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APPROXIMATIONS OF POSITIVE OPERATORS AND CONTINUITY OF THE SPECTRAL RADIUS III

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Abstract

We prove estimates on the speed of convergence of the 'peripheral eigenvalues' (and principal eigenvectors) of a sequence T_n of positive operators on a Banach lattice E to the peripheral eigenvalues of its limit operator T on E which is positive, irreducible and such that the spectral radius $r(T)$ of T is a Riesz point of the spectrum of T (that is, a pole of the resolvent of T with a residuum of finite rank) under some conditions on the kind of approximation of T_n to T. These results sharpen results of convergence obtained by the authors in previous papers.

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1. Introduction

In our papers [1, 2], we studied the convergence of the peripheral eigenvalues and eigenvectors of positive approximations of positive operators to the eigenelements of the (positive) limit problem. Let us recall our main results in [1, 2]. Let $0 \leq T_n, T$ be bounded linear operators on a WSC (weakly sequentially complete) Banach lattice E such that T_n order converges to T (or converges uniformly on the order intervals of E) and $\|(T_n - T)^+\| \longrightarrow 0$ as $n \longrightarrow \infty$. Let us suppose that T is an irreducible operator on E such that $r(T)$ is a Riesz point of $\sigma(T)$. Suppose that $T = T_1 + T_2$ with $T_2 > 0$ being an abstract kernel operator. Then $r(T_n) \longrightarrow r(T)$, the 'peripheral spectrum' of T_n converges to the peripheral spectrum of T and a similar statement is true for the eigenvectors ([1, 2]). The purpose of this paper is to give estimates on the speed of convergence of the 'peripheral eigenelements' of T_n to the peripheral eigenelements of T in the context of our previous papers. The estimates look like the usual ones in the approximation theory of linear operators ([7, 3]).

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Let us give some final words on the organisation of the paper. In Section 2 we recall the main results of [1, 2]. In Sections 3 and 4 we prove estimates on the speed of convergence of the peripheral eigenvalues and principal eigenvectors respectively.

2. Convergence to the peripheral eigenelements

We follow the same conventions on notation and terminology as in [1, 2]. General references on Banach lattices and positive operators are in [11, 12]. Let us only recall some definitions from spectral theory. Let T be a bounded linear operator on the Banach space E, $T \in \mathcal{L}(E)$. The spectrum of T, that is the set of $z \in \mathbb{C}$ such that $zI - T$ is not invertible in $\mathcal{L}(E)$, will be denoted by $\sigma(T)$. The spectral radius of T, $r(T)$, is the number $\sup\{|z| : z \in \sigma(T)\}$ $(= \lim_n \|T^n\|^{1/n})$. If $z \in \rho(T) := \mathbb{C} - \sigma(T)$, the resolvent of T, $R(z, T) := (z - T)^{-1}$ is an analytic function on $\rho(T)$. A complex number $\lambda \in \sigma(T)$ is called a Riesz point of $\sigma(T)$ if λ is a pole of the resolvent $R(z, T)$ with a residuum $P = (1/2\pi i) \int_C R(z, T)\, dz$ of finite rank (C is a positively oriented curve on the complex plane around λ containing λ as the only singularity of $R(z, T)$). Finally, let us recall that if E is a Banach lattice and $0 \le T \in \mathcal{L}(E)$ is a positive operator on E, then $r(T) \in \sigma(T)$. The peripheral spectrum of T, denoted by $\pi\sigma(T)$, is the set $\{z \in \sigma(T) : |z| = r(T)\}$.

Let us recall our main results in [1, 2].

THEOREM 2.1. ([1, Theorem 3.1]). *Let E be a Banach lattice. Let $0 \le T$, $T_n \in \mathcal{L}(E)$ be such that $T_n x \longrightarrow Tx$ for all $x \in E$ and $\|(T_n - T)^+\| \longrightarrow 0$. Suppose that $r(T)$ is a Riesz point of $\sigma(T)$. Then, $r(T_n) \longrightarrow r(T)$.*

Our next statement is more general than the ones given in [1, 2]. We include a proof of it at the end of this section.

THEOREM 2.2. *Let E be a Banach lattice. Let $0 \le T$, $T_n \in \mathcal{L}(E)$ be such that $T_n x \longrightarrow Tx$ for all $x \in E$ and $\|(T_n - T)^+\| \longrightarrow 0$. Suppose that $r(T)$ is a Riesz point of $\sigma(T)$. Then $r(T_n)$ is a Riesz point of $\sigma(T)$ for n sufficiently large.*

To get convergence to the peripheral eigenvectors we need more assumptions on E and T. We recall Theorem 3.6 in [1] (see also [2, Theorem 4.8]). To avoid a cumbersome statement we suppose that E is a reflexive Banach lattice.

THEOREM 2.3. *Let E be a reflexive Banach lattice. Let $0 \le T \in \mathcal{L}(E)$ be an irreducible operator such that $T = T_1 + T_2$ with $0 \le T_1$, $0 < T_2$ and T_2 being an abstract kernel operator. Suppose that $r(T)$ is a Riesz point of $\sigma(T)$. Let $0 \le T_n \in \mathcal{L}(E)$ be such that $T_n \longrightarrow T$ in order and $\|(T_n - T)^+\| \longrightarrow 0$. Let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$ containing the Fréchet filter and let $z_n \in E$, $\|z_n\| = 1$, $\alpha_n \in \mathbb{C}$,*

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$\alpha_n \longrightarrow \alpha$ with $|\alpha| = r(T)$ be such that $\lim_{\mathscr{U}} \|T_n z_n - \alpha_n z_n\| = 0$. *Then $\alpha \in \pi\sigma(T)$ and* $\lim_{\mathscr{U}} \|z_n - z\| = 0$ *where z is the unique (up to a sign) normalized solution of $Tz = \alpha z$.*

REMARKS. (a) The above statement holds also if E is a dual Banach lattice with order continuous norm and $0 \leq T \in \mathscr{L}(E)$ is a dual operator but we preferred it to simplify the already cumbersome statement.

