t \cdot x = 0$; therefore $x = 0$ and $(H^2_{\mathfrak{m}}(\mathfrak{b}))_n = 0$ for all $n \leqslant d$. (this is essentially the proof of Lemma 1.2 of [30]). Therefore, from (5) and (4) it follows $H^2_{\mathfrak{m}}(\mathfrak{b}) = uH^2_{\mathfrak{m}}(\mathfrak{b})$. But $u \in h - \operatorname{Rad}(R)$, hence $H^2_{\mathfrak{m}}(\mathfrak{b}) = 0$ by Step 2 and the homogeneous Nakayama's lemma.

*Remarks*.

1) Since $\hat{R}$ is a local ring we can derive the finite generation of

$H^2_{\mathfrak{m}}(\hat{\mathfrak{b}})$ over $\hat{R}$, also from [15], Exposé $V$, Corollaire 3.6. Since $\hat{R}$ is flat over $R$, $u$ is a regular element for $\hat{R}$. Therefore: $\mathrm{Gr}(\hat{R}, u\hat{R}) \simeq \hat{R}/u\hat{R}[T] \simeq \hat{G}[T]$, where $T$ is an indeterminate over $\hat{R}/u\hat{R}$ and $\hat{G} = \widehat{(G, G_+)}$. It follows that $\hat{R}$ is a normal integral domain (see § 3, Proposition 6, a)).

2) With the same notations of Theorem 4, but without the hypothesis (1), we can prove the following result:

For every ideal $J$ of $A$, let $\mathfrak{c} = JA[T, u] \cap R$. Then $(H^2_{\mathfrak{m}}(\mathfrak{c}))_n = 0$ for all $n \leqslant 0$. Let $\{u, x, y\}$ be a homogeneous system of parameters in $R$ with $\deg(x) = \deg(y) = 1$; thus the Čech complex of $\mathfrak{c}$ is given by:

$$C^{\cdot}(u, x, y; \mathfrak{c});\ 0 \xrightarrow{d_0} \mathfrak{c}_u \oplus \mathfrak{c}_x \oplus \mathfrak{c}_y \xrightarrow{d_1} \mathfrak{c}_{ux} \oplus \mathfrak{c}_{uy} \oplus \mathfrak{c}_{xy} \xrightarrow{d_2} \mathfrak{c}_{uxy} \longrightarrow 0$$

All the modules have a natural grading and the maps $d_i$ are as usual. Assume $\sigma = (d/(ux)^p;\ e/(uy)^p;\ f/(xy)^p) \in \ker(d_2)$, (i.e. $-dy^p + ex^p - fu^p = 0$) with $d, e, f$ homogeneous elements of $c$, and $\deg(d) = \deg(e) = \deg(f) - 2p = n \leqslant 0$.

We prove that there exists $\rho = (a/u^p;\ b/x^p;\ c/y^p)$, with $a, b, c$ homogeneous elements of $\mathfrak{c}$, such that $d_1(\rho) = \sigma$ i.e. $ax^p - bu^p = d$ and $ay^p - cu^p = e$ (the third equation $by^p - cx^p = f$ is dependent upon the others). If $p \leqslant 0$, the proof is trivial. Let $p > 0$. Since $(u^p, x^p, y^p)$ is an $R$-regular sequence, we have $d \in (x^p, u^p)$, $e \in (y^p, u^p)$. On the other hand, one can easily prove that $\oplus_{n \leqslant 0}(x^p, u^p)_n$ and $\oplus_{n \leqslant 0}(y^p, u^p)_n$ are included in $\oplus_{n \leqslant 0}(u^p)_n$; hence $d, e \in (u^p)$.

We now recall that $u^p$ is regular for $R/\mathfrak{c}$; therefore the system

$$\begin{cases} ax^p - bu^p = d \\ ay^p - cu^p = e \end{cases}$$

has solutions if we take $a = 0$.

