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Multiplicity Results for Nonlinear Neumann Problems

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Abstract. In this paper we study nonlinear elliptic problems of Neumann type driven by the p-Laplacian differential operator. We look for situations guaranteeing the existence of multiple solutions. First we study problems which are strongly resonant at infinity at the first (zero) eigenvalue. We prove five multiplicity results, four for problems with nonsmooth potential and one for problems with a C^1-potential. In the last part, for nonsmooth problems in which the potential eventually exhibits a strict super-p-growth under a symmetry condition, we prove the existence of infinitely many pairs of nontrivial solutions. Our approach is variational based on the critical point theory for nonsmooth functionals. Also we present some results concerning the first two elements of the spectrum of the negative p-Laplacian with Neumann boundary condition.

1 Introduction

In this paper we study the following Neumann problem with nonsmooth potential:

$$\begin{cases} -\operatorname{div}\big(\|\nabla x(z)\|_{\mathbb{R}^N}^{p-2}\nabla x(z)\big) \in \partial j\big(z, x(z)\big) & \text{for a.a. } z \in Z \\ \frac{\partial x}{\partial n} = 0 \quad \text{for } z \in \partial Z, \end{cases} \tag{1.1}$$

with $p \in (1, +\infty)$. Here $Z \subseteq \mathbb{R}^N$ denotes a bounded domain with a C^2-boundary ∂Z, n is the outward unit normal on the boundary and the boundary condition is interpreted in the sense of trace. In the present work we are interested in multiplicity results for problem (1.1). In contrast to the Dirichlet problem, the Neumann problem has not been studied so extensively and only recently there have been some multiplicity results by Binding–Drabek–Huang [3] and Faraci [10]. We should also mention the earlier works on ordinary differential equations by Harris [11] and Hart–Lazer–McKenna [12]. In Binding–Drabek–Huang [3], the authors consider a particular right-hand side nonlinearity of the form

$$f(z,\zeta) = \lambda a(z)|\zeta|^{p-2}\zeta + \beta(z)|\zeta|^{p^*-2}\zeta$$

(p^* being the critical Sobolev exponent), while in Faraci [10], the nonlinearity has the form

$$f(z,\zeta) = a(z)g(\zeta) - \lambda(z)|\zeta|^{p-2}\zeta,$$

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with $a \in L^\infty(Z)_+$, $\lambda \in L^\infty(Z)_+$, $\lambda(z) \geq c > 0$ for almost all $z \in Z$ and $N < p$ (low dimensional problem). So both works exclude from their analysis the case of strong resonance at infinity at the first (zero) eigenvalue of the negative p-Laplacian with Neumann boundary condition. We recall that according to the terminology coined by Bartolo–Benci–Fortunato [2], a problem is strongly resonant at the first (zero) eigenvalue if for the right-hand side nonlinearity $f(\zeta)$ and the corresponding potential function

$$F(\zeta) := \int_0^\zeta f(r)\,dr,$$

we have $f(\zeta) \to 0$ as $|\zeta| \to +\infty$ and $\lim_{\zeta\to\pm\infty} F(\zeta) = \xi_\pm \in \mathbb{R}^N$. As will become evident by our analysis, strongly resonant problems exhibit a certain lack of compactness, which complicates things. We should also say that of the works on ordinary differential equations mentioned earlier, Harris [11] considers problems with asymmetric nonlinearities, while Hart–Lazer–McKenna [12] assume a C^1 nonlinearity $f(\zeta)$ with a certain convenient behavior at $\pm\infty$ of f'. We emphasize that in all the aforementioned problems, the nonlinearity $f(z,\zeta)$ is at least a Caratheodory function, which of course means that corresponding potential function

$$F(z,\zeta) := \int_0^\zeta f(z,r)\,dr$$

is C^1 in $\zeta \in \mathbb{R}$ (smooth problem). In contrast, in our problem the potential function $j(z,\zeta)$ is only locally Lipschitz in $\zeta \in \mathbb{R}$ (nonsmooth problem). So our tools are different and are based on the nonsmooth critical point theory as this was developed originally by Chang [5] and extended recently by Kourogenis–Papageorgiou [16]. Also in Section 3, we present some simple but nevertheless useful observations about the spectrum of the negative p-Laplacian with Neumann boundary condition.

In the next section, for the convenience of the reader, we present some basic definitions and facts from the subdifferential theory of locally Lipschitz functions, which is the main analytical tool in the nonsmooth critical point theory, and also recall some notions and results from the nonsmooth critical point theory, which we shall need in the sequel. For further information on locally Lipschitz functions and their subdifferential theory, we refer to Clarke [7] and Denkowski–Migórski–Papageorgiou [8].

2 Mathematical Background

Let X be a Banach space and X^* its topological dual. By $\|\cdot\|_X$ we denote the norm of X and by $\langle\cdot,\cdot\rangle_X$ the duality brackets for the pair (X, X^*). A function $\varphi\colon X \mapsto \mathbb{R}$ is said to be *locally Lipschitz*, if for every $x \in X$, we can find an open set $U \subseteq X$ with $x \in U$, and a constant $k_U > 0$ depending on U, such that $|\varphi(z) - \varphi(y)| \leq k_U\|z-y\|_X$ for all $z, y \in U$. From convex analysis we know that a proper, convex and lower semicontinuous function $\psi\colon X \mapsto \bar{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ is locally Lipschitz in the interior of its effective domain $\operatorname{dom}\psi := \{x \in X : \psi(x) < +\infty\}$ (see Denkowski–Migórski–Papageorgiou [8, Proposition 5.2.10, p. 532]). In particular, an $\mathbb{R}$-valued, convex and lower semicontinuous function is locally Lipschitz. Moreover, if X is

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finite dimensional, then every convex and $\mathbb{R}$-valued function defined on X is locally Lipschitz.

In analogy with the directional derivative of a convex function, we define the *generalized directional derivative* of a locally Lipschitz function $\varphi\colon X \to \mathbb{R}$ at $x \in X$ in the direction $h \in X$ by

$$\varphi^0(x;h) := \limsup_{\substack{x' \to x \\ t \searrow 0}} \frac{\varphi(x'+th) - \varphi(x')}{t}.$$

The function $X \ni h \mapsto \varphi^0(x;h) \in \mathbb{R}$ is sublinear, continuous and by the Hahn–Banach theorem it is the support function of a nonempty, convex and w^*-compact subset of X^*, defined by

$$\partial\varphi(x) := \{x^* \in X^* : \langle x^*, h\rangle_X \le \varphi^0(x;h) \text{ for all } h \in X\}.$$

The multifunction $X \ni x \mapsto \partial\varphi(x) \in 2^{X^*} \setminus \{\varnothing\}$ is known as the *Clarke* (or *generalized*) *subdifferential* of φ at x. This multifunction is upper semicontinuous form X with the norm topology into X^* with the w^*-topology, *i.e.,* for all w^*-closed sets $C \subseteq X^*$, we have that $\partial\varphi^-(C)$ is strongly closed in X, where

$$\partial\varphi^-(C) = \{x \in X : \partial\varphi(x) \cap C \neq \varnothing\}$$

(see Denkowski–Migórski–Papageorgiou [8, p. 36, p. 407]). In particular, then Gr $\partial\varphi$ is closed in $X \times X^*_{w^*}$, where

$$\operatorname{Gr} \partial\varphi = \{(x, x^*) \in X \times X^* : x^* \in \partial\varphi(x)\}.$$

If $\varphi, \psi\colon X \mapsto \mathbb{R}$ are two locally Lipschitz functions, then

$$\partial(\varphi + \psi)(x) \subseteq \partial\varphi(x) + \partial\psi(x) \quad \forall x \in X$$

and

$$\partial(t\varphi)(x) = t\partial\varphi(x) \quad \forall x \in X,\ t \in \mathbb{R}.$$

If $\varphi\colon X \mapsto \mathbb{R}$ is continuous, convex (thus locally Lipschitz as well), then for all $x \in X$, the generalized subdifferential introduced above coincides with the subdifferential of φ in the sense of convex analysis, given by

$$\partial\varphi(x) := \{x^* \in X^* : \langle x^*, y - x\rangle_X \le \varphi(y) - \varphi(x) \text{ for all } y \in X\}.$$

If φ is strictly differentiable at x (in particular if φ is continuously Gâteaux differentiable at x), then $\partial\varphi(x) = \{\varphi'(x)\}$.

A point $x \in X$ is a *critical point* of the locally Lipschitz function $\varphi\colon X \to \mathbb{R}$, if $0 \in \partial\varphi(x)$. If $x \in X$ is a critical point, the value $c = \varphi(x)$ is a *critical value* of φ. It is easy to check that, if $x \in X$ is a local extremum of φ (*i.e.,* a local minimum or a local maximum), then $0 \in \partial\varphi(x)$ (*i.e.,* $x \in X$ is a critical point).

In the classical (smooth) theory, a compactness-type condition known as the *Palais–Smale condition* plays the crucial role. In the present nonsmooth setting this condition takes the following form:

A locally Lipschitz function $\varphi\colon X \to \mathbb{R}$ satisfies the *nonsmooth Palais–Smale condition at level* $c \in \mathbb{R}$ (PS_c*-condition* for short), if every sequence $\{x_n\}_{n\geq 1} \subseteq X$ such that $\varphi(x_n) \to c$ as $n \to +\infty$ and

$$m_\varphi(x_n) \to 0 \quad \text{as } n \to +\infty,$$

where

$$m_\varphi(x_n) := \min\{\|x^*\|_{X^*} : x^* \in \partial\varphi(x_n)\}$$

has a strongly convergent subsequence. If this is true for every level $c \in \mathbb{R}$, then we simply say that φ satisfies the *nonsmooth Palais–Smale condition* (*PS-condition* for short).

We shall also use a more general version of this compactness-type condition:

A locally Lipschitz function $\varphi\colon X \to \mathbb{R}$ satisfies the *nonsmooth C-condition at level $c \in \mathbb{R}$ (nonsmooth C_c-condition* for short), if every sequence $\{x_n\}_{n\geq 1} \subseteq X$ such that $\varphi(x_n) \to c$ and

$$(1 + \|x_n\|)m_\varphi(x_n) \to 0 \quad \text{as } n \to +\infty,$$

has a strongly convergent subsequence. If this is true for every level $c \in \mathbb{R}$, then we simply say that φ satisfies the *nonsmooth C-condition*.