(b) We considered here convergence with respect to an ultrafilter $\mathscr{U}$. This obliged us to suppose that E is reflexive or dual with order continuous norm. Our result in [2, Theorem 4.8] holds for weakly sequentially complete Banach lattices E but we consider only convergence with respect to the Fréchet filter. Unfortunately, we need the above stated version below, but recall that Theorem 2.3 holds for the Fréchet filter in the place of $\mathscr{U}$.

Before going into the proof of Theorem 2.2, let us recall two of the main results in [6] which are the key-stones in the proof of it and will also play an important role in the next section.

Let us fix an ultrafilter $\mathscr{U}$ on $\mathbb{N}$ containing the Fréchet filter and let E be a Banach space. The ultrapower of E with respect to $\mathscr{U}$, denoted by $\hat{E}_{\mathscr{U}}$, or simply by $\hat{E}$, is defined by $l^\infty(E)/C_{\mathscr{U}}(E)$ where $l^\infty(E) := \{(x_n)_{\mathscr{U}} : x_n \in E, \sup_n \|x_n\| < \infty\}$ and $C_{\mathscr{U}}(E) := \{(x_n) \in l^\infty(E) : \lim_{\mathscr{U}} \|x_n\| = 0\}$. The ultrapower $\hat{E}'$ of E' with respect to $\mathscr{U}$ is isometrically isomorphic to a closed subspace of the dual $(\hat{E})'$ of $\hat{E}$. If E is a Banach lattice, then $\hat{E}$ is again a Banach lattice. It is easy to construct a projection $m_{E''}$ from $\hat{E}$ onto E''. Let $\hat{x} \in \hat{E}$, $\varphi \in E'$. Then $\langle m_{E''}(\hat{x}), \varphi\rangle = \lim_{\mathscr{U}} \langle x_n, \varphi\rangle$ defines the desired projection $m_{E''}$. If E is a dual Banach lattice $E = F'$, we can define $m_E : \hat{E} \longrightarrow E$ by $\langle m_E(\hat{x}), \varphi\rangle := \lim_{\mathscr{U}} \langle x_n, \varphi\rangle$, $\hat{x} \in \hat{E}$, $\varphi \in F$. If E is a dual Banach lattice, then $m_{E''}$ (and m_E) are positive projections. Operators on E can be lifted to operators on $\hat{E}$ by $\hat{T}\hat{x} = (Tx_n)_{\mathscr{U}}$, $\hat{x} \in \hat{E}$, $T \in \mathscr{L}(E)$, in such a way that $\sigma(\hat{T}) = \sigma(T)$ ([11, Theorem V.1.4]). Notice that the approximate spectrum of T is converted into the point spectrum of $\hat{T}$. Now, we recall the following result from [6].

THEOREM 2.4. *Let E be a Banach space and let $T \in \mathscr{L}(E)$. Let $\partial_\infty\sigma(T)$ be the exterior boundary of $\sigma(T)$ (=the boundary of the unbounded connected component of $\rho(T)$). Then*

$$\partial_\infty\sigma(T) \cap \sigma_{ess}(T) = \partial_\infty\sigma(T) \cap \{z \in \mathbb{C} : \dim\ker(z - \hat{T}) \text{ is infinite}\}.$$

If E is a dual Banach space and T is a dual operator, both sets coincide with

$$\partial_\infty(T) \cap \{z \in \mathbb{C} : \text{ there exists } \hat{y} \in \hat{E},\ \hat{y} \neq 0,\ m_E(\hat{y}) = 0 \text{ and } \hat{T}\hat{y} = z\hat{y}\}.$$

This result means that the eigenspace associated to a Riesz point on the exterior boundary of $\sigma(T)$ is contained in E and cannot be enlarged by going to $\hat{E}$.

The next result will be used in the proof of Theorem 2.2 and also in the next section.

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THEOREM 2.5. *Let E be a Banach lattice. Let $S, T \in \mathscr{L}(E)$ be such that $0 \le S \le T$ and $r(S) = r(T)$. If $r(T)$ is a Riesz point of $\sigma(T)$, then $r(S)$ is a Riesz point of $\sigma(S)$.*

Theorem 2.2 is a consequence of the following result:

PROPOSITION 2.6. *Let E be a Banach lattice. Let $0 \le T$, $T_n \in \mathscr{L}(E)$ be such that $T_n x \longrightarrow Tx$ for all $x \in E$ and $\|(T_n - T)^+\| \longrightarrow 0$. Suppose that $r(T)$ is a Riesz point of $\sigma(T)$. Let $\lambda_n \in \partial_\infty \sigma(T_n)$ be such that $\lambda_n \longrightarrow \lambda \in \pi\sigma(T)$. Then, for n large enough, λ_n is a Riesz point of $\sigma(T_n)$.*

