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Abstract

Let (X,B) be a projective log canonical pair such that B is a $\mathbb{Q}$-divisor, and that there is a surjective morphism $f : X \to Z$ onto a normal variety Z satisfying $K_X + B \sim_{\mathbb{Q}} f^*M$ for some big $\mathbb{Q}$-divisor M, and the augmented base locus $\mathbf{B}_+(M)$ does not contain the image of any log canonical centre of (X,B). We will show that (X,B) has a good log minimal model. An interesting special case is when f is the identity morphism.

1. Introduction

1.1 Main results of this paper

We work over an algebraically closed field k of characteristic zero. For simplicity we will prove our results in the absolute projective case, but they can be formulated and proved similarly in the relative case.

For a $\mathbb{Q}$-divisor M on a normal projective variety Z, the stable base locus is denoted by $\mathbf{B}(M)$ and the augmented base locus by $\mathbf{B}_+(M)$. The latter is defined as $\mathbf{B}_+(M) = \bigcap_A \mathbf{B}(M-A)$ where the intersection runs over all ample $\mathbb{Q}$-divisors A.

Concerning the augmented base loci of divisors related to log canonical (lc) pairs we have the following statement which is one of the main results of this paper.

Theorem 1.1. *Let (X,B) be a projective lc pair such that B is a $\mathbb{Q}$-divisor, and that there is a surjective morphism $f : X \to Z$ onto a normal projective variety Z satisfying:*

- *$K_X + B \sim_{\mathbb{Q}} f^*M_Z$ for some big $\mathbb{Q}$-divisor M_Z;*
- *$\mathbf{B}_+(M_Z)$ does not contain the image of any lc centre of (X,B).*

Then (X,B) has a good log minimal model. In particular, the log canonical algebra $R(K_X + B)$ is finitely generated over k.

The proof of the theorem is given in § 5. The main difficulties of the proof are due to the presence of lc singularities. Perhaps this is a good place to emphasize that trying to understand lc pairs rather than Kawamata log terminal (klt) pairs (or just smooth varieties) is not simply for the sake of generality. It is often the case that failure to prove a statement about lc pairs of dimension d comes from failure to understand certain aspects of smooth varieties in dimension $d-1$. For example, if we cannot prove finite generation of lc rings of lc pairs of dimension d it is because we do not know how to prove abundance for varieties of dimension $d-1$. To be more precise, finite generation of lc rings of lc pairs of dimension d is equivalent to the existence of

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good minimal models of $\mathbb{Q}$-factorial pseudo-effective dlt pairs of dimension $\leqslant d-1$ (see Fujino and Gongyo [FG13] for more details).

Corollary 1.2. *Let (X,B) be a projective lc pair such that K_X+B is a big $\mathbb{Q}$-divisor and that $\mathbf{B}_+(K_X+B)$ does not contain any lc centre of (X,B). Then, (X,B) has a good log minimal model. In particular, the log canonical algebra $R(K_X+B)$ is finitely generated over k.*

The corollary follows immediately from Theorem 1.1. The klt case of the corollary follows easily from [BCHM10]. However, passing from klt to lc is often very subtle, as pointed above.

One may wonder whether the corollary still holds if we assume that (X,B) is log big instead of assuming that $\mathbf{B}_+(K_X+B)$ does not contain lc centres of (X,B). Here log big means that K_X+B is big and $(K_X+B)|_S$ is also big for any lc centre S. Example 5.2 below shows that one gets into serious trouble very quickly.

The corollary implies the following result which was conjectured in [Cac14] and first proved in [BH13, Theorem 1.6]: both papers put the extra assumption that K_X+B+P birationally has a CKM-Zariski decomposition. Recall that an lc polarized pair $(X,B+P)$ consists of an lc pair (X,B) together with a nef divisor P.

Corollary 1.3. *Let $(X,B+P)$ be a projective lc polarized pair such that B,P are $\mathbb{Q}$-divisors, K_X+B+P is big, and the augmented base locus $\mathbf{B}_+(K_X+B+P)$ does not contain any lc centre of (X,B). Then, the log canonical algebra $R(K_X+B+P)$ is finitely generated over k.*

Sketch of the proof of Theorem 1.1. Roughly speaking, we follow the ideas of the proof of [Bir12, Theorem 1.4]. The flexibility allowed in the statement of Theorem 1.1 enables us to do induction on dimension. By comparison it seems hard to apply such arguments to prove Corollary 1.2. Let (X,B) and f be as in the theorem. We can take a dlt blow-up, and hence assume that (X,B) is $\mathbb{Q}$-factorial dlt. If there is a component S of $\lfloor B\rfloor$ such that S is mapped onto Z, then we can use induction to show that the algebra of $(K_X+B)|_S$ is finitely generated and from this we can derive finite generation of $R(K_X+B)$. Once we have this finite generation we can find a good log minimal model of (X,B) (see Proposition 3.2). We can then assume that no component of $\lfloor B\rfloor$ is mapped onto Z.

By assumptions, we can write $M_Z\sim_{\mathbb{Q}} A_Z+L_Z$ where $A_Z\geqslant 0$ is ample and $L_Z\geqslant 0$, and $\operatorname{Supp}(A_Z+L_Z)$ does not contain the image of any lc centre of (X,B). Let $A=f^*A_Z$ and $L=f^*L_Z$. After replacing (X,B) and f, we can write $A\sim_{\mathbb{Q}} G+P$ where $G\geqslant 0$ is semi-ample containing no lc centre of (X,B), $P\geqslant 0$, and $\operatorname{Supp} P=\operatorname{Supp}\lfloor B\rfloor$. Moreover, $K_X+B+bG+aL$ is nef if $b-a\gg 0$ and $a\geqslant 0$ (see Lemma 4.1).

Let $\epsilon_1>\epsilon_2>\cdots$ be a sequence of sufficiently small rational numbers with $\lim\epsilon_j=0$. For each ϵ_j, we can run an LMMP on $K_X+B+\epsilon_jG+\epsilon_jL$ with scaling of αG for some large α such that the LMMP ends up with a good log minimal model X'_j (see Lemma 4.2). We can make sure that the X'_j are all isomorphic in codimension one.

We can run an LMMP on $K_{X'_1}+B'_1$ with scaling of $\epsilon_1G'_1+\epsilon_1L'_1$ such that $\lim\lambda_i=0$ where λ_i are the numbers appearing in the LMMP (see Lemma 4.3). Moreover, after some modifications and redefining the notation, we can assume that each X'_j appears in some step of the latter LMMP. By applying special termination and induction we show that the LMMP terminates near $\lfloor B'_1\rfloor$ (see Lemma 4.4). This implies that the LMMP terminates everywhere because of the choice of P,G,L. This essentially gives the minimal model we are looking for because of our choice of the ϵ_j and the construction. □

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1.2 Organization of the paper

In § 2 we bring together some basic definitions, conventions and notation. In § 3 we prove the existence of log minimal models for certain pairs. We will need these for the arguments presented in the later sections. In § 4 we prove several lemmas in order to prepare for the proof of Theorem 1.1. Finally, in § 5, we give the proof of Theorem 1.1 and Corollaries 1.2 and 1.3.