Next we quote the result of Szulkin [18, Lemma 3.1, p. 81], which will be needed in what follows.

Theorem 2.1 *If X is a Banach space, $\chi\colon X \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is a convex lower semicontinuous function with $\chi(0) = 0$ and*

$$-\|h\|_X \leq \chi(h) \quad \forall h \in X,$$

then there exists $v^* \in X^*$ such that $\|v^*\|_{X^*} \leq 1$ and

$$\langle v^*, h\rangle_X \leq \chi(h) \quad \forall h \in X.$$

Now, let us recall the following geometrical notion from critical point theory (see Struwe [17, p. 116] and Denkowski–Migórski–Papageorgiou [9, p. 178]).

Definition 2.2 Let Y be a Hausdorff topological space and E_1, D two nonempty closed sets. We say that E_1 and D *link in Y*, if

(a) $E_1 \cap D = \varnothing$,

(b) there exists a closed set $E \supseteq E_1$ such that for any $\eta \in C(E;Y)$, with $\eta|_{E_1} = \mathrm{id}_{E_1}$, we have that $\eta(E) \cap D \neq \varnothing$.

Finally, we recall the abstract minimax principle due to Kourogenis–Papageorgiou [16, Theorem 5, p. 253]. In fact the result of Kourogenis–Papageorgiou [16] is more general. However, the formulation that follows suffices for our purposes here.

Theorem 2.3 *If X is a reflexive Banach space, E_1 and D are nonempty subsets of X with D closed, E_1 and D link in X, $\sup_{E_1} \varphi < \inf_D \varphi$, $\varphi\colon X \to \mathbb{R}$ is locally Lipschitz, satisfies the nonsmooth C_c-condition with*

$$c := \inf_{\eta\in\Gamma} \sup_{v\in E} \varphi\bigl(\eta(v)\bigr),$$

where

$$\Gamma := \{\eta \in C(E;X) : \eta|_{E_1} = \mathrm{id}_{E_1}\}$$

and $E \supseteq E_1$ is as in the definition of linking sets, then $c \geq \inf_D \varphi$ and c is a critical value of φ, i.e., there exists a critical point $x_0 \in X$ of φ such that $\varphi(x_0) = c$. Moreover, if $c = \inf_D \varphi$, then $x_0 \in D$.

3 Spectral Properties of the p-Laplacian

In this section we develop some results concerning the beginning of the spectrum of the negative p-Laplacian with Neumann boundary condition. So we deal with the following nonlinear eigenvalue problem:

$$\text{(3.1)} \qquad \begin{cases} -\operatorname{div}\bigl(\|\nabla x(z)\|_{\mathbb{R}^N}^{p-2}\nabla x(z)\bigr) = \lambda|x(z)|^{p-2}x(z) & \text{for a.a. } z \in Z \\ \frac{\partial x}{\partial n} = 0 \quad \text{on } \partial Z, \end{cases}$$

with $p \in (1, +\infty)$. We say that $\lambda \in \mathbb{R}^N$ is an eigenvalue of the p-Laplacian with Neumann boundary condition (henceforth denoted by $\bigl(-\Delta_p, W^{1,p}(Z)\bigr)$), provided that problem (3.1) has a nontrivial solution, which is known as an eigenfunction corresponding to the eigenvalue λ. From nonlinear regularity theory (see *e.g.* Anane [1]), we know that every eigenfunction belongs to $C^{1,\beta}(\bar{Z})$ with $\beta \in (0,1)$. Remark that $\lambda = 0$ is an eigenvalue with the constant functions as eigenfunctions. More precisely we have the following Proposition.

Proposition 3.1 *$\lambda_0 = 0$ is the first eigenvalue of $\bigl(-\Delta_p, W^{1,p}(Z)\bigr)$ and is isolated and simple.*

Proof First we remark that problem (3.1) cannot have negative eigenvalues. Indeed, if $\lambda < 0$ is an eigenvalue with a corresponding eigenfunction x, if we multiply with $x(z)$ and integrate on Z, via the Green identity, we obtain

$$\|\nabla x\|_p^p = \lambda\|x\|_p^p,$$

which cannot be true for $\lambda < 0$.

The simplicity of $\lambda_0 = 0$ is a direct consequence of the fact that

$$0 = \inf_{\substack{x\in W^{1,p}(Z)\\ x\neq 0}} \frac{\|\nabla x\|_p^p}{\|x\|_p^p}.$$

Finally suppose that $\lambda_0 = 0$ is not isolated. So we can find a sequence of nonzero eigenvalues $\{\lambda_n\}_{n\geq 1}$ such that $\lambda_n \searrow 0$ as $n \to +\infty$. Consider a sequence of associated eigenfunctions $\{x_n\}_{n\geq 1} \subseteq C^1(\bar{Z})$ with $\|x_n\|_p = 1$ for $n \geq 1$. We have

$$\lambda_n = \frac{\|\nabla x_n\|_p^p}{\|x_n\|_p^p} = \|\nabla x_n\|_p^p \searrow 0 \quad \text{as } n \to +\infty$$

and so the sequence $\{x_n\}_{n\geq 1} \subseteq W^{1,p}(Z)$ is bounded. By passing to a subsequence if necessary, we may assume that

$$x_n \to x \quad \text{weakly in } W^{1,p}(Z),$$
$$x_n \to x \quad \text{in } L^p(Z),$$

with some $x \in W^{1,p}(Z)$. We have that $\|x\|_p = 1$ and $\|\nabla x\|_p = 0$, so

$$x = \frac{\pm 1}{|Z|_N^{\frac{1}{p}}},$$

with $|\cdot|_N$ denoting the Lebesgue measure on $\mathbb{R}^N$. Using as a test function $u \equiv 1$, we obtain

$$\int_Z |x_n(z)|^{p-2} x_n(z)\, dz = 0$$

and so by passing to the limit as $n \to +\infty$, we obtain

$$\int_Z |x(z)|^{p-2} x(z)\, dz = 0,$$

a contradiction. ■

Next we shall characterize the first nonzero element of the spectrum of $\left(-\Delta_p, W^{1,p}(Z)\right)$. Suppose that $\lambda > 0$ is a nonzero eigenvalue of (3.1) and u is an associate eigenfunction. Integrating (3.1) and using the Green identity (see Kenmochi [15] and Casas–Fernandez [4] or Hu–Papageorgiou [13, p. 884]), we obtain

$$\int_Z |u(z)|^{p-2} u(z)\, dz = 0.$$

So naturally, we are led to the consideration of the following set

$$C(p) := \left\{ x \in W^{1,p}(Z) : \|x\|_p = 1 \int_Z |x(z)|^{p-2} x(z)\, dz = 0 \right\}.$$

More precisely, let $\psi_p \colon W^{1,p}(Z) \to \mathbb{R}$ be the strictly convex C^1-map, defined by

$$\psi_p(x) := \|\nabla x\|_p^p \quad \forall x \in W^{1,p}(Z)$$

and consider the following minimization problem:

$$\inf_{x \in C(p)} \psi_p(x) = \lambda_1(p). \tag{3.2}$$

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Proposition 3.2 *Problem* (3.2) *has a solution* $\lambda_1 = \lambda_1(p) > 0$ *which is attained in* $C(p)$.

Proof Consider a minimizing sequence $\{x_n\}_{n\geq 1} \subseteq C(p)$, *i.e.*, $\psi_p(x_n) \searrow \lambda_1$. Evidently the sequence $\{x_n\}_{n\geq 1} \subseteq W^{1,p}(Z)$ is bounded and so, passing to a subsequence if necessary, we may assume that

$$\begin{aligned} x_n &\to x \quad \text{weakly in } W^{1,p}(Z),\\ x_n &\to x \quad \text{in } L^p(Z),\\ x_n(z) &\to x(z) \quad \text{for a.a. } z \in Z\\ |x_n(x)| &\leq k(z) \quad \text{for a.a. } z \in Z, \end{aligned}$$

with $k \in L^p(Z)$. Note that the sequence $\{|x_n(\cdot)|^{p-2}x_n(\cdot)\}_{n\geq 1} \subseteq L^{p'}(Z)$ (with $\frac{1}{p} + \frac{1}{p'} = 1$) is bounded and

$$|x_n(z)|^{p-2}x_n(z) \to |x(z)|^{p-2}x(z) \quad \text{for a.a. } z \in Z.$$

So it follows that

$$|x_n(\cdot)|^{p-2}x_n(\cdot) \to |x(\cdot)|^{p-2}x(\cdot) \quad \text{in } L^{p'}(Z),$$

hence

$$\int_Z |x(z)|^{p-2}x(z)\,dz = 0 \quad \text{and} \quad \|x\|_p = 1,$$

i.e., $x \in C(p)$. Also from the weak lower semicontinuity of the norm functional, we have that $\|\nabla x\|_p^p \leq \lambda_1$, hence

$$\|\nabla x\|_p^p = \lambda_1.$$

Since $x \in C(p)$, then x is a nonconstant element in $W^{1,p}(Z)$ and so $\lambda_1 > 0$. ∎

An immediate consequence of Proposition 3.2, is the following Poincaré–Wirtinger type inequality.

Corollary 3.3 *If* $x \in W^{1,p}(Z)$ *and*

$$\int_Z |x(z)|^{p-2}x(z)\,dz = 0,$$

then

$$\lambda_1\|x\|_p^p \leq \|\nabla x\|_p^p.$$

In fact for $p \geq 2$, we can show that $\lambda_1 > 0$ is the first nonzero eigenvalue of $\left(-\Delta_p, W^{1,p}(Z)\right)$.

Proposition 3.4 *If* $p \geq 2$, *then the number* λ_1 *is the first nonzero eigenvalue of* $\left(-\Delta_p, W^{1,p}(Z)\right)$.