PROOF. Let $\mathscr{U}$ be a free ultrafilter on $\mathbb{N}$ containing the Fréchet filter. If our assertion is not true, there exists a subsequence of T_n, call it again (T_n), such that λ_n is not a Riesz point of $\sigma(T_n)$. Let $\hat{T}_n$ be the canonical extension of T_n to E and let $F_n = \ker(\lambda_n - \hat{T}_n)$. By Theorem 2.4, $\dim F = \infty$. Let $\hat{\hat{E}}$ be the ultrapower of $\hat{E}$ with respect to the ultrafilter $\mathscr{U}$. The space $H = l^\infty(F_n)/C_{\mathscr{U}}(F_n)$ with $C_{\mathscr{U}}(F_n) = \{(\hat{x}_n) \in l^\infty(F_n) : \lim_{\mathscr{U}} \|\hat{x}_n\| = 0\}$ can be identified with a subspace of $\hat{\hat{E}}$. We notice that $\dim H = \infty$ ([4, Theorem 3.1]). Let $\hat{\hat{S}} : \hat{\hat{E}} \longrightarrow \hat{\hat{E}}$ be given by $\hat{\hat{S}}\hat{\hat{x}} = (\hat{T}_n \hat{x}_n)_{\mathscr{U}}$, $\hat{\hat{x}} = (\hat{x}_n)_{\mathscr{U}} \in \hat{\hat{E}}$. From $\|(T_n - T)^+\| \longrightarrow 0$, it follows that $0 \le \hat{\hat{S}} \le \hat{\hat{T}}$. Hence $r(\hat{\hat{S}}) \le r(T)$. Let $\hat{\hat{x}} = (\hat{x}_n)_{\mathscr{U}}$ with $\hat{x}_n \in \ker(r(T_n) - \hat{T}_n)$, $\|\hat{x}_n\| = 1$. Hence $\hat{\hat{S}}\hat{\hat{x}} = (\hat{T}_n \hat{x}_n) = (r(T_n)\hat{x}_n) = r(T)\hat{\hat{x}}$ (by Theorem 2.1). Thus $r(\hat{\hat{S}}) = r(T)$. Since $r(T)$ is a Riesz point of $\sigma(T)$, $r(T)$ is a Riesz point of $\hat{\hat{T}}$ (Theorem 2.4). By Theorem 2.5, $r(\hat{\hat{S}})$ is a Riesz point of $\sigma(\hat{\hat{S}})$. Now, we observe that $\lambda \in \sigma(\hat{\hat{S}})$, $|\lambda| = r(\hat{\hat{S}}) = r(T)$ and $H \subseteq \ker(\lambda - \hat{\hat{S}})$. By [11, Theorem V.5.5], λ is a Riesz point of $\sigma(\hat{\hat{S}})$ (see also [6, Lemma 4.4]). Hence $\ker(\lambda - \hat{\hat{S}}) < \infty$ (Theorem 2.4) a contradiction to the fact that $H \subseteq \ker(\lambda - \hat{\hat{S}})$.

3. Estimates on the speed of convergence to the peripheral eigenvalues

First we give some estimates on the speed of convergence of the peripheral eigenvalues. We first review the standard approach—with the standard set of assumptions-to this question taken from [7, Theorem 6.7] or [3, Theorem 5.2].

Let X be a complex Banach space with norm $\|\cdot\|$; $F : X \longrightarrow X$ a bounded linear operator and $\{F_n\}_{n=1}^\infty$ a family of bounded linear operators on X such that for $g \in X$,

$$\|Fg - F_n g\| \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty. \tag{3.1}$$

We assume that λ is an isolated eigenvalue of F with index ν and finite algebraic multiplicity $m \ge \nu$. Then there exist a circle Γ in the complex plane centered at λ which separates λ from $\sigma(F) \sim \{\lambda\}$. We denote by $P(\lambda, F)$ the projection $(1/2\pi i)\int_\Gamma (z-F)^{-1}\,dz$ associated with the eigenspace

$$X(\lambda, F) = \ker(\lambda - F)^\nu,$$

and let $E(\lambda, F) = \text{Range}\,(P(\lambda, F))$ be the corresponding generalized eigenspace. It is easy to verify that

$$E = E(\lambda, F) = X(\lambda, F), \qquad \dim E(\lambda, F) = m,$$
$$(\lambda - F)^\nu P(\lambda, F) = 0 \qquad \text{and} \qquad (\lambda - F)^{\nu-1} P(\lambda, F) \neq 0$$

([7, Chapters 5 and 8, page 573]). Now, let us assume that there is a constant C and an integer n_0 such that for $n \geq n_0$

$$\|(z - F_n)^{-1}\| \leq C \qquad \text{for all } z \in \Gamma. \tag{3.2}$$

Considering (3.2) we may define the projection operator

$$P(\lambda, F_n) = \frac{1}{2\pi i}\int_\Gamma (z - F_n)^{-1}\,dz$$

associated with the eigenspace

$$E_n = E(\sigma_n, F_n) = \ker(\lambda_1 - F_n)^{\nu_1} \oplus \cdots \oplus \ker(\lambda_r - F_n)^{\nu_r},$$

where $\sigma_n = \sigma_n(F_n) \cap B(\lambda, \Gamma)$, $B(\lambda, F)$ is the disc centered at λ with $\partial B(\lambda, F) = \Gamma$ and $\lambda_j \in \sigma_n$ are the eigenvalues of F_n with algebraic multiplicities m_j and indices ν_j. Finally, we assume that for n large enough:

$$m = \dim E(\lambda, F) = \dim E(\sigma_n, F_n) = \textstyle\sum_{j=1}^r m_j. \tag{3.3}$$

Now, the following general result holds ([7, Theorem 6.7] and [3, Theorem 5.2])

THEOREM 3.1. *Let λ be a Riesz point of $F : X \longrightarrow X$ with finite algebraic multiplicity m and assume that* (3.1), (3.2), (3.3) *hold. Then, there exist exactly m eigenvalues, counted with multiplicity in $\sigma_n = \{\lambda_1, \ldots, \lambda_m\}$ and a constant C such that*

$$\max_{j=1,\ldots,m} |\lambda - \lambda_j|^{1/\nu} \leq C\|F - F_n\|_{E(\lambda,F)} \tag{3.4}$$

where $\|\cdot\|_{E(\lambda,F)}$ denotes the operator norm restricted to $E(\lambda, F)$.