2. Preliminaries

Let k be an algebraically closed field of characteristic zero fixed throughout the paper. All the varieties will be over k unless stated otherwise.

2.1 Divisors

First we introduce some notation. Let $X \dashrightarrow X'$ be a birational map between normal projective varieties whose inverse does not contract divisors. If M is an $\mathbb{R}$-divisor on X we denote its birational transform on X' by M' or by $M_{X'}$. notation we use is When we have a birational map $X_1 \dashrightarrow X_2$ whose inverse does not contract divisors, if M_1 is a divisor on X_1 we denote its birational transform on X_2 by M_2. The notation will be clear from the context.

Now let X be a normal projective variety and M a $\mathbb{Q}$-divisor on X. The *stable base locus* of M is defined as $\mathbf{B}(M) = \bigcap_N \operatorname{Supp} N$ where N ranges over all effective $\mathbb{Q}$-divisors satisfying $N \sim_{\mathbb{Q}} M$. On the other hand, the *augmented base locus* of M is defined as

$$\mathbf{B}_+(M) = \bigcap_A \mathbf{B}(M - A)$$

where A ranges over all ample $\mathbb{Q}$-divisors. It is not difficult to see that $\mathbf{B}_+(M) = \mathbf{B}(M - \epsilon A)$ for some sufficiently small $\epsilon > 0$ if we fix the ample divisor A.

The *divisorial algebra* associated to M is defined as

$$R(X, M) = \bigoplus_{m \geqslant 0} H^0(X, \lfloor mM \rfloor).$$

For a given surjective morphism $f\colon X \to Z$ we say that M is *very exceptional* if M is vertical/Z, that is, $\operatorname{Supp} M$ is not mapped onto Z, and if for any prime divisor P on Z there is a prime divisor Q on X which is not a component of M but $f(Q) = P$.

2.2 Pairs and polarized pairs

A *pair* (X, B) consists of a normal quasi-projective variety X and an $\mathbb{R}$-divisor B on X with coefficients in $[0, 1]$ such that $K_X + B$ is $\mathbb{R}$-Cartier. In this paper we mostly deal with pairs with B being a $\mathbb{Q}$-divisor. For a prime divisor D on some birational model of X with a nonempty centre on X, $a(D, X, B)$ denotes the log discrepancy. For definitions and standard results on singularities of pairs we refer to [KM98].

A *polarized pair* $(X, B + P)$ consists of a projective pair (X, B) and a nef $\mathbb{R}$-divisor P. We say that $(X, B + P)$ is lc when (X, B) is lc.

A projective pair (X, B) is called *log big* when $K_X + B$ is big and for any lc centre S of (X, B) the pullback of $K_X + B$ to S^ν is big where $S^\nu \to S$ is the normalization. On the other hand, we say that a projective pair (X, B) is *log abundant* when $K_X + B$ is abundant and, for any lc centre S of (X, B), the pullback of $K_X + B$ to S^ν is abundant. Recall that a divisor M is said to be abundant when $\kappa(M) = \kappa_\sigma(M)$ where κ_σ is the numerical Kodaira dimension defined by Nakayama. Although one can make sense of this for $\mathbb{R}$-divisors, we only use the notion when M is a $\mathbb{Q}$-divisor.

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2.3 Log minimal, weak lc, and log smooth models

A projective pair (Y, B_Y) is a *log birational model* of a projective pair (X, B) if we are given a birational map $\phi\colon X \dashrightarrow Y$ and $B_Y = B^\sim + E$ where $B^\sim$ is the birational transform of B and E is the reduced exceptional divisor of ϕ^{-1}, that is, $E = \sum E_j$ where E_j are the exceptional/X prime divisors on Y. A log birational model (Y, B_Y) is a *weak lc model* of (X, B) if:

- $K_Y + B_Y$ is nef; and
- for any prime divisor D on X which is exceptional/Y, we have

$$a(D, X, B) \leqslant a(D, Y, B_Y).$$

A weak lc model (Y, B_Y) is a *log minimal model* of (X, B) if:

- (Y, B_Y) is $\mathbb{Q}$-factorial dlt;
- the above inequality on log discrepancies is strict.

Let (X, B) be an lc pair, and let $f\colon W \to X$ be a log resolution. Let $B_W \geqslant 0$ be a boundary on W so that

$$K_W + B_W = f^*(K_X + B) + E$$

where $E \geqslant 0$ is exceptional/X and the support of E contains each prime exceptional/X divisor D on W if $a(D, X, B) > 0$. We call $(W/Z, B_W)$ a *log smooth model* of $(X/Z, B)$. Note that the coefficients of the exceptional/X prime divisors in B_W are not necessarily 1.

2.4 LMMP with scaling

Let $(X_1, B_1 + C_1)$ be an lc pair such that (X_1, B_1) is $\mathbb{Q}$-factorial dlt, $K_{X_1} + B_1 + C_1$ is nef, and $C_1 \geqslant 0$. Now, by [Bir10, Lemma 3.1], either $K_{X_1} + B_1$ is nef or there is an extremal ray R_1 such that $(K_{X_1} + B_1) \cdot R_1 < 0$ and $(K_{X_1} + B_1 + \lambda_1 C_1) \cdot R_1 = 0$ where

$$\lambda_1 := \inf\{t \geqslant 0 \mid K_{X_1} + B_1 + tC_1 \text{ is nef}\}.$$

If $K_{X_1} + B_1$ is nef or if R_1 defines a Mori fibre structure, we stop. Otherwise R_1 gives a divisorial contraction or a log flip $X_1 \dashrightarrow X_2$. We can now consider $(X_2, B_2 + \lambda_1 C_2)$ where $B_2 + \lambda_1 C_2$ is the birational transform of $B_1 + \lambda_1 C_1$ and continue. By continuing this process, we obtain a sequence of numbers λ_i and a special kind of LMMP which is called the *LMMP on $K_{X_1} + B_1$ with scaling of C_1*. Note that, by definition, $\lambda_i \geqslant \lambda_{i+1}$ for every i, and we usually put $\lambda = \lim_{i\to\infty} \lambda_i$.

3. Minimal models and termination for certain pairs

In this section, we will prove some results on minimal models and termination that we will need for the proof of Theorem 1.1. The arguments in this section are similar to those of [Bir12, § 5], but since we cannot directly refer to the results of [Bir12, § 5], we will write detailed proofs.

Remark 3.1. In this and later sections, we will apply [Bir12, Theorem 1.4] in several places. That theorem assumes the ACC for lc thresholds which is a theorem of [HMX12]. The ACC is not needed in the klt case.