Proof Let $x \in C(p)$ be a solution of problem (3.2). Then by virtue of the Lagrange multiplier rule we can find $a, b, c \in \mathbb{R}$, not all of them equal to zero, such that for all $v \in W^{1,p}(Z)$, we have

$$\begin{aligned} ap \int_Z \|\nabla x(z)\|_{\mathbb{R}^N}^{p-2} \big(\nabla x(z), \nabla v(z)\big)_{\mathbb{R}^N} + bp \int_Z |x(z)|^{p-2} x(z) v(z)\, dz \\ + c(p-2) \int_Z |x(z)|^{p-2} v(z)\, dz + c \int_Z |x(z)|^{p-2} v(z)\, dz = 0. \end{aligned} \tag{3.3}$$

Taking $v \equiv c$ and recalling that $\int_Z |x(z)|^{p-2} x(z)\, dz = 0$ (since $x \in C(p)$), we obtain

$$[c^2(p-2) + c^2] \int_Z |x(z)|^{p-2}\, dz = 0,$$

so $c = 0$. Thus (3.3) becomes

$$\begin{aligned} a \int_Z \|\nabla x(z)\|_{\mathbb{R}^N}^{p-2} \big(\nabla x(z), \nabla v(z)\big)_{\mathbb{R}^N} \\ + b \int_Z |x(z)|^{p-2} x(z) v(z)\, dz = 0 \quad \forall v \in W^{1,p}(Z). \end{aligned}$$

Suppose that $a = 0$. Then we have

$$b \int_Z |x(z)|^{p-2} x(z) v(z)\, dz = 0 \quad \forall v \in W^{1,p}(Z).$$

Taking $v = x$, we obtain

$$b \|x\|_p^p = 0,$$

hence $b = 0$, a contradiction to the fact that the Lagrange multipliers cannot be all equal to zero. So $a \neq 0$ and without any loss of generality, we may assume that $a = 1$. So we have

$$\begin{aligned} \int_Z \|\nabla x(z)\|_{\mathbb{R}^N}^{p-2} \big(\nabla x(z), \nabla v(z)\big)_{\mathbb{R}^N} \\ + b \int_Z |x(z)|^{p-2} x(z) v(z)\, dz = 0 \quad \forall v \in W^{1,p}(Z). \end{aligned}$$

Using as a test function $v = x$, we infer that $b = -\lambda_1$. Then via the Green identity, we conclude that x solves (3.1) with $\lambda = \lambda_1$. Clearly from the definition of $\lambda_1 > 0$, we see that we cannot have an eigenvalue $\lambda \in (0, \lambda_1)$. ■

4 Multiple Solutions

In this section we prove two multiplicity results for problem (1.1), under conditions of strong resonance at infinity at $\lambda_0 = 0$ eigenvalue and without assuming any symmetry.

Our hypothesis on the nonsmooth potential function $j(z, \zeta)$ are the following:

$H(j)_1$ $\quad j\colon Z \times \mathbb{R} \to \mathbb{R}$ is a function such that

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(i) for every $\zeta \in \mathbb{R}$, $j(\cdot\,,\zeta)$ is measurable;
(ii) for almost all $z \in Z$, $j(z,\,\cdot)$ is locally Lipschitz;
(iii) for almost all $z \in Z$, all $\zeta \in \mathbb{R}$ and all $u \in \partial j(z,\zeta)$, we have

$$|u| \leq a(z) + c|\zeta|^{r-1}$$

with $a \in L^{r'}(Z)_+$, $c > 0$, $r \in [1, p^*)$ and $\frac{1}{r} + \frac{1}{r'} = 1$;
(iv) there exist functions $j_\pm \in L^1(Z)$ such that

$$\lim_{\zeta \to \pm\infty} j(z,\zeta) = j_\pm(z) \quad \text{uniformly for a.a. } z \in Z, \int_Z j_\pm(z)\,dz \leq 0$$

and if $\{x_n\}_{n\geq 1} \subseteq W^{1,p}(Z)$, $\{v_n\}_{n\geq 1} \subseteq L^{r'}(Z)$ are sequences such that $|x_n(z)| \to +\infty$ a.e. on Z and $v_n(z) \in \partial j\big(z, x_n(z)\big)$ a.e. on Z, then $\int_Z v_n(z)x_n(z)\,dz \to 0$ as $n \to \infty$;
(v) there exists $\delta > 0$ such that

$$j(z,\zeta) \geq 0 \quad \text{for a.a. } z \in Z \text{ and all } |\zeta| \leq \delta$$

and

$$j(z,\zeta) \leq \frac{\lambda_1}{p}|\zeta|^p \quad \text{for a.a. } z \in Z \text{ and all } \zeta \in \mathbb{R}.$$

We consider the nonsmooth energy functional $\varphi\colon W^{1,p}(Z) \to \mathbb{R}$, defined by

$$\varphi(x) := \frac{1}{p}\|\nabla x\|_p^p - \int_Z j\big(z, x(z)\big)\,dz \quad \forall x \in W^{1,p}(Z).$$

We know that φ is locally Lipschitz (see Hu–Papageorgiou [13, p. 313]).

The next proposition exhibits a characteristic feature of strongly resonant problems, namely the lack of compactness, *i.e.*, the nonsmooth PS-condition is satisfied only at certain levels.

Proposition 4.1 *If hypotheses $H(j)_1$ hold, then φ satisfies the nonsmooth C_c-condition for all $c \neq -\int_Z j_\pm(z)\,dz$.*

Proof Let $c \neq -\int_Z j_\pm(z)\,dz$. Let us consider a sequence $\{x_n\}_{n\geq 1} \subseteq W^{1,p}(Z)$ such that

$$\varphi(x_n) \to c \quad \text{and} \quad (1 + \|x_n\|)m_\varphi(x_n) \to 0.$$

Let $x_n^* \in \partial\varphi(x_n)$ be such that $m_\varphi(x_n) = \|x_n^*\|_{(W^{1,p}(Z))^*}$ for $n \geq 1$. The existence of such elements follows from the fact that $\partial\varphi(x_n) \subseteq \big(W^{1,p}(Z)\big)^*$ is weakly compact and the norm functional is weakly lower semicontinuous in a Banach space. We have

$$x_n^* = A(x_n) - u_n^* \quad \forall n \geq 1$$

with $A\colon W^{1,p}(Z) \to \big(W^{1,p}(Z)\big)^*$ being the nonlinear operator defined by

$$\langle A(x), y\rangle_{W^{1,p}(Z)} := \int_Z \|\nabla x(z)\|_{\mathbb{R}^N}^{p-2}\big(\nabla x(z), \nabla y(z)\big)_{\mathbb{R}^N}\,dz \quad \forall x, y \in W^{1,p}(Z)$$

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and $u_n^* \in L^{r'}(Z)$ with $u_n^*(z) \in \partial j\big(z, x_n(z)\big)$ for almost all $z \in Z$ (see Clarke [7, p. 80] and Denkowski–Migórski–Papageorgiou [8, p. 617]). The operator A is monotone, demicontinuous, thus maximal monotone (see Denkowski–Migórski–Papageorgiou [9, p. 37]).

We claim that the sequence $\{x_n\}_{n\geq 1} \subseteq W^{1,p}(Z)$ is bounded. Suppose that this is not true. Then by passing to a suitable subsequence if necessary, we may assume that

$$\|x_n\|_{W^{1,p}(Z)} \longrightarrow +\infty.$$

Let us set

$$y_n := \frac{x_n}{\|x_n\|_{W^{1,p}(Z)}} \quad \forall n \geq 1.$$

Passing to a next subsequence if necessary, we can say that

$$\begin{aligned} y_n &\longrightarrow y \quad \text{weakly in } W^{1,p}(Z), \\ y_n &\longrightarrow y \quad \text{in } L^p(Z), \end{aligned}$$

for some $y \in W^{1,p}(Z)$. From the choice of the sequence $\{x_n\}_{n\geq 1} \subseteq W^{1,p}(Z)$, we have

$$\frac{\varphi(x_n)}{\|x_n\|^p_{W^{1,p}(Z)}} = \frac{1}{p}\|\nabla y_n\|_p^p - \int_Z \frac{j\big(z, x_n(z)\big)}{\|x\|^p_{W^{1,p}(Z)}}\, dz \leq \frac{M_1}{\|x_n\|^p_{W^{1,p}(Z)}} \quad \forall n \geq 1, \tag{4.1}$$

for some $M_1 > 0$. By virtue of hypothesis $H(j)_1$(iv), we can find $M_2 > 0$, such that

$$|j(z,\zeta)| \leq \vartheta(z) + 1 \quad \text{for a.a. } z \in Z \text{ and all } |\zeta| > M_2,$$

with $\vartheta := \max\{j_+, j_-\} \in L^1(Z)$. On the other hand by virtue of the mean value theorem for locally Lipschitz functions (see Clarke [7, p. 41] and Denkowski–Migórski–Papageorgiou [8, p. 609]) and because of hypothesis $H(j)_1$(iii), we have

$$|j(z,\zeta)| = a(z)|\zeta| + c|\zeta|^r \leq \beta(z) \quad \text{for a.a. } z \in Z \text{ and all } |\zeta| \leq M_2,$$

with $\beta \in L^1(Z)_+$(note that because of hypothesis $H(j)_1$(v), we have that $j(z,0) = 0$ for almost all $z \in Z$). So finally, we can say that

$$|j(z,\zeta)| \leq \beta_1(z) \quad \text{for a.a. } z \in Z \text{ and all } \zeta \in \mathbb{R},$$

with some $\beta_1 \in L^1(Z)_+$. Therefore, if in (4.1) we use this bound, we have that

$$\frac{1}{p}\|\nabla y_n\|_p^p \leq \frac{M_1}{\|x_n\|^p_{W^{1,p}(Z)}} + \frac{\|\beta_1\|_1}{\|x_n\|^p_{W^{1,p}(Z)}},$$

so $\|\nabla y\|_p = 0$, *i.e.*, $y = \xi \in \mathbb{R}$.

If $\xi = 0$, then $\|\nabla y_n\|_p \to 0$ and so

$$y_n \longrightarrow y = \xi = 0 \quad \text{in } W^{1,p}(Z),$$

a contradiction to the fact that $\|y_n\|_p = 1$ for all $n \geq 1$.