We want to prove a similar result in the context of our Theorem 2.3 above. The proof is slightly more delicate here because we are not assuming (3.2) and (3.3). Some version of them will be proved. In fact we prove that (3.3) holds and we are able to get around the difficulty of not supposing (3.2). We follow as closely as possible the proof of this result as it is given in [3, Theorem 5.2].

In our case, we shall prove the following:

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THEOREM 3.2. *Let E be a reflexive Banach lattice. Let $0 \leq T \in \mathscr{L}(E)$ be an irreducible operator such that $T = T_1 + T_2$ with $0 \leq T_1$, $0 < T_2$ and T_2 being an abstract kernel operator. Suppose that $r(T)$ is a Riesz point of $\sigma(T)$. Let $0 \leq T_n \in \mathscr{L}(E)$ be such that $T_n \longrightarrow T$ in order and $\|(T_n - T)^+\| \longrightarrow 0$ as $n \longrightarrow \infty$. Then, for any $\lambda \in \pi\sigma(T)$, there exist a constant $k > 0$ and a sequence $\lambda_n \in \sigma(T_n)$ such that*

$$|\lambda_n - \lambda| \leq k\|T_n\varphi - T\varphi\| \tag{3.5}$$

where $\varphi \in E$ is the unique (up to a sign) normalized solution of $T\varphi = \lambda\varphi$. Moreover any sequence $\lambda_n \in \sigma(T_n)$ converging to λ satisfies an estimate like (3.5).

In what follows, let $\mathscr{U}$ be a fixed ultrafilter on $\mathbb{N}$ containing the Fréchet filter.

Let us identify E, T and T_n in Theorem 3.2 with X, F and F_n in Theorem 3.1. It is clear that (3.1) holds since E is a reflexive Banach lattice. (Hence, it has order continuous norm) and $T_n \longrightarrow T$ in order. We have to deal with the fact that we are not assuming (3.2) and (3.3) in the present situation. For that purpose we prove the following lemmas:

LEMMA 3.3. *Let E, T and T_n be as in Theorem 3.2. Let $\hat{S} : \hat{E} \longrightarrow \hat{E}$ be given by $\hat{S}\hat{x} = (T_n x_n)_{\mathscr{U}}$, $\hat{x} = (x_n)_{\mathscr{U}} \in \hat{E}$. Then*

(i) $r(\hat{S}) = r(T)$,

(ii) *$\pi\sigma(\hat{S}) = \pi\sigma(T)$ consists of Riesz points whose algebraic and geometric multiplicity is one.*

Moreover, $\ker(\lambda - \hat{S}) = \ker(\lambda - T)$ *for every* $\lambda \in \pi\sigma(\hat{S})$.

PROOF. Let $\hat{T}$ be the extension of T to $\hat{E}$. From $\|(T_n - T)^+\| \longrightarrow 0$ it follows that $0 \leq \hat{S} \leq \hat{T}$. Then $r(\hat{S}) \leq r(T)$. By Theorem 2.2, for n large enough, there exists $u_n \in E_+$, $\|u_n\| = 1$ such that $T_n u_n = r(T_n)u_n$. Let $\hat{u} = (u_n)_{\mathscr{U}} \in \hat{E}$. By Theorem 2.1, $\hat{S}\hat{u} = r(T)\hat{u}$. Hence $r(\hat{S}) = r(T)$. Since $r(T)$ is a Riesz point of $\sigma(T)$, $r(\hat{T})$ is a Riesz point of $\sigma(\hat{T})$ (for instance, using Theorem 2.4). From $0 \leq \hat{S} \leq \hat{T}$ and Theorem 2.5, it follows that $r(\hat{S})$ is a Riesz point of $\sigma(\hat{S})$. Using [11, Theorem V.5.5] (see also [6, Lemma 4.4]) $\pi\sigma(\hat{S})$ consist of Riesz points. Again, using $0 \leq \hat{S} \leq \hat{T}$, it follows that $0 \leq R(z, \hat{S}) \leq R(z, \hat{T})$ for all $z > r(T)$. Since T is irreducible, $r(T)$ is a simple pole of the resolvent and $\{(z - r(T))R(z, \hat{T}) : z > r(T)\}$ is bounded ([11, Theorem V.5.2]). It follows that $\{(z - r(\hat{S}))R(z, \hat{S}) : z > r(\hat{S})\}$ is also bounded. Hence, $r(\hat{S})$ is a simple pole of $R(z, \hat{S})$, thus it has index one and its algebraic and geometric multiciplities coincide. By [11, Corollary of Theorem V.5.1], the same is true for all points in $\pi\sigma(\hat{S})$. Now, let $\lambda \in \pi\sigma(\hat{S})$ and let $\hat{x} \in \ker(\lambda - \hat{S})$, $\hat{x} = (x_n)_{\mathscr{U}}$, $\|\hat{x}\| = 1$. Then $\lim_{\mathscr{U}} \|T_n x_n - \lambda x_n\| = 0$. By Theorem 2.3, $\lambda \in \pi\sigma(T)$ and $\lim_{\mathscr{U}} \|x_n - x\| = 0$ where x is the unique (up to a sign) normalized solution of $Tz = \lambda z$. It follows that $\pi\sigma(\hat{S}) \subseteq \pi\sigma(T)$ and $\ker(\lambda - \hat{S}) \subseteq \ker(\lambda - T)$. On the other hand, it is easy to

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check that $\pi\sigma(T) \subseteq \pi\sigma(\hat{S})$. Moreover, since for all $\lambda \in \pi\sigma(T)$, $\ker(\lambda - T)$ is one dimensional, it follows that $\ker(\lambda - \hat{S}) = \ker(\lambda - T)$ for any $\lambda \in \pi\sigma(\hat{S})$. Our lemma is proved.