Proposition 3.2. *Let (X, B) be a projective lc pair with B a $\mathbb{Q}$-divisor, and $f\colon X \dashrightarrow Z$ a rational map onto a normal projective variety Z so that we have:*

- *f is a projective morphism with connected fibres over some nonempty open subset $U \subseteq Z$;*
- *$K_X + B \sim_{\mathbb{Q}} 0$ over U;*

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- $\kappa(K_X + B) \geqslant \dim Z$; *and*
- *the algebra* $R(K_X + B)$ *is finitely generated over* k.

Then (X, B) *has a good log minimal model.*

Proof. Since $R(K_X+B)$ is a finitely generated k-algebra, there exist a log resolution $g\colon W \to X$, a contraction $h\colon W \to T$, and a decomposition $g^*(K_X + B) = A + E$ where $E \geqslant 0$, A is the pullback of some ample $\mathbb{Q}$-divisor on T, and for every sufficiently divisible integer $m > 0$ we have $\mathrm{Fix}(mg^*(K_X + B)) = mE$.

Let C be a rational boundary so that (W, C) is a log smooth model of (X, B) (as defined in § 2.3). We can write

$$K_W + C = g^*(K_X + B) + F = A + E + F$$

where $F \geqslant 0$ is exceptional/X. In particular, for every sufficiently divisible integer $m > 0$, we have

$$\mathrm{Fix}(m(K_W + C)) = mE + mF.$$

Moreover, $K_W + C \sim_{\mathbb{Q}} E + F/T$.

We may assume that the rational map $W \dashrightarrow Z$ is a morphism. Since $K_X + B \sim_{\mathbb{Q}} 0$ over U, $g^*(K_X + B) \sim_{\mathbb{Q}} 0$ over U. So $A \sim_{\mathbb{Q}} 0$ and $E \sim_{\mathbb{Q}} 0$ on the general fibres of $W \to Z$, which implies that such general fibres are contracted to points by $W \to T$. Therefore, by comparing the dimensions of the fibres of $W \to Z$ and $W \to T$ and by the other assumptions we get

$$\dim Z \leqslant \kappa(g^*(K_X + B)) = \dim T \leqslant \dim Z$$

which implies that

$$\kappa(K_W + C) = \kappa(g^*(K_X + B)) = \dim T = \dim Z.$$

Thus, we have an induced birational map $\psi\colon Z \dashrightarrow T$. The birationality comes from the fact that $W \to Z$ and $W \to T$ both have connected fibres. Perhaps after shrinking U we can assume that $\psi|_U$ is an isomorphism.

Now run an LMMP/T on $K_W + C$ with scaling of some ample divisor. Since $K_X + B \sim_{\mathbb{Q}} 0$ over U, (W, C) has a log minimal model over $\psi(U)$ by [Bir12, Corollary 3.7] hence the LMMP terminates over $\psi(U)$ by [Bir12, Theorem 1.9]. So we arrive at a model W' on which

$$K_{W'} + C' \sim_{\mathbb{Q}} E' + F' \sim_{\mathbb{Q}} 0$$

over $\psi(U)$ (recall that we use E' to denote the birational transform of E; we use similar notation for other divisors). In particular, since $E' + F' \geqslant 0$, this means that $E' + F'$ is vertical/T. Moreover, since $W \dashrightarrow W'$ is a partial LMMP on $K_W + C$, for every sufficiently divisible integer $m > 0$ we have

$$\mathrm{Fix}(m(K_{W'} + C')) = mE' + mF'$$

which implies that $E' + F'$ is very exceptional/T by [Bir12, Lemma 3.2]. Therefore, by [Bir12, Theorem 3.4], we can run an LMMP/T on $K_{W'} + C'$ so that it contracts $E' + F'$ and so it terminates with a model W'' on which $K_{W''} + C'' \sim_{\mathbb{Q}} 0/T$. In fact, $K_{W''} + C'' \sim_{\mathbb{Q}} A''$ is semi-ample since A'' is the pullback of an ample $\mathbb{Q}$-divisor on T. The pair (W'', C'') is a good log minimal model of (W, C) hence also a good log minimal model of (X, B) by [Bir12, Remark 2.8]. □

Proposition 3.3 (cf. [GL13]). *Let* (X, B) *be a projective klt pair and* $f\colon X \to Z$ *a contraction such that* $K_X + B \sim_{\mathbb{R}} f^*M_Z$ *for some big* $\mathbb{R}$*-Cartier divisor* M_Z*. Then,* (X, B) *has a good log minimal model.*

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Proof. By applying the canonical bundle formula [Amb05, FG12], we can find a boundary B_Z on Z so that (Z, B_Z) is klt and

$$K_X + B \sim_{\mathbb{R}} f^*(K_Z + B_Z).$$

By assumptions, $K_Z + B_Z$ is big, hence by [BCHM10] (Z, B_Z) has a good log minimal model (T, B_T). Let M_T on T be the pushdown of M_Z which is semi-ample by construction. There is a common resolution $d\colon V \to Z$ and $e\colon V \to T$ such that $d^*M_Z - e^*M_T$ is effective and exceptional/T.

Now take a log resolution $g\colon W \to X$ so that $h\colon W \dashrightarrow V$ is a morphism, and let $G = h^*(d^*M_Z - e^*M_T)$. Let C be a boundary so that (W, C) is a klt log smooth model of (X, B) as defined in § 2.3. We can write

$$K_W + C = g^*(K_X + B) + F \sim_{\mathbb{R}} g^*f^*M_Z + F = h^*d^*M_Z + F = h^*e^*M_T + G + F$$

where $F \geqslant 0$ is exceptional/X.

Let $U \subset Z$ be the largest open subset such that $\psi|_U$ is an isomorphism where ψ is the birational map $Z \dashrightarrow T$. Note that the codimension of $T \backslash \psi(U)$ is at least 2, and $G = 0$ over U. Now run an LMMP/T on $K_W + C$ with scaling of some ample divisor. Since $K_X + B \sim_{\mathbb{R}} 0/Z$, the LMMP terminates over $\psi(U)$ and F is contracted over $\psi(U)$. So we arrive at a model W' on which $K_{W'} + C' \sim_{\mathbb{R}} G' + F'/T$ where $G' + F' = 0$ over $\psi(U)$. Actually, by construction, $G' + F'$ is mapped into $T \backslash \psi(U)$, in particular, $G' + F'$ is very exceptional/T. Therefore, by [Bir12, Theorem 3.4], we can run an LMMP/T on $K_{W'} + C'$ so that it contracts $G' + F'$ and so it terminates with a model W'' on which $K_{W''} + C''$ is $\mathbb{R}$-linearly equivalent to the pullback of M_T, hence it is semi-ample. Now (W'', C'') is a good log minimal model of both (W, C) and (X, B). □

Proposition 3.4. *Let $(X, \Delta + C)$ be a projective $\mathbb{Q}$-factorial klt pair where $\Delta, C \geqslant 0$ are $\mathbb{Q}$-Cartier. Let $f\colon X \dashrightarrow Z$ be a rational map onto a normal projective variety Z so that we have:*

- *f is a projective morphism with connected fibres over some nonempty open subset $U \subseteq Z$;*
- *$K_X + \Delta \sim_{\mathbb{Q}} 0$ over U, and $C = 0$ over U;*
- *$K_X + \Delta + C$ is nef; and*
- *$\kappa(K_X + \Delta) \geqslant \dim Z$.*

Then, for any real number $0 \leqslant t \leqslant 1$, we can run an LMMP on $K_X + \Delta + tC$ with scaling of $(1-t)C$ which terminates with a good log minimal model of $(X, \Delta + tC)$.