So $\xi \neq 0$ and first assume that $\xi > 0$. Then

$$x_n(z) \longrightarrow +\infty \quad \text{for a.a. } z \in Z.$$

From the choice of the sequence $\{x_n\}_{n\geq 1} \subseteq W^{1,p}(Z)$, we have

$$\Big| \langle A(x_n), x_n \rangle - \int_Z u_n^* x_n \, dz \Big| \leq \varepsilon_n, \quad \text{with } \varepsilon_n \searrow 0.$$

Because of hypothesis $H(j)_1$(iv), we have that $\int_Z u_n^* x_n \, dz \longrightarrow 0$. So $\langle A(x_n), x_n \rangle = \|Dx_n\|_p^p \longrightarrow 0$ as $n \to \infty$. Since $\varphi(x_n) \longrightarrow c$, given $\varepsilon > 0$, we can find $n_0 = n_0(\varepsilon) \geq 1$ such that

$$\begin{aligned} &|\varphi(x_n) - c| \leq \varepsilon \ \ \forall n \geq n_0 \\ &\qquad\qquad \Longrightarrow \quad c - \varepsilon \leq \varphi(x_n) = \frac{1}{p}\|Dx_n\|_p^p - \int_Z j\big(z, x_n(z)\big)\, dz \leq c + \varepsilon. \end{aligned}$$

Since $x_n(z) \longrightarrow +\infty$ a.e. on Z, by virtue of hypothesis $H(j)_1$(iv) and the Lebesgue dominated convergence theorem, we have that $\int_Z j\big(z, x_n(z)\big)\, dz \longrightarrow \int_Z j_+(z)\, dz$. So in the limit as $n \to \infty$, we obtain

$$c - \varepsilon \leq -\int_Z j_+(z)\, dz \leq c + \varepsilon.$$

Because $\varepsilon > 0$ was arbitrary, we let $\varepsilon \searrow 0$ to conclude that

$$c = -\int_Z j_+(z)\, dz,$$

a contradiction. Similarly, if we assume that $\xi < 0$, then we reach the contradiction that $c = -\int_Z j_-(z)\, dz$.

Therefore the sequence $\{x_n\}_{n\geq 1} \subseteq W^{1,p}(Z)$ is bounded and so, passing to a subsequence if necessary, we may assume that

$$\begin{aligned} x_n &\longrightarrow x \quad \text{weakly in } W^{1,p}(Z), \\ x_n &\longrightarrow x \quad \text{in } L^p(Z). \end{aligned}$$

for some $x \in W^{1,p}(Z)$. Note that, because of hypothesis $H(j)_1$(iii), the sequence $\{u_n^*\}_{n\geq 1} \subseteq L^{r'}(Z)$ is bounded and so

$$\int_Z u_n^*(z)(x_n - x)(z)\, dz \longrightarrow 0.$$

From the choice of the sequence $\{x_n\}_{n\geq 1} \subseteq W^{1,p}(Z)$, we have

$$\langle A(x_n), x_n - x \rangle_{W^{1,p}(Z)} \leq \varepsilon_n \|x_n - x\|_{W^{1,p}(Z)} + \int_Z u_n^*(z)(x_n - x)(z)\, dz \quad \forall n \geq 1,$$

with $\varepsilon_n \searrow 0$, so

$$\limsup_{n\to+\infty}\langle A(x_n), x_n - x\rangle_{W^{1,p}(Z)} \leq 0.$$

Because A is maximal monotone, it is generalized pseudomonotone (see Denkowski–Migórski–Papageorgiou [9, p. 60]) and so

$$\|\nabla x_n\|_p = \langle A(x_n), x_n\rangle_{W^{1,p}(Z)} \longrightarrow \langle A(x), x\rangle_{W^{1,p}(Z)} = \|\nabla x\|_p.$$

Also recall that $\nabla x_n \longrightarrow \nabla x$ weakly in $L^p(Z;\mathbb{R}^N)$. Since $L^p(Z;\mathbb{R}^N)$ is uniformly convex, it has the Kadec–Klee property (see Denkowski–Migórski–Papageorgiou [8, p. 309]). Therefore

$$\nabla x_n \longrightarrow \nabla x \quad \text{in } L^p(Z;\mathbb{R}^N)$$

and so

$$x_n \longrightarrow x \quad \text{in } W^{1,p}(Z).$$

■

Using this Proposition, we can have the first multiplicity result for problem (1.1).

Theorem 4.2 *If hypotheses $H(j)_1$ hold, then problem* (1.1) *has at least two nontrivial solutions.*

Proof Let

$$U_+ := \Big\{x \in W^{1,p}(Z) : \int_Z |x(z)|^{p-2}x(z)\,dz > 0\Big\}$$
$$U^- := \Big\{x \in W^{1,p}(Z) : \int_Z |x(z)|^{p-2}x(z)\,dz < 0\Big\}.$$

These are two nonempty, open cones in $W^{1,p}(Z)$. Let

$$m_+ := \inf_{U_+}\varphi.$$

Recall that $j(z,0) = 0$ for almost all $z \in Z$ (see hypothesis $H(j)_1$(v)) and so $\varphi(0) = 0 \geq m_+$.

If $m_+ = 0$, then because of hypothesis $H(j)_1$(v), we have that

$$\varphi(\xi) = 0 = m_+ \quad \forall \xi \in (0,\delta]$$

(with $\delta > 0$ as in hypothesis $H(j)_1$(v)) and so, we have a continuum of nontrivial minimizers of φ on U_+, which implies that $0 \in \partial\varphi(\xi)$ (since U_+ is open) and from this we infer that these minimizers are solutions of (1.1).

If $m_+ < 0$, then because of hypothesis $H(j)_1$(iv) and Proposition 4.1, we have that φ satisfies the nonsmooth PS_{m_+}-condition.

Let $\hat{\varphi}\colon W^{1,p}(Z) \to \bar{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$, be defined by

$$\hat{\varphi}(x) := \begin{cases} \varphi(x) & \text{if } x \in \bar{U}_+ \\ +\infty & \text{otherwise.} \end{cases}$$

Evidently $\hat{\varphi}$ is lower semicontinuous and bounded below. Using the generalized Ekeland variational principle (see Denkowski–Migórski–Papageorgiou [9, p. 97]), we can find a sequence $\{x_n\}_{n\geq 1} \subseteq U_+$ such that

$$\hat{\varphi}(x_n) = \varphi(x_n) \searrow m_+ \text{ and } \hat{\varphi}(x_n) \leq \hat{\varphi}(y) + \frac{\|x_n - y\|}{n(1 + \|x_n\|)} \quad \forall y \in W^{1,p}(Z).$$

Let $y = x_n + \lambda h$, with $\lambda > 0$, $h \in W^{1,p}(Z)$. Since $x_n \in U_+$, we can find $\delta_1 > 0$ such that

$$y = x_n + \lambda h \in \bar{U}_+ \quad \forall \lambda \in (0, \delta_1].$$

So, we have

$$-\frac{\lambda\|h\|}{n(1 + \|x_n\|)} \leq \hat{\varphi}(x_n + \lambda h) - \hat{\varphi}(x_n) = \varphi(x_n + \lambda h) - \varphi(x_n),$$

so

$$-\frac{\|h\|}{n(1 + \|x_n\|)} \leq \frac{\varphi(x_n + \lambda h) - \varphi(x_n)}{\lambda} \quad \forall \lambda \in (0, \delta_1]$$

and

$$-\frac{\|h\|}{n(1 + \|x_n\|)} \leq \varphi^0(x_n; h).$$

Invoking Theorem 2.1, we obtain $v_n^* \in \big(W^{1,p}(Z)\big)^*$, with $\|v_n^*\|_{(W^{1,p}(Z))^*} = 1$, such that

$$\langle v_n^*, h\rangle_{W^{1,p}(Z)} \leq n(1 + \|x_n\|)\varphi^0(x_n; h) \quad \forall h \in W^{1,p}(Z),$$

so

$$\frac{1}{n(1 + \|x_n\|)} v_n^* \in \partial\varphi(x_n).$$

From this it follows that $(1 + \|x_n\|)m_\varphi(x_n) \leq \frac{1}{n}$, hence $(1 + \|x_n\|)m_\varphi(x_n) \to 0$. So

$$x_n \to x \quad \text{in } W^{1,p}(Z)$$

(see Proposition 4.1) for some $x \in \bar{U}_+$. Also $\varphi(x_n) \to \varphi(x)$ and so $\varphi(x) = m_+$.

If $x \in \partial U_+$, then

$$\int_Z |x(z)|^{p-2}x(z)\,dz = 0$$

and because of hypothesis $H(j)_1$(v) and Corollary 3.3, we have

$$\begin{aligned} 0 > m_+ = \varphi(x) &= \frac{1}{p}\|\nabla x\|_p^p - \int_Z j\big(z, x(z)\big)\,dz \\ &\geq \frac{1}{p}\|\nabla x\|_p^p - \frac{\lambda_1}{p}\|x\|_p^p \geq 0, \end{aligned}$$

a contradiction. This means that $x \in U_+$ and so $x \neq 0$. Moreover, x is a local minimizer of φ, which means that $0 \in \partial\varphi(x)$.

Similarly, working on U_-, we obtain $y \in U_-$ such that

$$\varphi(y) = m_- = \inf_{U_-} \varphi.$$

Then $y \in W^{1,p}(Z)$ is a local minimizer of φ and so $0 \in \partial\varphi(y)$, $y \neq x$.