3) Theorem 4 has many sources; in particular see [8, 12, 3, 4]. Instead of condition (1) of Theorem 4 in these papers is used the equivalent condition:

$$(6) \qquad H^1(Y, \mathcal{O}_Y(n)) = 0 \text{ for all } n > 0$$

where $Y = \mathrm{Proj}(G)$ (see [16], Ch. III, Proposition 2.1.5.)

For the sake of completeness we briefly show how the geometrical techniques work to get results as in Theorem 4.

Let $\mathscr{R} = \oplus_{n \geqslant 0} Q^n$ be the blow-up algebra with respect to the ideal $Q$, and set: $X = \mathrm{Proj}(\mathscr{R})$, $Y = \mathrm{Proj}(G)$; let $\chi\colon Y \to X$ be the closed immersion deduced from the canonical map $\mathscr{R} \to \mathscr{R}/Q\mathscr{R} \simeq G$.

Since $A$ and $G$ are normal integral domains of dimension two, it follows easily that the canonical morphism $X \to \operatorname{Spec}(A)$ is a desingularization of $\operatorname{Spec}(A)$. In particular we get that the canonical morphism $\varphi$: $\operatorname{Pic}(X) \to \operatorname{Pic}(X - Y)$ is surjective (see [16] IV, 21. 6. 11). Moreover: $\operatorname{Ker}\varphi = [\Theta_X(1)]\cdot \boldsymbol{Z}$, and this is an infinite cyclic group. But $X - Y \simeq \operatorname{Spec}(A) - \{\mathfrak{m}\}$; therefore $\operatorname{Pic}(X - Y) \simeq \operatorname{Pic}(\operatorname{Spec}(A) - \{\mathfrak{m}\}) \simeq \operatorname{Cl}(A)$ (see [13], 18. 10) and we get the short exact sequence:

$$0 \longrightarrow \boldsymbol{Z} \longrightarrow \operatorname{Pic}(X) \longrightarrow \operatorname{Cl}(A) \longrightarrow 0$$

Another well known short exact sequence is the following:

$$0 \longrightarrow \boldsymbol{Z} \longrightarrow \operatorname{Pic}(Y) \longrightarrow \operatorname{Cl}(G) \longrightarrow 0$$

where the first morphism maps 1 to $[\Theta_Y(1)]$. Finally, we consider the morphism $\chi^*$: $\operatorname{Pic}(X) \to \operatorname{Pic}(Y)$ deduced from the closed immersion $\chi$: $Y \hookrightarrow X$. Putting all together we get the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \boldsymbol{Z} & \longrightarrow & \operatorname{Pic}(X) & \longrightarrow & \operatorname{Cl}(A) & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \chi^*} & & \downarrow{\scriptstyle \bar{\chi}} & & \\ 0 & \longrightarrow & \boldsymbol{Z} & \longrightarrow & \operatorname{Pic}(Y) & \longrightarrow & \operatorname{Cl}(G) & \longrightarrow & 0 \end{array}$$

where $\bar{\chi}$ is deduced from $\chi^*$. It is easily checked that $\bar{\chi} = i$. From the "snake-lemma" it follows that the maps $\chi^*$ and $\bar{\chi} = i$ have isomorphic kernels and cokernels. The geometrical techniques developed in [15, 9, 6, 3] allow a direct study of $\chi^*$. Now we sketch their use.

First of all, we can define for all $n \geqslant 0$ a graded ring $G_n = \oplus_{i\geqslant 0} Q^i/Q^{i+n+1}$; in particular we have $G_0 = G$. If $m \geqslant n$ we get an epimorphic map $G_m \to G_n$. Let $Y_n = \operatorname{Proj}(G_n)$; we have a closed immersion $Y_n \hookrightarrow Y_m$ whenever $m \geqslant n$.