Proof. We will show that we can run an LMMP on $K_X + \Delta$ with scaling of C which terminates. This shows in particular that $(X, \Delta + tC)$ has a log minimal model for each $t \in [0, 1]$. That fact that the log minimal model is good will follow from the construction and Propositions 3.2 and 3.3 and the finite generation result of [BCHM10].

Put $X_1 := X$, $\Delta_1 := \Delta$, and $C_1 := C$. Let $\lambda_1 \geqslant 0$ be the smallest number such that $K_{X_1} + \Delta_1 + \lambda_1 C_1$ is nef. By [Bir10, Lemma 3.1], λ_1 is a rational number. We may assume that $\lambda_1 > 0$. Since $R(K_{X_1} + \Delta_1 + \lambda_1 C_1)$ is a finitely generated k-algebra, and since $K_{X_1} + \Delta_1 + \lambda_1 C_1$ is nef, by Proposition 3.2, $K_{X_1} + \Delta_1 + \lambda_1 C_1$ is semi-ample hence it defines a contraction $X_1 \to V_1$. Note that V_1 is birational to Z since $\kappa(K_{X_1} + \Delta_1 + \lambda_1 C_1) = \dim Z$, which can be seen as in the proof of Proposition 3.2. In particular, $K_{X_1} + \Delta_1 \sim_{\mathbb{Q}} 0$ and $C_1 = 0$ over some nonempty open subset of V_1.

Run the LMMP/V_1 on $K_{X_1} + \Delta_1$ with scaling of an ample/V_1 divisor. This terminates with a good log minimal model X_2 of (X_1, Δ_1) over V_1 by [Bir12, Theorem 1.4]. So $K_{X_2} + \Delta_2$ is

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semi-ample/V_1. Now since $K_{X_2} + \Delta_2 + \lambda_1 C_2$ is the pullback of some ample divisor on V_1,

$$K_{X_2} + \Delta_2 + \lambda_1 C_2 + \delta(K_{X_2} + \Delta_2)$$

is semi-ample for some sufficiently small $\delta > 0$. In other words, $K_{X_2} + \Delta_2 + \tau C_2$ is semi-ample for some $\tau < \lambda_1$. We can consider $X_1 \dashrightarrow X_2$ as a partial LMMP on $K_{X_1} + \Delta_1$ with scaling of $\lambda_1 C_1$.

We can continue the process. That is, let $\lambda_2 \geqslant 0$ be the smallest number such that $K_{X_2} + \Delta_2 + \lambda_2 C_2$ is nef, and so on (note that $\lambda_1 > \tau \geqslant \lambda_2$). This process is an LMMP on $K_X + \Delta$ with scaling of C. The numbers λ_i that appear in the LMMP satisfy $\lambda := \lim_{i\to\infty} \lambda_i \neq \lambda_j$ for any j. The LMMP terminates by [Bir12, Theorem 1.9] if we show that $(X, \Delta + \lambda C)$ has a log minimal model.

Remember that $K_{X_2} + \Delta_2$ is semi-ample/V_1 hence it defines a contraction $X_2 \to W_2/V_1$ so that $K_{X_2} + \Delta_2 \sim_{\mathbb{Q}} 0/W_2$. Moreover, by construction, $W_2 \to V_1$ is birational and

$$K_{X_2} + \Delta_2 + \lambda_1 C_2 \sim_{\mathbb{Q}} 0/W_2.$$

Therefore,

$$K_{X_2} + \Delta_2 + \lambda C_2 \sim_{\mathbb{R}} 0/W_2$$

and, by Proposition 3.3, $(X_2, \Delta_2 + \lambda C_2)$ has a good log minimal model which is also a good log minimal model of $(X, \Delta + \lambda C)$. □

4. Preparations for the proof of Theorem 1.1

In this section we give the necessary preparations for the proof of Theorem 1.1. We recommend reading the sketch of the proof of Theorem 1.1 given in the introduction before continuing.

Lemma 4.1. *Let (X, B) and f be as in Theorem 1.1 with the extra assumption that every lc centre of (X, B) is vertical/Z. Then we can replace (X, B) and f so that we have the following additional properties:*

- *(X, B) is $\mathbb{Q}$-factorial dlt, and f is a contraction;*
- *$M_Z \sim_{\mathbb{Q}} A_Z + L_Z$ where $A_Z \geqslant 0$ is ample and $L_Z \geqslant 0$;*
- *$\operatorname{Supp}(A_Z + L_Z)$ does not contain the image of any lc centre of (X, B);*
- *letting $A = f^* A_Z$, $L = f^* L_Z$, we can write $A \sim_{\mathbb{Q}} G + P$ where $G \geqslant 0$ is semi-ample containing no lc centre of (X, B);*
- *$P \geqslant 0$, $\operatorname{Supp} P = \operatorname{Supp}\lfloor B \rfloor$; and*
- *$K_X + B + bG + aL$ is nef if $b \gg a \geqslant 0$.*

Proof. We can take a dlt blow-up and assume that (X, B) is $\mathbb{Q}$-factorial dlt. Since $K_X + B \sim_{\mathbb{Q}} f^* M_Z$ and $\mathbf{B}_+(M_Z)$ does not contain the image of any lc centre of (X, B), we can write $M_Z \sim_{\mathbb{Q}} R_Z + S_Z$ where $R_Z \geqslant 0$ is ample, $S_Z \geqslant 0$, and $\operatorname{Supp}(R_Z + S_Z)$ does not contain the image of any lc centre of (X, B). By taking the Stein factorization we may assume f is a contraction.

On the other hand, if $0 < \epsilon \ll 1$ is a rational number, then

$$K_X + \Delta := K_X + B - \epsilon \lfloor B \rfloor$$

is klt and $K_X + \Delta \sim_{\mathbb{Q}} 0$ over some nonempty open subset U of Z because $\lfloor B \rfloor$ is vertical/Z. Thus, by [Bir12, Theorem 1.4] we can run an LMMP/Z on $K_X + \Delta$ ending up with a good log minimal model X' over Z. Let $f' \colon X' \to Z'/Z$ be the contraction associated to $K_{X'} + \Delta'$ and write $K_{X'} + \Delta' \sim_{\mathbb{Q}} f'^* N_{Z'}$ for some $N_{Z'}$. Since $K_X + B \sim_{\mathbb{Q}} 0/Z$, $K_{X'} + B'$ is lc. Moreover,

$$K_{X'} + B' \sim_{\mathbb{Q}} \epsilon \lfloor B' \rfloor + f'^* N_{Z'}.$$

Now let (X'', B'') be a dlt blow-up of (X', B') and let $g\colon X'' \to X'$ be the corresponding morphism. Let $Q'' = g^* \epsilon \lfloor B' \rfloor$ and let $N'' = g^* f'^* N_{Z'}$. Then

$$K_{X''} + B'' \sim_{\mathbb{Q}} Q'' + N''$$

and, since $(X', B' - \epsilon \lfloor B' \rfloor)$ is klt, $\operatorname{Supp} Q'' = \operatorname{Supp} \lfloor B'' \rfloor$.