Let $w = x$ or $w = y$. We have seen that $0 \in \varphi(w)$. We shall show that this implies that w is a nontrivial solution of (1.1). From this inclusion we have that

$$A(w) = u^*, \tag{4.2}$$

with $u^* \in L^{r'}(Z)$, $u^*(z) \in \partial j\big(z, w(z)\big)$ for almost all $z \in Z$. So

$$\langle A(w), \vartheta \rangle_{W^{1,p}(Z)} = \int_Z u^*(z)\vartheta(z)\, dz \quad \forall \vartheta \in C_0^\infty(Z).$$

Remark that

$$-\mathrm{div}\left(\|\nabla w(\cdot)\|_{\mathbb{R}^N}^{p-2} \nabla w(\cdot) \right) \in W^{-1,p'}(Z) = \left(W_0^{1,p}(Z) \right)^*$$

(see Denkowski–Migórski–Papageorgiou [8, p. 362]). After integration by parts, we have

$$\left\langle -\mathrm{div}\left(\|\nabla w(\cdot)\|_{\mathbb{R}^N}^{p-2} \nabla w(\cdot) \right), \vartheta \right\rangle_{W_0^{1,p}(Z)} = \langle u^*, \vartheta \rangle_{W_0^{1,p}(Z)} \quad \forall \vartheta \in C_0^\infty(Z).$$

Since the embedding $C_0^\infty \subseteq W_0^{1,p}(Z)$ is dense, we obtain

$$-\mathrm{div}\left(\|\nabla w(\cdot)\|_{\mathbb{R}^N}^{p-2} \nabla w(\cdot) \right) = u^*(z) \in \partial j\big(z, w(z)\big) \quad \text{for a.a. } z \in Z. \tag{4.3}$$

Using the Green identity (see Kenmochi [15] and Casas–Fernandez [4]), for all $v \in W^{1,p}(Z)$, we have

$$\int_Z \|\nabla w\|_{\mathbb{R}^N}^{p-2} \big(\nabla w(z), \nabla v(z) \big)_{\mathbb{R}^N} dz + \int_Z \mathrm{div}\left(\|\nabla w(z)\|_{\mathbb{R}^N}^{p-2} \nabla w(z) \right) v(z)\, dz = \left\langle \frac{\partial w}{\partial n_p}, \gamma_0(v) \right\rangle_{\partial Z},$$

where $\frac{\partial w}{\partial n_p}(z) := \|\nabla x(z)\|_{\mathbb{R}^N}^{p-2} \big(\nabla x(z), n(z) \big)_{\mathbb{R}^N}$, by $\langle \cdot, \cdot \rangle_{\partial Z}$, we denote the duality brackets for the pair $\big(W^{\frac{1}{p'},p}(\partial Z), W^{-\frac{1}{p'},p'}(\partial Z) \big)$, and γ_0 is the trace map. Using (4.2) and (4.3), we obtain

$$\left\langle \frac{\partial w}{\partial n_p}, \gamma_0(v) \right\rangle_{\partial Z} = 0 \quad \forall v \in W^{1,p}(Z).$$

But $\gamma_0\big(W_0^{1,p}(Z)\big) = W^{-\frac{1}{p'},p'}(\partial Z)$ (see John–Kufner–Fučik [14]). So it follows that

$$\frac{\partial w}{\partial n_p} = 0 \quad \text{on } \partial Z,$$

i.e., $w = x$ and $w = y$ are two distinct nontrivial solutions of (1.1). ∎

We can have another such multiplicity result by employing a modified version of hypotheses $H(j)_1$.

$H(j)_1'$ $\quad j\colon Z \times \mathbb{R} \to \mathbb{R}$ is a function such that

(i) for every $\zeta \in \mathbb{R}$, $j(\cdot,\zeta)$ is measurable;
(ii) for almost all $z \in Z$, $j(z,\cdot)$ is locally Lipschitz;
(iii) for almost all $z \in Z$, all $\zeta \in \mathbb{R}$ and all $u \in \partial j(z,\zeta)$, we have

$$|u| \le a(z) \quad \text{with } \alpha \in L^{r'}(Z),\ \frac{1}{r} + \frac{1}{r'} = 1,\ r \in [1,p^*);$$

(iv) there exist functions $j_\pm \in L^1(Z)$ such that

$$\lim_{\zeta \to \pm\infty} j(z,\zeta) = j_\pm(z) \text{ uniformly for a.a. } z \in Z,\ \int_Z j_\pm(z)\,dz \le 0;$$

(v) there exists $\delta > 0$ such that

$$j(z,\zeta) \ge 0 \quad \text{for a.a. } z \in Z \text{ and all } |\zeta| \le \delta,$$
$$j(z,\zeta) \le \frac{\lambda_1}{p}|\zeta|^p \quad \text{for a.a. } z \in Z \text{ and all } \zeta \in \mathbb{R}.$$

Proposition 4.3 *If hypotheses $H(j)_1'$ hold, then φ satisfies the nonsmooth PS_c-condition for $c \ne -\int_Z j_\pm(z)\,dz$.*

Proof As in the proof of Proposition 4.1 (keeping the notation introduced there), we have

$$y_n \xrightarrow{w} y = \xi \quad \text{in } W^{1,p}(Z),\ \xi \in \mathbb{R},\ \xi \ne 0.$$

Consider the direct sum decomposition $W^{1,p}(Z) = \mathbb{R} \oplus V$, with $V = \{v \in W^{1,p}(Z)\colon \int_Z v(z)\,dz = 0\}$. If $x \in W^{1,p}(Z)$, then we can write in a unique way $x = \bar{x} + \hat{x}$ with $\bar{x} \in \mathbb{R}$, $\hat{x} \in V$. From the choice of the sequence $\|x_n\|_{n\ge1} \subseteq W^{1,p}(Z)$, we have

$$|\langle A(x_n),\hat{x}_n\rangle - \int_Z u_n^* \hat{x}_n\,dz| \le \varepsilon_n \|\hat{x}_n\|,$$
$$\Rightarrow \|D\hat{x}_n\|_p^p \le c_1(1 + \|\hat{x}_n\|^r) \quad \text{for some } c_1 > 0 \text{ (see hypothesis } H(j)_1'\text{(iii))}.$$

By virtue of the Poincaré–Wirtinger inequality (see Denkowski–Migórski–Papageorgiou [8, p. 357]) and since $r < p$, we infer that $\{x_n\}_{n\geq 1} \subseteq W^{1,p}(Z)$ is bounded. So we may assume that

$$\begin{gathered}
\hat{x}_n \to \hat{x} \quad \text{weakly in } W^{1,p}(Z),\\
\hat{x}_n \to \hat{x} \quad \text{in } L^p(Z),\\
\hat{x}_n(z) \to \hat{x}(z) \quad \text{a.e. on } Z,\\
|\hat{x}_n(z)| \leq k(z) \quad \text{a.e. on } Z \text{ with } k \in L^p(Z).
\end{gathered}$$

By the Egorov and Lusin theorems, given $\delta > 0$, we can find $Z_\delta \subseteq Z$ closed subset with $|Z \setminus Z_\delta|_N < \delta$ (by $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$) such that $\hat{x}_n(z) \to \hat{x}(z)$ uniformly on Z and $\hat{x}|_{Z_\delta}$ is continuous. By definition for almost all $z \in Z$ and all $n \geq 1$, we have

$$u_n(z)\hat{x}_n(z) \leq j^0\big(z, x_n(z); \hat{x}_n(z)\big) := \limsup_{\substack{w_n \to x_n(z)\\ \varepsilon \searrow 0}} \frac{j\big(z, w_n + \varepsilon\hat{x}_n(z)\big) - j(z, w_n)}{\varepsilon}.$$

Since $x_n(z) \to +\infty$ a.e. on Z, we have $w_n \to +\infty$. Therefore given $\varepsilon > 0$, we can find $n_0(\varepsilon) \geq 1$ such that for almost all $z \in Z_\delta$ and all $n \geq n_0$, we have

$$\begin{gathered}
j_+(z) - \varepsilon^2 \leq j\big(z, w_n + \varepsilon\hat{x}_n(z)\big) \leq j_+(z) + \varepsilon^2,\\
j_+(z) - \varepsilon^2 \leq j(z, w_n) \leq j_+(z) + \varepsilon^2.
\end{gathered}$$

Since for almost all $z \in Z$ and all $x, h \in \mathbb{R}$, we have $j^0(z, x; -h) = (-j)^0(z, x; h)$, it follows that

$$\begin{aligned}
&|u_n^*(z)\hat{x}_n(z)| \leq \frac{2\varepsilon^2}{\varepsilon} = 2\varepsilon \quad \text{for a.a. } z \in Z_\delta\\
&\qquad \Longrightarrow \quad u_n^*(z)\hat{x}_n(z) \to 0 \quad \text{uniformly for a.a. } z \in Z_\delta\\
&\qquad \Longrightarrow \quad \int_{Z_\delta} |u_n^*(z)\hat{x}_n(z)|\, dz \to 0.
\end{aligned}$$

On the other hand, since $|u_n^*(z)\hat{x}_n(z)| \leq \alpha(z)k(z) = \beta(z)$ a.e. on Z, with $\beta \in L^1(Z)$, we can choose $\delta > 0$ small enough that

$$\int_{Z\setminus Z_\delta} |u_n^*\hat{x}_n|\, dz \leq \varepsilon \quad \text{for all } n \geq 1.$$

Since $\varepsilon > 0$ was arbitrary, finally we conclude that

$$\int_Z |u_n^*\hat{x}_n|\, dz \to 0$$

and since $\|Dx_n\|_p^p \le \varepsilon_n\|\hat{x}_n\| + \int_Z u_n^*\hat{x}_n\,dz$, it follows that $\|Dx_n\|_p \to 0$ as $n \to \infty$. The rest of the proof is the same as the last part of the proof of Proposition 4.1. ∎

So we can obtain the following existence theorem. The proof is the same as that of Theorem 4.2, using this time the usual and not the generalized Ekeland variational principle (see Denkowski–Migórski–Papageorgiou [9, p. 93]).