LEMMA 3.4. *Let E, T and T_n be as in Theorem 3.2. Then for every $\lambda \in \pi\sigma(T)$, there exists a $\rho > 0$ such that for n large enough*

$$\sigma(T_n) \cap B(\lambda, \rho) = \{\lambda_n\}, \qquad \lambda_n \longrightarrow \lambda \tag{3.6}$$

and if $C = \partial B(\lambda, \rho)$, then

$$\operatorname{dist}(z, \sigma(T_n)) > \rho/2, \qquad \operatorname{dist}(z, \sigma(T)) > \rho/2 \tag{3.7}$$

for all $z \in C$.

PROOF. Recall that, under the assumptions of Theorem 3.2, $r(T) > 0$ ([5, Theorem V.5.5]). Let $\lambda \in \pi\sigma(T)$. Then λ is a Riesz point of $\sigma(T)$ and $\ker(\lambda - T) = 1$ ([11, Theorem V.5.5]). Let us prove that there exist some $\lambda_n \in \sigma(T_n)$ such that $\lambda_n \longrightarrow \lambda$ as $n \longrightarrow \infty$. If λ is not an accumulation point of a sequence $\lambda_n \in \sigma(T_n)$, then there exists an open disc D around λ in $\mathbb{C}$ such that $z - T_n$ is invertible for all $z \in D$ and all $n \geq n_0$ (for some $n_0 \in \mathbb{N}$). Since $\|(T_n - T)^+\| \longrightarrow 0$ and the invertible operators are an open subset of $\mathscr{L}(E)$, $z - T_n \wedge T$ is invertible for all $z \in D$ and all $n \geq n_1$ for some $n_1 \in \mathbb{N}$ sufficiently large. But $0 \leq T_n \wedge T \leq T$. Using Moustaka' result [1, Theorem 3.5] and [10, Satz 3.2], we know that $r(T_n \wedge T)\,(\pi\sigma(T)/r(T)) \subseteq \pi\sigma(T_n - T)$. Therefore, $(\lambda/r(T))\,r(T_n \wedge T) \in \pi\sigma(T_n \wedge T)$. But $\mu_n := (r(T_n \wedge T)/r(T)) \longrightarrow 1$ (Theorem 2.1). Hence $\lambda\mu_n \in \pi\sigma(T_n \wedge T) \cap D$ for n sufficiently large. For such n, $\lambda\mu_n - T_n \wedge T$ is not invertible, contradicting our assertion above. It follows that there exists a sequence $\lambda_n \in \sigma(T_n)$ such that $\lambda_n \longrightarrow \lambda$ as $n \longrightarrow \infty$. It is easy to observe that we can take $\lambda_n \in \partial_\infty\sigma(T)$. Now, we claim that

$$\liminf_n \operatorname{dist}(\lambda, \sigma(T_n) \sim \{\lambda_n\}) > 0.$$

Otherwise, $\liminf_n \operatorname{dist}(\lambda, \sigma(T_n) \sim \{\lambda_n\}) = 0$. Hence, there exists a sequence $\alpha_{nj} \in \sigma(T_{nj}) \sim \{\lambda_{nj}\}$ such that $\alpha_{nj} \longrightarrow \lambda$ as $j \longrightarrow \infty$. No confusion arises if we write again α_n instead of α_{nj}. Observe that for n large enough, λ_n is a Riesz point of $\sigma(T_n)$ (Proposition 2.6). Hence we may take $\alpha_n \in \partial_\infty\sigma(T_n)$. Let us consider the sequence T_n and let $\hat{S} : \hat{E} \longrightarrow \hat{E}$ be given by $\hat{S}\hat{x} = (T_n x_n)_{\mathscr{U}}$, $\hat{x} = (x_n)_{\mathscr{U}}$. Let $F_n := \ker(\lambda_n - T_n)$, $G_n := \ker(\alpha_n - T_n)$ and let $H_n = F_n + G_n$. Let $H := l^\infty(H_n)/C_{\mathscr{U}}(H_n)$. Since $\dim H_n \geq 2$ (for n large enough), then $\dim H \geq 2$ ([4, Proposition 3.1]). Let us prove that $H \subseteq \ker(\lambda - \hat{S})$. Let $\hat{x} = (x_n)_{\mathscr{U}} \in H$. Let $y_n \in F_n$, $z_n \in G_n$ be such that $x_n = y_n + z_n$. Difficulties arise from the fact that we cannot guarantee that y_n, z_n are bounded. Compute:

$$\hat{S}\hat{x} = (T_n x_n)_{\mathscr{U}} = (T_n y_n + T_n z_n)_{\mathscr{U}} = (\lambda_n y_n + \alpha_n z_n)_{\mathscr{U}} = (\lambda_n x_n + (\alpha_n - \lambda_n) z_n)_{\mathscr{U}}.$$