If $(A, Q)$ is henselian, then the sequence $\{\operatorname{Pic}(Y_n)\}_n$ is essentially constant and $\operatorname{Pic}(X) = \varprojlim_n \operatorname{Pic}(Y_n)$ (see [6], Ch. IV, Proposition 6.2). By virtue of well-known Theorem of Mori ([13], Corollary 6.12), we can reduce to the case "$(A, Q)$ henselian" by replacing $A$ with $\hat{A} = \widehat{(A, Q)}$. Moreover, for all $n \geqslant 0$ we have a short exact sequence of abelian sheaves on the topological space of $Y$:

$$(7) \qquad 0 \longrightarrow (i_{n+1})_* \Theta_Y(n+1) \longrightarrow (\Theta_{Y_{n+1}})^* \longrightarrow (\Theta_{Y_n})^* \longrightarrow 1$$

where $i_{n+1}$: $Y \to Y_{n+1}$ is the canonical closed immersion. Since $\dim(Y) = 1$

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the long exact sequence deduced from (7) is:

$$H^1(Y, \Theta_Y(n+1)) \longrightarrow \operatorname{Pic}(Y_{n+1}) \longrightarrow \operatorname{Pic}(Y_n) \longrightarrow 0 \tag{8}$$

If condition (6) holds, we get $\chi^*\colon \operatorname{Pic}(X) \simeq \operatorname{Pic}(Y)$ and Theorem 4 follows. On the other hand, without any hypothesis on $G$, we get easily the following:

PROPOSITION 5. *If* $\operatorname{char}(k) = p > 0$, *then* $\ker(\chi^*) = \ker(i)$ *is a $p$-torsion group.*

With the same notations of Theorem 4, let $\mathfrak{b}$ be a homogeneous, proper, divisorial ideal of $R$, $\mathfrak{b} \not\subseteq uR$ such that $[\mathfrak{b}]_R \in \ker(j)$. If $\mathfrak{b} = P_1^{(n_1)} \cap \cdots \cap P_r^{(n_r)}$ is the primary decomposition of $\mathfrak{b}$, put $\mathfrak{b}^{(pm)} = P_1^{(n_1 p^m)} \cap \cdots \cap P_r^{(n_r p^m)}$ for $m > 0$. Then Proposition 5 means that $H^2_{\mathfrak{m}}(\mathfrak{b}^{(p^m)}) = 0$ for some $m \geqslant 0$. The authors were unable to prove directly this fact.

## § 3. Concluding remarks

1) As the following counterexample shows, the hypothesis (1) of Theorem 4 does not suffice to deduce the injectivity of $j\colon \operatorname{Cl}(R) \to \operatorname{Cl}(G)$ when $R \to R/uR = G$ is a general hypersurface section. Let $k$ be an algebraically closed field. Let $R = k[X, Y, Z, W]/(XY - ZW) = k[x, y, z, w]$ and let $G = R/(x - y)$. $G$ is the homogeneous coordinate ring of a smooth conic in $\boldsymbol{P}_k^2$, hence $G$ satisfies the hypothesis (1) of Theorem 4. But $\operatorname{Cl}(R) \simeq \boldsymbol{Z}$ and $\operatorname{Cl}(G) \simeq \boldsymbol{Z}/2\boldsymbol{Z}$.

2) If a form ring $G$ is given, we can consider two rings of special relevance for our problem: $G_{G_+}$ and $\hat{G} = \widehat{(G, G_+)}$. This relevance is partially explained by the properties collected in the following:

PROPOSITION 6. *Let $G = G(A, I)$ be a normal integral domain, where $I \subseteq \operatorname{Rad}(A)$ as always. Then we have:*

a) $\operatorname{Gr}(\hat{G}, \hat{G}_+) \simeq \operatorname{Gr}(G, G_+) \simeq G$, *hence $\hat{G}$ is a normal integral domain.*