Theorem 4.4 *If hypotheses* $H(j)_1'$ *hold, then problem* (1.1) *has at least two nontrivial solutions.*

If $N = 1$ (*i.e.*, we have an ordinary differential equation with $\bar{Z} = T = [0, b]$), then we can be more general and assume the following:

$H(j)_1''$ $\quad j\colon T \times \mathbb{R} \to \mathbb{R}$ is a function such that

(i) for every $\zeta \in \mathbb{R}$, $j(\cdot,\zeta)$ is measurable;

(ii) for almost all $t \in T$, $j(t,\cdot)$ is locally Lipschitz;

(iii) for almost all $t \in T$, all $\zeta \in \mathbb{R}$ and all $u \in \partial j(t,\zeta)$, we have

$$|u| \le a(t) + c(t)|x|^{r-1}$$

with $a, c \in L^1(T)_+ 1 \le r < \infty$;

(iv) there exist functions $j_\pm \in L^1(T)$ such that

$$\lim_{\zeta \to \pm\infty} j(t,\zeta) = j_\pm(t) \quad \text{uniformly for a.a. } t \in T, \text{ and}$$

$$\int_0^b j_\pm(t)\,dt \le 0;$$

(v) there exists $\delta > 0$ such that

$$j(t,\zeta) \ge 0 \quad \text{for a.a. } t \in T \text{ and all } |\zeta| \le \delta,$$

$$j(t,\zeta) \le \frac{\lambda_1}{p}|\zeta|^p \quad \text{for a.a. } t \in T \text{ and all } \zeta \in \mathbb{R}.$$

In this case the energy functional $\varphi\colon W^{1,p}(T) \to \mathbb{R}$ is given by

$$\varphi(x) = \frac{1}{p}\|x'\|_p^p - \int_0^b j(t,x(t))\,dt$$

Proposition 4.5 *If hypotheses* $H(j)_1''$ *hold, then* φ *satisfies the nonsmooth* PS_c*-condition for all* $c \neq -\int_Z j_\pm(z)\,dz$.

Proof The proof is the same as that of Proposition 4.3. We remark that in this case no appeal to Egorov and Lusin theorems is necessary, since $\{\hat{x}_n\}_{n\ge1} \subseteq C(T)$ is relatively compact (recall that $W^{1,p}(T)$ is embedded compactly in $C(T)$). So the pointwise estimates are valid for all $t \in T$. ∎

So using this Proposition and as for Theorem 4.2, by means of the usual Ekeland variational principle, we have the following multiplicity result for the ordinary differential equation:

Theorem 4.6 *If hypotheses $H(j)_1''$ hold, then problem* (1.1) *(with $\bar{Z} = T = [0, b]$) has at least two nontrivial solutions.*

We can guarantee the existence of at least three solutions, if we impose an extra condition on $j(z, \zeta)$.

$H(j)_2$ $j\colon Z \times \mathbb{R} \to \mathbb{R}$ is a function satisfying hypotheses $H(j)_1$ or $H(j)_1'$ or $H(j)_1''$ (with $\bar{Z} = T = [0, b]$) without the condition that $\int_Z j_\pm(z)\, dz \leq 0$ and in addition

(vi) there exists $\xi_0 > 0$ such that

$$0 > \int_Z j(z, \pm\xi_0)\, dz > \int_Z j_\pm(z)\, dz.$$

Theorem 4.7 *If hypotheses $H(j)_2$ hold, then problem* (1.1) *has at least three solutions.*

Proof Arguing as in the proof of Theorem 4.2 (see also Theorems 4.4 and 4.6), we can produce two nontrivial solutions located in U_+ and U_- respectively. Note that now we have

$$m_+ \leq -\int_Z j(z, \xi_0)\, dz < -\int_Z j_+(z)\, dz,$$
$$m_- \leq -\int_Z j(z, -\xi_0)\, dz < -\int_Z j_-(z)\, dz.$$

So φ satisfies the nonsmooth $\mathrm{PS}_{m_\pm}$-condition (see Propositions 4.1, 4.3 and 4.5). Let

$$E := \{y \in W^{1,p}(Z) : -\xi_0 \leq y(z) \leq \xi_0\},$$
$$E_1 := \{\pm\xi_0\},$$
$$D := \Big\{ y \in W^{1,p}(Z) : \int_Z |y(z)|^{p-2} y(z)\, dz = 0 \Big\}.$$

We claim that E_1 and D link in $W^{1,p}(Z)$ (see Definition 2.2). To this end note that $E_1 \cap D = \varnothing$. Let

$$\Gamma := \big\{ \eta \in C\big(E; W^{1,p}(Z)\big) : \eta|_{E_1} = \mathrm{id}_{E_1} \big\}$$

and let $\eta \in \Gamma$. Consider the function $\gamma_p\colon W^{1,p}(Z) \to \mathbb{R}$, defined by

$$\gamma_p(x) := \int_Z |x(z)|^{p-2} x(z)\, dz \quad \forall x \in W^{1,p}(Z).$$

Evidently γ_p is continuous. Let us set

$$\hat{\gamma}_p := \gamma_p \circ \eta.$$

Then

$$\hat{\gamma}_p(-\xi_0) < 0 < \hat{\gamma}_p(\xi_0)$$

and so by the intermediate value theorem, we can find $x \in E$ such that $\hat{\gamma}_p(x) = 0$. Then $\eta(x) \in D$, which means that $\eta(E) \cap D \neq \varnothing$ and so we conclude that E_1 and D link in $W^{1,p}(Z)$. From Proposition 4.1, we know that φ satisfies the PS_c-condition with

$$c := \inf_{\eta \in \Gamma} \sup_{v \in E} \varphi\big(\eta(v)\big).$$

Also, from hypothesis $H(j)_2$(vi), we have that

$$-\int_Z j_{\pm}(z)\, dz, \quad \sup_{E_1} \varphi < 0 = \inf_D \varphi \leq c.$$

Thus we can apply Theorem 2.3 (the abstract minimax principle) and obtain $w \in W^{1,p}(Z)$ such that

$$\varphi(w) \geq \inf_D \varphi \geq 0 > m_{\pm},$$

($m_{\pm}$ are as in the proof of Theorem 4.2, so $w \neq x$, $w \neq y$ where x and y are the solutions obtained in Theorem 4.2) and

$$0 \in \partial\varphi(w).$$

From the last inclusion, as in the proof of Theorem 4.2, we conclude that $w \in W^{1,p}(Z)$ is a third solution of (1.1) distinct from the other two. ∎

5 Problems with Smooth Potential

In this section we prove a third multiplicity result based on the so called second deformation theorem (see Chang [6, p. 23]). Again it concerns problems which are strongly resonant at infinity at the first (zero) eigenvalue of $\big(-\Delta_p, W^{1,p}(Z)\big)$. However, since the second deformation theorem exists only for smooth (*i.e.*, C^1) energy functionals, in this case we consider an elliptic problem with a Caratheodory right-hand side nonlinearity. It is an interesting open problem to extend the second deformation theorem to the case of nonsmooth, locally Lipschitz functions.

Let us start by recalling the statement of the second deformation theorem (see Chang [6, p. 23]). In what follows for a Banach space X and $\varphi \in C^1(X)$, we introduce the following sets:

$$K^{\varphi} := \{x \in X : \varphi'(x) = 0\},$$

the set of critical points of φ,

$$K_c^{\varphi} := \{x \in X : \varphi'(x) = 0, \varphi(x) = c\},$$

the set of critical points of φ with critical value c, and

$$\varphi^c := \{x \in X : \varphi(x) \leq c\},$$

the sublevel set of φ at c.

Theorem 5.1 *If X is a Banach space, $\varphi \in C^1(X)$ satisfies the PS_c-condition for every $c \in [a, d]$, a is the only critical value of φ on $[a, d)$ and $\varphi^{-1}(\{a\}) \cap K_a^\varphi$ consists of isolated critical points, then there exists $h \in C\big([0, 1]_X \times (\varphi^d \setminus K_d^\varphi), X\big)$ such that*

(a) $h(t, \cdot)|_{\varphi^a} = \mathrm{id}_{\varphi^a}$ *for all* $t \in [0, 1]$,
(b) $h(0, \cdot) = \mathrm{id}$,
(c) $h(1, \varphi^d \setminus K_d^\varphi) \subseteq \varphi^a$.

In addition for all $t < s$ and all $x \in \varphi^d \setminus K_d^\varphi$, we have $\varphi\big(h(s, x)\big) \leq \varphi\big(h(t, x)\big)$.

Remark 5.2 The conclusion of this theorem, implies that φ^a is a strong deformation retract of $\varphi^d \setminus K_d^\varphi$. Also the last conclusion implies that h is φ-decreasing.

The problem that we study in this section is the following:

$$\text{(5.1)} \qquad \begin{cases} -\mathrm{div}\,\big(\|\nabla x(z)\|_{\mathbb{R}^N}^{p-2} \nabla x(z)\big) = f\big(z, x(z)\big) & \text{for a.a. } z \in Z \\ \frac{\partial x}{\partial n} = 0 \quad \text{on } \partial Z, \end{cases}$$

with $p \in (1, +\infty)$. In what follows, we set

$$F(z, \zeta) = \int_0^\zeta f(z, r)\, dr,$$

the potential function corresponding to the nonlinearity f. Also p^* is the Sobolev critical exponent, defined by

$$p^* := \begin{cases} \frac{Np}{N-p} & \text{if } N > p, \\ +\infty & \text{if } N \leq p. \end{cases}$$

Our hypotheses on the nonlinearity $f(z, \zeta)$ are the following:

$H(f)$ $\quad f\colon Z \times \mathbb{R} \to \mathbb{R}$ is a function such that

(i) for every $\zeta \in \mathbb{R}$, $f(\,\cdot\,, \zeta)$ is measurable;
(ii) for almost all $z \in Z$, $f(z, \,\cdot\,)$ is continuous;
(iii) for almost all $z \in Z$ and all $\zeta \in \mathbb{R}$, we have

$$|f(z, \zeta)| \leq a(z) + c|\zeta|^{r-1},$$

with $a \in L^{r'}(Z)_+$, $c > 0$, $r \in [1, p^*)$, $\frac{1}{r} + \frac{1}{r'} = 1$;

(iv) there exist functions $F_{\pm} \in L^1(Z)$ such that

$$\limsup_{\zeta \to \pm\infty} F(z,\zeta) \leq F_{\pm}(z),$$

uniformly for almost all $z \in Z$ and

$$\int_Z F_{\pm}(z) \leq 0;$$

(v) for almost all $z \in Z$ and all $\zeta \in \mathbb{R}$, we have that

$$F(z,\zeta) \leq \frac{\lambda_1}{p}|\zeta|^p;$$

(vi) there exists $\xi_0 > 0$ such that

$$\int_Z F(z, \pm\xi_0)\, dz > 0.$$

Remark 5.3 Compared with hypotheses $H(j)_1$, we relax the asymptotic conditions at $\pm\infty$ and we drop the local sign condition at $\zeta = 0$. Instead, we employ the extra condition from hypotheses $H(j)_2$. The potential function which for $\zeta \in [-2,2]$ has the form $F(r) = r^2 - r$ satisfies hypothesis $H(f)$(vi), but does not satisfy the local sign condition $H(j)_1$(v).