Since $\lambda_n x_n$ is bounded, $(\alpha_n - \lambda_n)z_n$ is also a bounded sequence. Let $\hat{w} = ((\alpha_n - \lambda_n)z_n)_{\mathscr{U}}$. Thus $\hat{S}\hat{x} = \lambda\hat{x} + \hat{w}$. Then

$$(\hat{S} - \lambda)^2\hat{x} = (\hat{S} - \lambda)\hat{w} = ((\alpha_n - \lambda_n)(T_n - \lambda)z_n)_{\mathscr{U}} = (\alpha_n - \lambda)(\alpha_n - \lambda_n)z_n)_{\mathscr{U}} = 0$$

since $\alpha_n \longrightarrow \lambda$ and $(\alpha_n - \lambda_n)z_n$ is bounded. Hence $\hat{x} \in \ker(\lambda - \hat{S})^2 = \ker(\lambda - \hat{S}) = \ker(\lambda - T)$ by Lemma 3.3. Hence $H \subseteq \ker(\lambda - \hat{S})$ and $1 = \dim\ker(\lambda - \hat{S}) \geq \dim H \geq 2$. This contradiction proves our claim above. The last assertion of the lemma follows easily for any $\rho > 0$ such that $\rho < \liminf_n \operatorname{dist}(\lambda, \sigma(T_n) \sim \{\lambda_n\})$,

LEMMA 3.5. *In the context of Lemma* 3.4, *there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$*

$$\ker(\lambda_n - T_n)^2 = 1. \tag{3.8}$$

Hence, for all $n \geq n_0$, λ_n has algebraic and geometric multiplicities one.

PROOF. If (3.8) were not true, we could find a sequence $n_i \longrightarrow \infty$ such that $\dim\ker(\lambda_{n_i} - T_{n_i})^2 \geq 2$. Let $\hat{S} : \hat{E} \longrightarrow \hat{E}$ be given by $\hat{S}\hat{x} = (T_{n_i}, x_i)_{\mathscr{U}}$, $\hat{x} = (x_i)_{\mathscr{U}} \in \hat{E}$. By Lemma 3.3, $\ker(\lambda - \hat{S}) = 1$. Let $F_i = \ker(\lambda_{n_i} - T_{n_i})^2$. Let $\hat{F} := l^\infty(F_i)/C_{\mathscr{U}}(F_i)$. Let $\hat{x} \in \hat{F}$. Then $(\lambda - \hat{S})^2\hat{x} = ((\lambda - T_{n_i})^2 x_i)_{\mathscr{U}} = 0$. Hence, $F \subseteq \ker(\lambda - \hat{S})^2 = \ker(\lambda - \hat{S})$ by Lemma 3.3. It follows that $1 = \dim\ker(\lambda - \hat{S}) \geq \dim F \geq 2$, a contradiction. Thus, (3.8) follows.

LEMMA 3.6. *Under the assumptions of Theorem* 3.2, *for any $\lambda \in \pi\sigma(T)$ there exists some $\rho > 0$ and some $n_0 \in \mathbb{N}$ such that*

$$B(\lambda, \rho) \cap \sigma(T_n) = \{\lambda_n\}, \qquad \forall n \geq n_0, \quad \lambda_n \longrightarrow \lambda \text{ as } n \longrightarrow \infty, \tag{3.9}$$

$$C = \partial B(\lambda, \rho) \subseteq \rho(T_n), \qquad \forall n \geq n_0, \quad C \subseteq \rho(T), \tag{3.10}$$

$$\lim_{\mathscr{U}} \|P_n - P\| = 0, \tag{3.11}$$

where $P_n = (1/2\pi i)\int_C R(z, T_n)\,dz$, $P = (1/2\pi i)\int_C R(z, T)\,dz$ are the spectral projections of T_n and T associated to the corresponding spectral sets $\{\lambda_n\}$ and $\{\lambda\}$.

PROOF. Let $\hat{S} : \hat{E} \longrightarrow \hat{E}$ be given by $\hat{S}\hat{x} = (T_n x_n)_{\mathscr{U}} \in \hat{E}$. By Lemma 3.3, λ is a Riesz point of $\sigma(\hat{S})$ of algebraic and geometric multiplicity one. Moreover, $\dim\ker(\lambda - \hat{S}) = 1$. This implies that $\dim\ker(\lambda - (\hat{S})') = 1$ where $(\hat{S})' : (\hat{E})' \longrightarrow (\hat{E})'$ is the adjoint of $\hat{S}$. If we combine these remarks with Lemma 3.4 we know that there exists some $\rho > 0$ such that (3.9), (3.10) hold for all n sufficiently large and such that $C \subseteq \rho(\hat{S})$. Let P_n, P be given by the above formulas. Let $\hat{P} = (1/2\pi i)\int_C R(z, \hat{S})\,dz$ be the spectral projection onto $\ker(\lambda - \hat{S})$. By Lemma 3.3, $\hat{P} = P$. Let $\varphi \in E$, $\psi \in E'$, $\|\varphi\| = \|\psi\| = 1$ be such that $T\varphi = \lambda\varphi$, $T'\psi = \lambda\psi$.