Since R_Z is ample and $N_{Z'}$ is ample$/Z$, if $\delta > 0$ is a sufficiently small rational number, then $R'' + \delta N''$ is semi-ample where R'' is the pullback of R_Z. Let S'' be the pullback of S_Z. Now from

$$Q'' + N'' \sim_{\mathbb{Q}} K_{X''} + B'' \sim_{\mathbb{Q}} R'' + S''$$

we obtain

$$\begin{aligned} K_{X''} + B'' &\sim_{\mathbb{Q}} \frac{1}{1+\delta}(R'' + S'' + \delta Q'' + \delta N'') \\ &= \frac{\delta}{1+\delta} Q'' + \frac{1}{1+\delta}(R'' + \delta N'') + \frac{1}{1+\delta} S''. \end{aligned}$$

By putting $P'' := (\delta/(1+\delta))Q''$, taking a general $G'' \sim_{\mathbb{Q}} (1/(1+\delta))(R'' + \delta N'')$, and letting $L'' := (1/(1+\delta))S''$, we get

$$K_{X''} + B'' \sim_{\mathbb{Q}} P'' + G'' + L''.$$

Let $L_Z = (1/(1+\delta))S_Z$ and let $A_Z \sim_{\mathbb{Q}} M_Z - L_Z$ be general. Then, A_Z is ample and $\operatorname{Supp}(A_Z + L_Z)$ does not contain the image of any lc centre of (X'', B''). Moreover, if we let A'' be the pullback of A_Z, then by construction, $A'' \sim_{\mathbb{Q}} P'' + G''$, G'' is semi-ample whose support does not contain any lc centre of (X'', B''), and $\operatorname{Supp} P'' = \operatorname{Supp} \lfloor B'' \rfloor$. Finally, since G'' is the pullback of an ample divisor on Z' and since $K_{X''} + B''$ and L'' are also pullbacks of certain divisors on Z', it is clear that $K_{X''} + B'' + bG'' + aL''$ is nef if $b \gg a \geqslant 0$. Now replace (X, B) with (X'', B'') and put $P = P''$, $G = G''$, $A = A''$, and $L = L''$. $\square$

Lemma 4.2. *Assume that (X, B) and f satisfy the assumptions and the properties listed in Lemma 4.1. Let $0 < \epsilon \ll 1$ and $\alpha \gg 0$ be rational numbers so that $K_X + B + \epsilon G + \epsilon L + \alpha G$ is nef. Then, we can run an LMMP on $K_X + B + \epsilon G + \epsilon L$ with scaling of αG which terminates with a good log minimal model of $(X, B + \epsilon G + \epsilon L)$.*

Proof. Since $0 < \epsilon \ll 1$, G is semi-ample, and $\operatorname{Supp} L$ does not contain any lc centre of (X, B), we can assume that $(X, B + \epsilon G + \epsilon L + \alpha G)$ is dlt. On the other hand, since

$$K_X + B \sim_{\mathbb{Q}} P + G + L$$

we can write

$$\begin{aligned} K_X + B + \epsilon G + \epsilon L &\sim_{\mathbb{Q}} K_X + B + \epsilon G + \epsilon L + \epsilon P - \epsilon P \\ &\sim_{\mathbb{Q}} (1+\epsilon)(K_X + B) - \epsilon P \\ &\sim_{\mathbb{Q}} (1+\epsilon)\left(K_X + B - \frac{\epsilon}{1+\epsilon} P\right). \end{aligned}$$

Thus, it is enough to run an LMMP on $K_X + \Delta := K_X + B - (\epsilon/(1+\epsilon))P$ with scaling of βG where $\beta = \alpha/(1+\epsilon)$. Since $\operatorname{Supp} P = \operatorname{Supp} \lfloor B \rfloor$, we can assume that $(X, \Delta + \beta G)$ is klt. Moreover, there is a nonempty open subset $U \subset Z$ such that $K_X + \Delta \sim_{\mathbb{Q}} 0$ over U and $G = 0$ over U. Now apply Proposition 3.4. $\square$

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Lemma 4.3. *Assume that (X, B) and f satisfy the assumptions and the properties listed in Lemma 4.1. Assume that X' is a log minimal model of $(X, B + \epsilon G + \epsilon L)$ for some rational number $0 < \epsilon \ll 1$ obtained as in Lemma 4.2. Then, we can run an LMMP on $K_{X'} + B'$ with scaling of $\epsilon G' + \epsilon L'$ such that $\lim \lambda_i = 0$ where λ_i are the numbers appearing in the LMMP with scaling.*

Proof. Let $\epsilon = a_1 > a_2 > a_3 > \cdots$ be a strictly decreasing sequence of rational numbers approaching zero. As in the proof of Lemma 4.2, we can write

$$K_X + B + a_2 G + a_2 L \sim_{\mathbb{Q}} r(K_X + \Delta)$$

and

$$K_X + B + \epsilon G + \epsilon L \sim_{\mathbb{Q}} r(K_X + \Delta + \tau G + \tau L)$$

where $r, \tau > 0$ are rational, $\Delta \geqslant 0$, and $(X, \Delta + \tau G + \tau L)$ and $(X', \Delta' + \tau G' + \tau L')$ are klt.

By Lemma 4.2, $X \dashrightarrow X'$ is obtained by an LMMP on $A + L + \epsilon G + \epsilon L$ with scaling of some αG. It is also an LMMP on $A + L + \epsilon L$. Let $U \subset Z$ be the complement of $\mathrm{Supp}(A_Z + L_Z) \cup f(G)$. Then, $X \dashrightarrow X'$ is an isomorphism when restricted to $f^{-1}U$. So $X' \dashrightarrow Z$ is a projective morphism with connected fibres over U. Moreover, $G = 0 = L$ and $K_X + \Delta \sim_{\mathbb{Q}} 0$ over U which implies that $G' = 0 = L'$ and $K_{X'} + \Delta' \sim_{\mathbb{Q}} 0$ over U. In addition,

$$\kappa(K_{X'} + \Delta') \geqslant \kappa(K_X + \Delta) = \dim Z$$

where the inequality follows from the fact that $X' \dashrightarrow X$ does not contract divisors and $K_{X'} + \Delta'$ is the pushdown of $K_X + \Delta$ (for each sufficiently divisible integer m we have an inclusion $H^0(X, m(K_X + \Delta)) \subseteq H^0(X', m(K_{X'} + \Delta'))$. Therefore, by Proposition 3.4, we can run an LMMP on $K_{X'} + \Delta'$ with scaling of $\tau G' + \tau L'$ which terminates. This corresponds to an LMMP on $K_{X'} + B' + a_2 G' + a_2 L'$ with scaling of $(a_1 - a_2)G' + (a_1 - a_2)L'$ which terminates on some model X''.