Let $\varphi\colon W^{1,p}(Z) \to \mathbb{R}$ be defined by

$$\varphi(x) := \frac{1}{p}\|\nabla x\|_p^p - \int_Z F\big(z, x(z)\big)\ dz \quad \forall x \in W^{1,p}(Z).$$

We know that $\varphi \in C^1\big(W^{1,p}(Z)\big)$. Arguing as in the proof of Proposition 4.1, (in fact the argument is simpler) we obtain the following result:

Proposition 5.4 *If hypotheses $H(f)$ hold, then φ satisfies the* PS_c*-condition for all* $c < \min\{-\int_Z F_+(z)\,dz, -\int_Z F_-(z)\,dz\}$.

This proposition combined with Theorem 5.1, leads to a multiplicity result for problem (5.1), which can be viewed as a partial extension of Theorem 4.2.

Theorem 5.5 *If hypotheses $H(f)$ hold, then problem* (5.1) *has at least two nontrivial solutions.*

Proof By virtue of hypothesis $H(f)$(iv), φ is bounded below. Moreover, hypothesis $H(f)$(vi) implies that

$$-\infty < m := \inf_{W^{1,p}(Z)} \varphi < 0.$$

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So φ satisfies the nonsmooth PS_m-condition (see hypothesis $H(f)$(iv) and Proposition 5.4). Hence, we can find $x_1 \in W^{1,p}(Z)$ such that

$$\varphi(x_1) = m < \varphi(0) = 0 \text{ and } \varphi'(x_1) = 0.$$

As before, we deduce that $x_1 \in W^{1,p}(Z)$ is a nontrivial solution of problem (5.1).

Now suppose that x_1 and 0 are the only critical points of φ. Recall that

$$B_r := \{x \in W^{1,p}(Z) : \|x\|_{W^{1,p}(Z)} < r\}.$$

From hypothesis $H(f)$(vi), we know that for $r = \xi_0 |Z|_N^{1/p}$ (where $|Z|_N$ denotes the Lebesgue measure of Z),we have

$$\beta := \sup_{x \in \partial B_r \cap \mathbb{R}} \varphi(x) < 0. \tag{5.2}$$

On the other hand, if

$$D := \left\{ x \in W^{1,p}(Z) : \int_Z |x(z)|^{p-2} x(z)\, dz = 0 \right\},$$

then because of hypothesis $H(f)$(v) and Corollary 3.3, we have

$$\inf_D \varphi = 0.$$

Let

$$S := \left\{ \eta \in C\big(\bar{B}_r \cap \mathbb{R}; W^{1,p}(Z)\big) : \eta|_{\partial B_r \cap \mathbb{R}} = \mathrm{id}_{\partial B_r \cap \mathbb{R}} \right\}$$

and consider the map $\eta_0 \colon \bar{B}_r \cap \mathbb{R} \to W^{1,p}(Z)$, defined by

$$\eta_0(x) := \begin{cases} x_1 & \text{if } \|x\|_{W^{1,p}(Z)} < \frac{r}{2} \\ h\big(\frac{2(r - \|x\|_{W^{1,p}(Z)})}{r}, \frac{rx}{\|x\|_{W^{1,p}(Z)}}\big) & \text{if } \|x\|_{W^{1,p}(Z)} \geq \frac{r}{2}, \end{cases}$$

where $h(t, x)$ is the homotopy postulated by Theorem 5.1 for the interval $[m, 0]$. Since by hypothesis, x_1 is the only minimizer of φ on $W^{1,p}(Z)$ (recall that $m < 0 = \varphi(0)$), we have that

$$\eta_0(x) = h(1, 2x) \quad \forall x \in \overline{B_r} \cap \mathbb{R},\ \|x\|_{W^{1,p}(Z)} = \frac{r}{2}.$$

Remark that $\|2x\|_{W^{1,p}(Z)} = r$ and so $\varphi(2x) < 0$, which by virtue of Theorem 5.1, implies that

$$\eta_0(x) = h(1, 2x) = x_1.$$

This proves that η_0 is continuous. Moreover,

$$\eta_0(x) = h(0, x) = x \quad \forall x \in \partial B_r \cap \mathbb{R}, \tag{5.3}$$

i.e., $\eta_0|_{\partial B_r \cap \mathbb{R}} = \mathrm{id}_{\partial B_r \cap \mathbb{R}}$. Therefore $\eta_0 \in S$ and because h is φ decreasing (see Theorem 5.1), we have

$$\varphi\big(h(s,x)\big) \le \varphi\big(h(t,x)\big) \quad \forall t,s \in [0,1],\ t < s,\ x \in \varphi^0 \setminus \{0\},$$

so from (5.2) and (5.3), we see that

$$\varphi\big(\eta_0(x)\big) < 0 \quad \forall x \in \bar{B}_r \cap \mathbb{R}. \tag{5.4}$$

Recall that $\partial B_r \cap \mathbb{R}$ and D link (see the proof of Theorem 4.7). So, we have that

$$\eta(\bar{B}_r \cap \mathbb{R}) \cap D \neq \varnothing \quad \forall \eta \in S.$$

Since $\inf_D \varphi = 0$, we have

$$\sup_{x \in \bar{B}_r \cap \mathbb{R}} \varphi\big(\eta(x)\big) \ge 0 \quad \forall \eta \in S \tag{5.5}$$

and

$$\sup_{x \in \bar{B}_r \cap \mathbb{R}} \varphi\big(\eta_0(x)\big) = \varphi\big(\eta_0(x^*)\big), \tag{5.6}$$

for some $x^* \in \bar{B}_r \cap \mathbb{R}$. Comparing (5.4), (5.5) and (5.6), we reach a contradiction. Therefore φ must have another critical point $x_2 \neq x_1$, $x_2 \neq 0$. Since $\varphi'(x_2) = 0$ as before, we infer that x_2 is the second nontrivial solution. ■

Remark 5.6 If in hypotheses $H(j)_2$(iii) and $H(f)$(iii), $a \in L^\infty(Z)_+$, then from the nonlinear regularity theory (see for example Anane [1]), the solutions obtained in Theorems 4.2, 4.4, 4.6, 4.7 and 5.5 belong in $C^{1,\beta}\big(\bar{Z}\big)$ (with $0 < \beta < 1$), and so the Neumann boundary condition can be interpreted in a pointwise sense.

6 Infinitely Many Solutions

Thus far the problems studied were strongly resonant, which roughly speaking means that the growth of the potential function is very slow. In contrast, in this section we examine problems with a potential exhibiting a strictly super-p-growth (*i.e.,* it is strictly superquadratic if $p = 2$ (semilinear case)). Imposing a symmetry condition, we obtain an infinity of pairs $(x, -x)$ of solutions of problem (1.1). This is done with the help of the so called nonsmooth symmetric mountain pass theorem.

Let us recall the nonsmooth symmetric mountain pass theorem. It can be proved as Theorem 4.4 of Szulkin [18, p. 95], using instead the nonsmooth deformation theorem (see Chang [6] and Kourogenis–Papageorgiou [16, Theorem 4, p. 250]).

Theorem 6.1 *Suppose X is a reflexive Banach space, $\varphi\colon X \to \mathbb{R}$ is a locally Lipschitz function satisfying the nonsmooth* PS*-condition, $\varphi(0) = 0$ and*

(a) *there exist a subspace V of X of finite codimension and numbers $\varrho > 0$ and $r > 0$ such that*

$$\varphi(x) \geq \varrho \quad \forall x \in V,\ \|x\|_X = r;$$

(b) *there exists a finite dimensional subspace Y of X,* $\dim Y > \operatorname{codim} V$, *such that*

$$\varphi(x) \to -\infty \quad \text{as } \|x\|_X \to +\infty,\ x \in X.$$

Then φ has at least $\dim Y - \operatorname{codim} V$ *distinct pairs $(x, -x)$ of nontrivial critical points. In particular, if condition* (b) *is replaced by*

(b′) *for each integer $k \geq 1$, there exists a subspace Y of X such that* $\dim Y = k$ *and it satisfies condition (b),*

then φ admits infinitely many distinct pairs $(x, -x)$ of nontrivial critical points.

Our hypotheses on the nonsmooth potential function $j(z, \zeta)$ are the following:

$H(j)_3$ $\quad j\colon Z \times \mathbb{R} \to \mathbb{R}$ is a function such that

(i) for every $\zeta \in \mathbb{R}$, $j(\cdot\,, \zeta)$ is measurable and $\int_Z j(z, 0)\, dz = 0$;
(ii) for almost all $z \in Z$, $j(z, \cdot)$ is locally Lipschitz and even;
(iii) for almost all $z \in Z$ and all $\zeta \in \mathbb{R}$ and all $u \in \partial j(z, \zeta)$, we have

$$|u| \leq a(z) + c|\zeta|^{r-1}$$

with $a \in L^\infty(Z)_+$, $c > 0$, $r \in [1, p^*)$ and $\frac{1}{r} + \frac{1}{r'} = 1$;
(iv) for almost all $z \in Z$ and all $\zeta \in \mathbb{R}$, we have that

$$\mu j(z, \zeta) + c_1|\zeta|^s - c_2 \leq -j^0(z, \zeta; -\zeta),$$

with $c_1, c_2 > 0$, $1 < s \leq p < \mu$;
(v) $\limsup_{\zeta \to 0} \frac{p\, j(z,\zeta)}{|\zeta|^p} < 0$ uniformly for almost all $z \in Z$;
(vi) there exists $M > 0$ such that for almost all $z \in Z$ and all $|\zeta| \geq M$, we have

$$j(z, \zeta) \geq \beta(z)|\zeta|^\vartheta,$$

with $\beta \in L^\infty(Z)_+$, $\beta \neq 0$, $\vartheta > p$.