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Then $\hat{P} = P = (\psi \otimes \varphi)/\langle \psi, \varphi \rangle$. Now let $\varphi_n \in E$, $\psi_n \in E'$, $\|\varphi_n\| = \|\psi_n\| = 1$ be such that $T_n \varphi_n = \lambda_n \varphi_n$, $T_n' \psi_n = \lambda_n \psi_n$. Let $\hat{\varphi} = (\varphi_n)_{\mathscr{U}} \in E$, and $\hat{\psi} = (\psi_n)_{\mathscr{U}} \in \hat{E}' \subseteq (\hat{E})'$. Obviously, $\hat{\varphi} \in \ker(\lambda - \hat{S})$, $\hat{\psi} \in \ker(\lambda - (\hat{S})')$, $\|\hat{\varphi}\| = \|\hat{\psi}\| = 1$. Hence $\hat{\varphi} = \varphi$ and $\hat{\psi} = \psi$. This implies that $\lim_{\mathscr{U}} \|\varphi_n - \varphi\| = 0$ and $\lim_{\mathscr{U}} \|\psi_n - \psi\| = 0$. Hence $\lim_{\mathscr{U}} \langle \psi_n, \varphi_n \rangle = \langle \psi, \varphi \rangle \neq 0$. Using Lemma 3.5 we have that $P_n = (\psi_n \otimes \varphi_n)/\langle \psi_n, \varphi_n \rangle$. Now, it is easy to check that $\lim_{\mathscr{U}} \|P_n - P\| = 0$.

Now, it is easy to give the proof of Theorem 3.2.

PROOF OF THEOREM 3.2. Let φ, ψ, φ_n and ψ_n be the normalized vectors defined during the proof of Lemma 3.6. Using (3.11) it follows that there exist subsequences P_{n_i}, φ_{n_i} and ψ_{n_i} with $P_{n_i} = (\psi_{n_i} \otimes \varphi_{n_i})/\langle \psi_{n_i}, \varphi_{n_i} \rangle$ converging to P, φ and ψ respectively. Hence given any subsequence (n_j) of $\mathbb{N}$, there exists a further subsequence $(n_{j(k)})$ of (n_j) such that $P_{n_{j(k)}}$, $\varphi_{n_{j(k)}}$, $\psi_{n_{j(k)}}$ converge to P, φ and ψ respectively. It follows that $P_n \longrightarrow P$, $\varphi_n \longrightarrow \varphi$ and $\psi_n \longrightarrow \psi$ as $n \longrightarrow \infty$, respectively. Now, use $\langle v \rangle$ to denote the one dimensional vector space generated by any vector v. Define $B_n : \langle \varphi \rangle \longrightarrow \langle \varphi_n \rangle$ by $B_n \varphi = P_n \varphi = \langle \psi_n, \varphi \rangle / \langle \psi_n, \varphi_n \rangle \varphi_n$. Then for some constant $C > 0$ and n large enough (say $n > n_0$)

$$\|B_n^{-1}\| = \left| \frac{\langle \psi_n, \varphi_n \rangle}{\langle \psi_n, \varphi \rangle} \right| < C.$$

Let $A : \langle \varphi \rangle \longrightarrow \langle \varphi \rangle$, $A_n : \langle \varphi \rangle \longrightarrow \langle \varphi \rangle$ be given by $A\varphi = T\varphi = \lambda\varphi$, $A_n \varphi = B_n^{-1} T_n B_n \varphi$ $(n \geq n_0)$ Check that $A_n \varphi = \lambda_n \varphi$. Then:

$$\begin{aligned} |\lambda - \lambda_n| &= \|(A - A_n)\varphi\| = \|B_n^{-1} B_n T \varphi - B_n^{-1} P_n T_n \varphi\| = \|B_n^{-1} P_n T \varphi - B_n^{-1} P_n T_n \varphi\| \\ &\leq \|B_n^{-1}\| \, \|P_n\| \, \|T\varphi - T_n \varphi\| \leq k \|T\varphi - T_n \varphi\| \end{aligned}$$

holds for some constant k and all $n \geq n_0$. The last assertion of the theorem is obvious from the above lemmas.

REMARKS. Once the effort was made in the previous lemmas, the proof of Theorem 3.2 is standard ([3, 8]).

Let us finish this section with the following corollary.

COROLLARY 3.7. *Let E be a reflexive Banach lattice and let $\pi_n \geq 0$ be a sequence of operators on E converging strongly to the identity. Suppose that T, $T_n = \pi_n T \pi_n$ satisfy the assumptions of Theorem* 3.2. *Then, for any $\lambda \in \pi\sigma(T)$, there exists a constant $k \geq 0$ and a sequence $\lambda_n \in \sigma(T_n)$ such that*

$$|\lambda - \lambda_n| \leq k \, \|\pi_n \varphi - \varphi\|. \tag{3.12}$$

Moreover, any sequence $\lambda_n \in \sigma(T_n)$ converging to λ satisfies an estimates like (3.12).

4. Estimates on the speed of convergence of the principal eigenvectors

Our main result, Theorem 4.2 will be a consequence of the following result of Ivo Marek [9, Theorem 3].