Next, using the same arguments as above, we can run an LMMP on $K_{X''} + B'' + a_3 G'' + a_3 L''$ with scaling of $(a_2 - a_3)G'' + (a_2 - a_3)L''$ which terminates. Continuing this process gives the desired LMMP on $K_{X'} + B'$ with scaling of $\epsilon G' + \epsilon L'$ such that $\lim \lambda_i = 0$ where λ_i are the numbers appearing in the LMMP. □

The next lemma will be used to do induction on dimension in the proof of Theorem 1.1.

Lemma 4.4. *Assume that Theorem 1.1 holds in dimension up to and including $d - 1$. Let (X, B) and f satisfy the assumptions and the properties listed in Lemma 4.1 where $d = \dim X$. Let $U \subset Z$ be a nonempty open set and $\phi \colon X \dashrightarrow X'$ a birational map satisfying:*

- *ϕ^{-1} does not contract divisors;*
- *$\phi|_{f^{-1}U}$ is an isomorphism;*
- *the generic point of each lc centre of (X, B) is in $f^{-1}U$;*
- *(X', B') is $\mathbb{Q}$-factorial dlt;*
- *the generic point of each lc centre of (X', B') is in $\phi(f^{-1}U)$.*

Let S be an lc centre of (X, B), S' its birational transform on X', and by adjunction define $K_{S'} + B'_{S'} := (K_{X'} + B')|_{S'}$. Then $(S', B'_{S'})$ has a good log minimal model.

Proof. Let $\psi \colon S \dashrightarrow S'$ be the induced birational map. Let $g \colon S \to V$ be the contraction given by the Stein factorization of $S \to f(S)$, and let $U_V \subset V$ be the inverse image of U. By assumptions, ψ is an isomorphism when restricted to $g^{-1}U_V$. Moreover, by construction, the generic point of each lc centre of $(S', B'_{S'})$ is inside $\psi(g^{-1}U_V)$.

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Let $W \to X$ and $W \to X'$ be a common log resolution so that it induces a common log resolution $h\colon T \to S$ and $e\colon T \to S'$ where $T \subset W$ is the birational transform of S. Such W exists because (X, B) is $\mathbb{Q}$-factorial dlt and $X \dashrightarrow X'$ is an isomorphism near the generic point of S. Let B'_T be a rational boundary so that (T, B'_T) is a log smooth model of $(S', B'_{S'})$, as defined in § 2.3, and so that each lc centre of (T, B'_T) maps onto an lc centre of $(S', B'_{S'})$. This ensures that the generic point of each lc centre of (T, B'_T) is mapped into U_V.

Now run an LMMP$/V$ on $K_T + B'_T$ with scaling of an ample divisor. By adjunction define $K_S + B_S = (K_X + B)|_S$. Since ψ is an isomorphism on $g^{-1}U_V$ and since $K_S + B_S \sim_{\mathbb{Q}} 0/V$, the pair $(S', B'_{S'})$ is a weak lc model of (T, B'_T) over U_V. Thus, the LMMP terminates over U_V by [Bir12, Corollary 3.7 and Theorem 1.9] and we reach a model $\overline{T}$ on which $K_{\overline{T}} + B'_{\overline{T}} \sim_{\mathbb{Q}} 0$ over U_V. Moreover, the generic point of each lc centre of $(\overline{T}, B'_{\overline{T}})$ is mapped into U_V. Therefore, by [Bir12, Theorem 1.4], we can run an LMMP$/V$ on $K_{\overline{T}} + B'_{\overline{T}}$ which terminates with a good log minimal model of (T, B'_T) over V. Replacing $\overline{T}$ with the minimal model, we may assume that $K_{\overline{T}} + B'_{\overline{T}}$ is semi-ample$/V$. Let $\overline{g}\colon \overline{T} \to \overline{V}/V$ be the contraction defined by $K_{\overline{T}} + B'_{\overline{T}}$.

Replacing M_Z by $A_Z + L_Z$ enables us to assume that $M_Z = A_Z + L_Z$. Then

$$K_S + B_S \sim_{\mathbb{Q}} g^* M_V = g^*(A_V + L_V)$$

where M_V, A_V, and L_V are the pullbacks on V of M_Z, A_Z, and L_Z respectively. On the other hand,

$$K_{S'} + B'_{S'} = (K_{X'} + B')|_{S'} \sim_{\mathbb{Q}} (A' + L')|_{S'}$$

and

$$K_T + B'_T = e^*(K_{S'} + B'_{S'}) + E_T$$

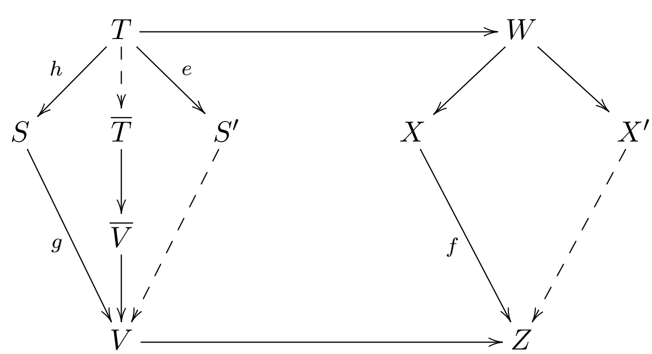
for some $E_T \geqslant 0$ which is exceptional$/S'$. Writing

$$M'_T = e^*(A' + L')|_{S'} + E_T,$$

we have

$$K_T + B'_T \sim_{\mathbb{Q}} M'_T \quad \text{and} \quad K_{\overline{T}} + B'_{\overline{T}} \sim_{\mathbb{Q}} M'_{\overline{T}}.$$

The following diagram shows some of the objects and maps we have constructed so far:



By construction, $E_{\overline{T}}$ is mapped into $V \backslash U_V$ since the above LMMP contracts any component of E_T whose generic point is mapped into U_V. Moreover, over U_V, $M'_{\overline{T}}$ is nothing but the pullback of $M_V = A_V + L_V$. Therefore, we can write

$$K_{\overline{T}} + B'_{\overline{T}} \sim_{\mathbb{Q}} M'_{\overline{T}} = \overline{g}^* M'_{\overline{V}}$$

such that $M'_{\overline{V}} \geqslant 0$ and $M'_{\overline{V}} = p^* M_V$ over U_V if we denote $\overline{V} \to V$ by p. Note that p is birational and it is an isomorphism over U_V.