b) *Let $\mathfrak{c}$ be a homogeneous ideal of $G$; then $\operatorname{In}_{\hat{G}_+}(\mathfrak{c}\hat{G}) \simeq \mathfrak{c}$ where the isomorphism is that of* a). *In particular, let $m\colon \operatorname{Cl}(G) \to \operatorname{Cl}(\hat{G})$ the homomorphism deduced from the flat extension $G \to \hat{G}$,* i.e. $m([\mathfrak{c}]_G) = [\mathfrak{c}\hat{G}]_{\hat{G}}$ *for every integral divisorial ideal $\mathfrak{c}$ of $G$. Then $i \cdot m = 1_{\operatorname{Cl}(G)}$.*

c) $\operatorname{Gr}(G_{G_+}, G_+G_{G_+}) \simeq G \otimes_{G/G_+} K$ *(graded isomorphism) where $K$ is the quotient field of $A/I$. In particular, if $I$ is maximal, then $G$ is a $h$-local ring and $\operatorname{Gr}(G_{G_+}, G_+G_{G_+}) \simeq G$.*

d) *Let $\mathfrak{c}$ be a homogeneous ideal of $G$; then $\operatorname{In}_{G_+G_{G_+}}(\mathfrak{c}_{G_+})$ is graded iso-*

*morphic to* $\mathfrak{c} \otimes_{G/G_+} K$, *where* $K$ *is the residue field of* $G_{G_+}$. *In particular, if* $I$ *is maximal, we have* $\mathrm{In}_{G_+G_{G+}}(\mathfrak{c}_{G_+}) \simeq \mathfrak{c}$; *if we consider* $\mathrm{Cl}(G) \xrightarrow{\sigma} \mathrm{Cl}(G_{G_+}) \xrightarrow{\bar{i}} \mathrm{Cl}(G)$ *where* $\sigma$ *is the canonical isomorphism* ([13], *Corollary* 10.3), *we have* $\bar{i} \cdot \sigma = 1_{\mathrm{Cl}(G)}$ *and consequently* $\bar{i} = \sigma^{-1}$.

*Proof.* Easy calculations.

Now let $G = \oplus_{n \geqslant 0} G_n$ be a graded two dimensional normal domain such that $G_0$ is a field, $G = G_0[G_1]$ and $G$ is a finitely generated algebra over $G_0$. Since Danilov's condition DCG (i.e. $\mathrm{Cl}(G) \simeq \mathrm{Cl}(G[[T]])$ is equivalent to $(H^1_{G_+}(G))_n = 0$ for all $n \geqslant 0$ (see [12], Satz 4.4), from DCG condition it trivially follows that $\mathrm{Cl}(G) \simeq \mathrm{Cl}(\hat{G})$, since this is equivalent to $(H^1_{G_+}(G))_n = 0$ for all $n > 0$. (see [12], Theorem 4.1). Moreover there exist factorial graded rings as $G$ such that $\mathrm{Cl}(G) \simeq \mathrm{Cl}(\hat{G})$ but not satisfying the DCG condition (see [9], page 128).

However, condition (1) is not necessary for $j$ to be injective as the following example shows.

Let $G = \boldsymbol{Q}[X, Y, Z]/(X^4 + Y^4 - Z^4)$ where $\boldsymbol{Q}$ is the field of rational numbers; $\mathrm{Cl}(G)$ is finite (see [11]); but $\mathrm{Cl}(G) \neq \mathrm{Cl}(\hat{G})$, since $\mathrm{Cl}(\hat{G}) \simeq \mathrm{Cl}(G) \oplus \boldsymbol{Q}$ (see [12]); then take $A = G_{G_+}$. Since $\bar{i}$ is an isomorphism (see Proposition 6.d)), $j$ is surjective by definition of $\bar{i}$. Morevoer $\mathrm{Cl}(R)$, $\mathrm{Cl}(A)$ and $\mathrm{Cl}(G)$ are finite sets with the same number of elements; hence $j$ is injective.

3) The authors do not know the existence of factorial graded ring $G$ satisfying the general above-mentioned hypotheses and such that $\mathrm{Cl}(G) \not\simeq \mathrm{Cl}(\hat{G})$.

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