THEOREM 4.1. *Let E be a complex Banach lattice. Let $0 \leq T \leq \mathscr{L}(E)$ be an irreducible operator such that $r(T)$ is a Riesz point of $\sigma(T)$. Let $0 \leq \pi_n \in \mathscr{L}(E)$ be a sequence of operators such that $\pi_n \longrightarrow I$ strongly. Let $T_n = \pi_n T \pi_n$. Suppose that*

(i) *$r(T_n)$ is also a Riesz point of algebraic and geometric multiplicity one. Let x_n be the unique normalized positive solution of $T_n x_n = r(T_n)x_n$.*
(ii) *$|r(T_n) - r(T)| \leq k\,\epsilon_n$ with k a constant independent of n.*
(iii) *There is a constant $0 < \rho < r(T)$ and $n_0 \in \mathbb{N}$ such that*
   (a) *$r(T_n) > \rho, \forall n \geq n_0$*
   (b) *$\sigma(T) = \sigma_1 \cup \sigma_2$ with $\sigma_1 \subseteq B(0, \rho)$, $\sigma_2 = \pi\sigma(T) = \{\lambda_1, \ldots, \lambda_k\}$*
   (c) *$\sigma(T_n) = \sigma_{n(1)} \cup \sigma_{n(2)}$ with $\sigma_{n(1)} \subseteq B(0, \rho)$, $\sigma_{n(2)} = \{\lambda_{n(1)}, \ldots, \lambda_{n(k)}\}$ and $\lambda_{n(i)} \longrightarrow \lambda_i$ as $n \longrightarrow \infty$, $i = 1, 2, \ldots, k$, $\forall n \geq n_0$.*

Let x_0 be the unique normalized positive solution of $Tx_0 = r(T)x_0$. Then $\|x_n - x_0\| \leq k \max\{\epsilon_n, \|\pi_n x_n - x_n\|\}$ where k is some constant independent of n.

THEOREM 4.2. *Let E be a reflexive Banach lattice. Let $0 \leq T \in \mathscr{L}(E)$ be an irreducible operator such that $T = T_1 + T_2$ with $0 \leq T_1$, T_2 and $T_2 > 0$ being an abstract kernel operator. Suppose that $r(T) > 0$ is a Riesz point of $\sigma(T)$. Let $x_0 \in E_+$, $\|x_0\| = 1$, be such that $Tx_0 = r(T)x_0$. Let $0 \leq \pi_n \in \mathscr{L}(E)$ such that $\pi_n \longrightarrow I$ strongly. Let $T_n = \pi_n T \pi_n$. Suppose that T_n is irreducible, $T_n \longrightarrow T$ in order and $\|(T_n - T)^+\| \longrightarrow 0$ as $n \longrightarrow \infty$. Let $x_n \in E_+$, $\|x_n\| = 1$, be such that $Tx_n = r(T)x_n$. Then*

$$\|x_n - x_0\| \leq k\|\pi_n x_0 - x_0\| \tag{4.1}$$

for some constant k independent of n.

PROOF. By Theorem 2.2, for some $n_0 \in \mathbb{N}$ and $n \geq n_0$, $r(T_n)$ is a Riesz point of $\sigma(T_n)$. Since the T_n are irreducible, $r(T_n)$ has algebraic and geometric multiplicity one ([11, Theorem V.5.2]): the x_n's are the corresponding eigenvectors. By Corollary 3.7, (i), (ii) in Theorem 4.1 hold with $\epsilon_n = \|\pi_n x_0 - x_0\|$. Let us prove (iii). Since $r(T)$ is a Riesz point of $\sigma(T)$, $\sigma(T) = \sigma_1 \cup \sigma_2$ with $\sigma_1 \subseteq B(0, \rho_1)$, $0 < \rho_1 < r(T)$, $\sigma_2 = \pi\sigma(T) = \{\lambda_1, \ldots, \lambda_p\}$. By Lemma 3.4, there exist $\delta > 0$, $n_1 \in \mathbb{N}$ such that $\sigma(T_n) \cap B(\lambda_i, \delta) = \{\lambda_{n(i)}\}$ with $\lambda_{n(i)} \longrightarrow \lambda$ as $n \longrightarrow \infty$, $i = 1, 2, \ldots, p$, $\forall n \geq n_0$. Let $\sigma_{n(2)} = \{\lambda_{n(1)}, \ldots, \lambda_{n(p)}\}$, $\sigma_{n(1)} = \sigma(T_n) \sim \sigma_{n(2)}$. We claim that there exists some $\rho > 0$ and $n_0 \in \mathbb{N}$ such that $\sigma_{n(1)} \subseteq B(0, \rho_2)$, $\forall n \geq n_0$. Otherwise, there exists $\rho_k > 0$, $\rho_k \longrightarrow r(T)$, $(n_k) \subseteq \mathbb{N}$, $n_k < n_{k+1}$, $\alpha_k \in \partial_\infty\sigma(T_{n_k})$, $\rho_k \leq |\alpha_k|$, $\alpha_k \in \sigma_{n_k(2)}$.

Since α_k is in the approximate point spectrum of T_n, there exists a sequence $z_k \in E_+$, $\|z_k\| = 1$ such that $\|T_{n_k} z_k - \alpha_k z_k\| \longrightarrow 0$. Without loss of generality we may suppose that $\alpha_k \longrightarrow \alpha$. Hence $|\alpha| = r(T)$. By Theorem 2.3 and the remarks following it, $\alpha \in \pi\sigma(T)$ and z_k converges to the unique normalized solution of $Tz = \alpha z$. Thus $\alpha = \lambda_j$ for some $j \in \{1, \ldots, p\}$. Since $\alpha_k \longrightarrow \lambda_j$ as $k \longrightarrow \infty$, $\alpha_k \in B(\lambda_j, \delta)$ for k large enough, say $k \geq k_0$. Thus $\alpha_k = \lambda_{n_k(j)} \in \sigma_{n_k(2)}$, a contradiction. Our claim is proved. Now taking $\rho > \max(\rho_1, \rho_2)$, $\rho < r(T)$, (iii) (b), (c) hold with this ρ. (iii) (a) follows easily since $r(T_n) \longrightarrow r(T)$ (Theorem 2.1). Thus, (4.1) is an immediate translation of the conclusion of Theorem 4.1.

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