Now let A_W on W be the pullback of A and, similarly, let A'_W be the pullback of A'. Since A_W is nef and ϕ^{-1} does not contract divisors, by the negativity lemma we have $A'_W \geqslant A_W$. By construction, $M'_T \geqslant A'_W|_T$ hence $M'_T \geqslant A_W|_T$ which in turn implies that $M'_{\overline{T}} \geqslant \overline{g}^*p^*A_V$ so $M'_{\overline{V}} \geqslant p^*A_V$. Let $N'_{\overline{V}} = M'_{\overline{V}} - p^*A_V$. Over U_V we have $N'_{\overline{V}} = p^*L_V$, and the generic point of each lc centre of $(\overline{T}, B'_{\overline{T}})$ is mapped into U_V hence $\operatorname{Supp} N'_{\overline{V}}$ does not contain the image of any lc centre of $(\overline{T}, B'_{\overline{T}})$.

Finally, by construction, $M'_{\overline{V}}$ is ample/V hence $N'_{\overline{V}}$ is also ample/V. So we can write $M'_{\overline{V}} \sim_{\mathbb{Q}} A'_{\overline{V}} + L'_{\overline{V}}$ where $A'_{\overline{V}} \sim_{\mathbb{Q}} p^*A_V + \delta N'_{\overline{V}}$ is ample and $L'_{\overline{V}} = (1-\delta)N'_{\overline{V}}$ for some small rational number $\delta > 0$. Choosing $A'_{\overline{V}}$ general makes sure that $\operatorname{Supp}(A'_{\overline{V}} + L'_{\overline{V}})$ hence $\mathbf{B}_+(M'_{\overline{V}})$ does not contain the image of any lc centre of $(\overline{T}, B'_{\overline{T}})$. Since we are assuming Theorem 1.1 in dimension less than d, the pair $(\overline{T}, B'_{\overline{T}})$ has a good log minimal model which is also a good log minimal model of $(S', B'_{S'})$. □

5. Proof of main results

In this section we will prove the main theorem and its corollaries.

Proof of Theorem 1.1. Assume that the theorem holds in dimension up to and including $d-1$. Let (X, B) and f be as in the theorem with $\dim X = d$. We can take a dlt blow-up, hence assume that (X, B) is $\mathbb{Q}$-factorial dlt. Moreover, by taking the Stein factorization of f, we can assume that f is a contraction.

First assume that there is an lc centre of (X, B) mapping onto Z. In this case, there is a component S of $\lfloor B \rfloor$ such that S is mapped onto Z. Let $g\colon S \to Z$ be the induced morphism. By adjunction define $K_S + B_S = (K_X + B)|_S$. Then (S, B_S) and g satisfy the assumptions of Theorem 1.1 since $K_S + B_S \sim_{\mathbb{Q}} g^*M_Z$ and no lc centre of (S, B_S) is mapped into $\mathbf{B}_+(M_Z)$. By induction, (S, B_S) has a good log minimal model. In particular, the algebra $R(K_S + B_S)$ is finitely generated over k. By [Bir12, Lemma 6.4], the algebra $R(M_Z)$ is also finitely generated which in turn implies that the algebra $R(K_X + B)$ is finitely generated as f is a contraction. Now by Proposition 3.2, (X, B) has a good log minimal model. So from now on we assume that every lc centre of (X, B) is vertical/Z. We will construct a log minimal model and use Lemma 5.1 below to deduce the existence of a good log minimal model.

We can replace (X, B) and f so that they satisfy the properties listed in Lemma 4.1. Let $\epsilon_1 > \epsilon_2 > \cdots$ be a sequence of sufficiently small rational numbers with $\lim \epsilon_j = 0$. By Lemma 4.2, for each ϵ_j, we can run an LMMP on $K_X + B + \epsilon_j G + \epsilon_j L$ with scaling of αG for some large α such that the LMMP ends up with a good log minimal model $(X'_j, B'_j + \epsilon_j G'_j + \epsilon_j L'_j)$ of $(X, B + \epsilon_j G + \epsilon_j L)$. Any prime divisor contracted by $X \dashrightarrow X'_j$ is a component of $A + L$ since the LMMP is also an LMMP on $A + L + \epsilon_j L$. Thus, perhaps after replacing the sequence $\epsilon_1, \epsilon_2, \ldots$ with a subsequence, we could assume that the maps $X \dashrightarrow X'_j$ contract the same prime divisors, hence assume that the X'_j are all isomorphic in codimension one. This also implies that $K_{X'_1} + B'_1$ is a limit of movable divisors.

Let $X' := X'_1$, $B' := B'_1$, $G' := G'_1$, and $L' := L'_1$, etc. By Lemma 4.3, we can run an LMMP on $K_{X'} + B'$ with scaling of $\epsilon_1 G' + \epsilon_1 L'$ such that $\lim \lambda_i = 0$ where λ_i are the numbers appearing in the LMMP. The LMMP does not contract any divisors because $K_{X'} + B'$ is a limit of movable divisors. For each ϵ_j, there is a model Y which appears in some step of the LMMP on $K_{X'} + B'$ and some i such that $\lambda_i \epsilon_1 \geqslant \epsilon_j \geqslant \lambda_{i+1} \epsilon_1$ and such that

$$K_Y + B_Y + \lambda_i \epsilon_1 G_Y + \lambda_i \epsilon_1 L_Y$$

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and

$$K_Y + B_Y + \lambda_{i+1}\epsilon_1 G_Y + \lambda_{i+1}\epsilon_1 L_Y$$

are both nef. Now since $X'_j \dashrightarrow Y$ is an isomorphism in codimension one, $(Y, B_Y + \epsilon_j G_Y + \epsilon_j L_Y)$ is also a log minimal model of $(X, B + \epsilon_j G + \epsilon_j L)$ so by replacing X'_j with Y we could assume that each of the X'_j appears as a model in the LMMP on $K_{X'} + B'$. By redefining the notation we assume that the LMMP on $K_{X'} + B'$ is as $X' = X'_1 \dashrightarrow X'_2 \dashrightarrow X'_3 \dashrightarrow \cdots$.

Assume that the LMMP on $K_{X'} + B'$ terminates. Then we arrive at a model X'' on which all the divisors $K_{X''} + B'' + \epsilon_j G'' + \epsilon_j L''$ are semi-ample when $j \gg 0$. For any prime divisor D on X we have

$$a(D, X, B + \epsilon_j G + \epsilon_j L) \leqslant a(D, X'', B'' + \epsilon_j G'' + \epsilon_j L'') \leqslant a(D, X'', B''),$$

hence taking limit gives

$$a(D, X, B) \leqslant a(D, X'', B'').$$

Therefore, (X'', B'') is a weak lc model of (X, B) from which we can construct a log minimal model by [Bir12, Corollary 3.7]. So it is enough to show that the LMMP terminates.

First we will show that the LMMP terminates near $\lfloor B' \rfloor$. Let $U = Z \setminus \operatorname{Supp}(A_Z + L_Z)$. By assumptions, the generic point of each lc centre of (X, B) belongs to $f^{-1}U$. Since $X \dashrightarrow X'$ is an LMMP on $A + L + \epsilon_1 L$, it is an isomorphism when restricted to $f^{-1}U$. On the other hand, since $X' = X'_1 \dashrightarrow X'_i$ is an LMMP on $A' + L'$, it is an isomorphism when restricted to the image of $f^{-1}U$ in X'. So the rational maps $X \dashrightarrow X'_i$ are all isomorphisms when restricted to $f^{-1}U$. Moreover, the generic point of each lc centre of (X'_i, B'_i) belongs to the image of $f^{-1}U$ in X'_i. This enables us to use Lemma 4.4. More precisely, let S be an lc centre of (X, B) and S'_i its birational transform on X'_i. By adjunction define $K_{S'_i} + B'_{S'_i} = (K_{X'_i} + B'_i)|_{S'_i}$. By Lemma 4.4, $(S'_i, B'_{S'_i})$ has a good log minimal model.

The LMMP on $K_{X'} + B'$ does not contract $S' = S'_1$. If S' is a minimal lc centre, then by [Fuj07, Sho03], the induced birational maps $S'_i \dashrightarrow S'_{i+1}$ are isomorphisms in codimension one for $i \gg 0$ and the LMMP induces an LMMP with scaling on some dlt blow-up of $(S'_j, B'_{S'_j})$ for some j (see [Bir12, Remark 2.10] for more information on this kind of reduction). Since $(S'_j, B'_{S'_j})$ has a log minimal model, the LMMP terminates by [Bir12, Theorem 1.9]. If S' is not a minimal lc centre, by induction, we can assume that the LMMP on $K_{X'} + B'$ terminates near any lc centre of $(S'_j, B'_{S'_j})$ for any large j. Thus, again by [Fuj07, Sho03], the induced birational maps $S'_i \dashrightarrow S'_{i+1}$ are isomorphisms in codimension one for $i \gg 0$ and we may assume that we get an induced LMMP with scaling on some dlt blow-up of $(S'_j, B'_{S'_j})$ for some j. The latter LMMP terminates for the same reasons as before. Therefore, we can assume that the LMMP on $K_{X'} + B'$ terminates near $\lfloor B' \rfloor$.

Outside $\lfloor B' \rfloor$ we have

$$G' + L' = P' + G' + L' \sim_{\mathbb{Q}} A' + L' \sim_{\mathbb{Q}} K_{X'} + B'.$$

Therefore, the LMMP on $K_{X'} + B'$ terminates everywhere, otherwise the extremal ray in each step of the LMMP intersects $K_{X'} + B'$ negatively but intersects $G' + L'$ positively, which is a contradiction. This completes the proof of the theorem. □

Lemma 5.1. *Assume that Theorem 1.1 holds in dimension up to an including $d - 1$. Let (X, B) and f satisfy the assumptions and properties listed in Lemma 4.1 where $d = \dim X$. Then any log minimal model of (X, B) is good.*

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Proof. Assume that (X, B) has a log minimal model. It is enough to show that (X, B) has one good log minimal model because then all the other log minimal models would be good. By [Bir12, Theorem 1.9], we can run an LMMP on $K_X + B$ with scaling of some ample divisor which ends up with a log minimal model X'. The LMMP is an LMMP on $A+L$. Let $U = Z \setminus \operatorname{Supp}(A_Z + L_Z)$. Then the rational map $\phi\colon X \dashrightarrow X'$ is an isomorphism when restricted to $f^{-1}U$. Moreover, the generic point of each lc centre of (X, B) belongs to $f^{-1}U$ which in turn implies that the generic point of each lc centre of (X', B') belongs to $\phi(f^{-1}U)$. In view of

$$\kappa(K_{X'} + B') = \kappa(K_X + B) = \kappa_\sigma(K_X + B) = \kappa_\sigma(K_{X'} + B')$$

and Lemma 4.4, the pair (X', B') is log abundant. Therefore, by [FG11, Theorem 4.2], $K_{X'} + B'$ is semi-ample, hence (X', B') is a good log minimal model. □

Proof of Corollary 1.2. This follows from Theorem 1.1 by taking $Z = X$ and f to be the identity morphism. □

The next example shows that in Corollary 1.2 we cannot simply weaken the assumption that $\mathbf{B}_+(K_X + B)$ does not contain any lc centre of (X, B) to (X, B) being log big.

Example 5.2. Take a smooth projective variety S with $\kappa(K_S) \geqslant 0$, let Z be the projective cone over S (with respect to some very ample divisor) and $X_2 \to Z$ the blow-up of the vertex. Identify S with the exceptional divisor of $X_2 \to Z$. Pick a smooth ample divisor H on S and let $\pi\colon X = X_1 \to X_2$ be the blow-up of X_2 along H. Now let $B = T + \epsilon E + \frac{1}{2}A$ where T is the birational transform of S, E is the exceptional divisor of $X \to X_2$, $\epsilon > 0$ is small, and A is the pullback of a sufficiently ample divisor on Z. Note that $T \to S$ and $E \cap T \to H$ are isomorphisms, hence $(K_X + B)|_T$ is big as it is identified with $K_S + \epsilon H$ via $T \to S$. Therefore, (X, B) is log big: indeed, $K_X + B$ is big, $(K_X + B)|_T$ is big, and T is the only lc centre of (X, B). Now run an LMMP on $K_X + B$ with scaling of some ample divisor. The LMMP would necessarily be over Z because of the presence of A. It is quite likely that the first step of the LMMP is the contraction $X \to X_2$. In that case termination of the LMMP is essentially equivalent to existence of a minimal model of S. Even if K_S is already nef, showing that $K_{X_2} + B_2$ is semi-ample amounts to showing semi-ampleness of K_S, which is the abundance problem.

Proof of Corollary 1.3. We can write

$$K_X + B + P \sim_{\mathbb{Q}} A + L$$

where $A \geqslant 0$ is ample, $L \geqslant 0$, and $\operatorname{Supp}(A + L)$ does not contain any lc centre of (X, B). For some small rational number $\epsilon > 0$, we can write

$$K_X + \Delta \sim_{\mathbb{Q}} K_X + B + P + \epsilon(A + L) \sim_{\mathbb{Q}} (1 + \epsilon)(K_X + B + P)$$

such that (X, Δ) is lc and any lc centre of (X, Δ) is also an lc centre of (X, B). The augmented base locus

$$\mathbf{B}_+(K_X + \Delta) = \mathbf{B}_+((1 + \epsilon)(K_X + B + P)) = \mathbf{B}_+(K_X + B + P)$$

does not contain any lc centre of (X, Δ). Now apply Corollary 1.2 to deduce that $R(K_X + \Delta)$ is finitely generated, which in turn implies that $R(K_X + B + P)$ is also finitely generated. □

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