As before we consider the locally Lipschitz energy functional $\varphi\colon W^{1,p}(Z) \to \mathbb{R}$, defined by

$$\varphi(x) := \frac{1}{p}\|\nabla x\|_p^p - \int_Z j\bigl(z, x(z)\bigr)\, dz \quad \forall x \in W^{1,p}(Z).$$

Proposition 6.2 *If hypotheses $H(j)_3$ hold, then φ satisfies the nonsmooth* PS*-condition.*

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Proof Let $\{x_n\}_{n\geq 1} \subseteq W^{1,p}(Z)$ be a sequence such that

$$|\varphi(x_n)| \leq M_1 \quad \text{for some } M_1 > 0 \text{ and } m_\varphi(x_n) \to 0.$$

Let $x_n^* \in \partial\varphi(x_n)$ be such that $m_\varphi(x_n) = \|x_n^*\|_{(W^{1,p}(Z))^*}$ for $n \geq 1$. We have

$$x_n^* = A(x_n) - u_n^* \quad \forall n \geq 1$$

with $A\colon W^{1,p}(Z) \to \big(W^{1,p}(Z)\big)^*$ being the nonlinear maximal monotone operator defined as in the proof of Proposition 4.1 and $u_n^* \in L^{r'}(Z)$ with $u_n^*(z) \in \partial j\big(z, x_n(z)\big)$ for almost all $z \in Z$ and all $n \geq 1$. We have

$$|\langle x_n^*, x_n\rangle_{W^{1,p}(Z)}| = \Big| \|\nabla x_n\|_p^p - \int_Z u_n^*(z)x_n(z)\,dz\Big| \leq \varepsilon_n\|x_n\|_{W^{1,p}(Z)},$$

with $\varepsilon_n \searrow 0$, so

$$-\|\nabla x_n\|_p^p + \int_Z u_n^*(z)x_n(z)\,dz \leq \varepsilon_n\|x_n\|_{W^{1,p}(Z)}$$

and so from the definition of j^0, we have that

$$-\|\nabla x_n\|_p^p - \int_Z j^0\big(x, x_n(x); -x_n(z)\big)\,dz \leq \varepsilon_n\|x_n\|_{W^{1,p}(Z)} \quad \forall n \geq 1. \tag{6.1}$$

Also, we have that

$$\frac{\mu}{p}\|\nabla x_n\|_p^p - \int_Z \mu j\big(z, x_n(z)\big)\,dz \leq \mu M_1 \quad \forall n \geq 1. \tag{6.2}$$

Adding (6.1) and (6.2), we obtain

$$\Big(\frac{\mu}{p} - 1\Big)\|\nabla x_n\|_p^p - \int_Z \Big(\mu j\big(z, x_n(z)\big) + j^0\big(z, x_n(z); -x_n(z)\big)\Big)\,dz \\ \leq \mu M_1 + \varepsilon_n\|x_n\|_{W^{1,p}(Z)}.$$

From hypothesis $H(j)_3$(iv), we get

$$\Big(\frac{\mu}{p} - 1\Big)\|\nabla x_n\|_p^p + c_1\|x_n\|_s^s - c_2|Z|_N \leq \mu M_1 + \varepsilon_n\|x_n\|_{W^{1,p}(Z)}.$$

Since $1 < s \leq p < \mu$, we have

$$c_3\|\nabla x_n\|_p^s + c_1\|x_n\|_s^s \leq c_4,$$

for some $c_3, c_4 > 0$. Thus the sequence $\{x_n\}_{n\geq 1} \subseteq W^{1,p}(Z)$ is bounded (see Denkowski–Migórski–Papageorgiou [8, p. 361]).

So, passing to a subsequence if necessary, we may assume that

$$x_n \to x \quad \text{weakly in } W^{1,p}(Z),$$
$$x_n \to x \quad \text{in } L^p(Z),$$

with some $x \in W^{1,p}(Z)$. Continuing as in the last part of Proposition 4.1, we can show that

$$x_n \to x \quad \text{in } W^{1,p}(Z). \qquad \blacksquare$$

Now we can have the multiplicity result.

Theorem 6.3 *If hypotheses $H(j)_3$ hold, then there exist infinitely many distinct pairs $(x, -x)$, $x \in W^{1,p}(Z)$, which solve problem* (1.1).

Proof Let $Y \subseteq W^{1,p}(Z)$ be a finite dimensional subspace, with $Y \subseteq L^\infty(Z)$. Let $\gamma > M$, where $M > 0$ is as in hypothesis $H(j)_3$(vi). Recalling that since $\dim Y < +\infty$, all norms on Y are equivalent, we introduce the quantity

$$\xi_0 := \inf_{\substack{y \in Y \\ \|y\|_\infty = \gamma}} \int_{\{|y(z)|>M\}} \beta(z)|y(z)|^\vartheta \, dz,$$

where β and ϑ are as in hypothesis $(j)_3(vi)$. For $y \in Y$, we have

$$\begin{aligned}
\varphi(y) &= \frac{1}{p}\|\nabla y\|_p^p - \int_Z j\big(z, y(z)\big)\, dz \\
&\le \eta\|y\|_\infty^p - \int_Z j\big(z, y(z)\big)\, dz \\
&= \eta\|y\|_\infty^p - \int_{\{|y(z)|>M\}} j\big(z, y(z)\big)\, dz - \int_{\{|y(z)|\le M\}} j\big(z, y(z)\big)\, dz \\
&\le \eta\|y\|_\infty^p - \int_{\{|y(z)|>M\}} \beta(z)|y(z)|^\vartheta \, dz + \eta_1, \qquad (6.3)
\end{aligned}$$

for some $\eta, \eta_1 > 0$ (since on Y all norms are equivalent). We have

$$\int_{\{|y(z)|>M\}} \beta(z)|y(z)|^\vartheta \, dz = \frac{\|y\|_\infty^\vartheta}{\gamma^\vartheta} \int_{\{|y(z)|>M\}} \beta(z)\gamma^\vartheta \left(\frac{|y(z)|}{\|y\|_\infty}\right)^\vartheta dz.$$

Since we shall send $\|y\|_{W^{1,p}(Z)}$ to $+\infty$, we have that $\|y\|_\infty$ goes to ∞ and so we may assume that $\|y\|_\infty > \gamma$. Therefore, we have

$$\{z \in Z : |y(z)| > M\} \supseteq \left\{ z \in Z : \frac{\gamma}{\|y\|_\infty}|y(z)| > M \right\}$$

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and so

$$
\begin{aligned}
-\int_{\{|y(z)|>M\}}\beta(z)|y(z)|^{\vartheta}\,dz &\le -\int_{\{\frac{\gamma}{\|y\|_\infty}|y(z)|>M\}}\beta(z)|y(z)|^{\vartheta}\,dz\\
&= -\frac{\|y\|_\infty^{\vartheta}}{\gamma^{\vartheta}}\int_{\{\frac{\gamma}{\|y\|_\infty}|y(z)|>M\}}\beta(z)\gamma^{\vartheta}\Big(\frac{|y(z)|}{\|y\|_\infty}\Big)^{\vartheta}\,dz\\
&\le -\frac{\xi_0\|y\|_\infty^{\vartheta}}{\gamma^{\vartheta}}.
\end{aligned}
$$

So returning to (6.3), we have

$$
\varphi(y)\le \eta\|y\|_\infty^{p}-\frac{\xi_0}{\gamma^{\vartheta}}\|y\|_\infty^{\vartheta}+\eta_1
$$

and from the facts that $\vartheta>p$ and all norms in Y are equivalent, we get that

$$
\varphi(y)\to-\infty\quad\text{as } \|y\|_{W^{1,p}(Z)}\to+\infty.
$$

So finally $\varphi|_Y$ is anticoercive.

On the other hand, by virtue of hypothesis $H(j)_3$(v), for a given $\varepsilon\in(0,1)$, we can find $\delta=\delta(\varepsilon)>0$ such that

$$
j(z,\zeta)\le-\frac{\varepsilon}{p}|\zeta|^p\quad\text{for a.a. } z\in Z \text{ and all } |\zeta|\le\delta.
$$

Also from the mean value theorem for locally Lipschitz functions (see Clarke [7, p. 41] and Denkowski–Migórski–Papageorgiou [8, p. 609]), we know that for almost all $z\in Z$ and all $|\zeta|>\delta$, we have

$$
\begin{aligned}
j(z,\zeta)&\le\big(a(z)+c|\zeta|^{r-1}\big)|\zeta|+j(z,0)\\
&\le\Big(\frac{a(z)}{\delta^{r-1}}+c\Big)|\zeta|^r+j(z,0)\le\gamma(z)|\zeta|^{\mu}+j(z,0),
\end{aligned}
$$

for some $\gamma\in L^\infty(Z)$ and $\max\{r,p\}<\mu<p^*$. So finally, we can say that for almost all $z\in Z$ and all $\zeta\in\mathbb{R}$, we have

$$
j(z,\zeta)\le-\frac{\varepsilon}{p}|\zeta|^p+\gamma_1(z)|\zeta|^{\mu}+j(z,0),
$$

for some $\gamma_1\in L^\infty(Z)_+$. Therefore, for $x\in W^{1,p}(Z)$, we have that

$$
\begin{aligned}
\varphi(x)&\ge\frac{1}{p}\|\nabla x\|_p^p+\frac{\varepsilon}{p}\|x\|_p^p-c_5\|x\|_{W^{1,p}(Z)}^{\mu}-\int_Z j(z,0)\,dz\\
&\ge\frac{\varepsilon}{p}\|x\|_{W^{1,p}(Z)}^{p}-c_5\|x\|_{W^{1,p}(Z)}^{\mu},
\end{aligned}
$$

for some $c_5 > 0$ (since $\int_Z j(z,0)\,dz = 0$ and $\varepsilon \in (0,1)$).

Because $\mu > p$, from the above inequality, we see that if we choose $r > 0$ small enough, we can have that

$$\varphi|_{\partial B_r} \geq \varrho > 0.$$

Because of these facts and Proposition 6.2, we can apply Theorem 6.1 and obtain the infinite number of distinct pairs $(x, -x)$ of nontrivial critical points of φ. As before, we check that each such pair is a pair of nontrivial solutions of problem (1.1). ■

Remark 6.4 The nonsmooth, locally Lipschitz function $j\colon \mathbb{R} \to \mathbb{R}$, defined by

$$j(\zeta) := \begin{cases} -|\zeta|^s & \text{if } |\zeta| < 1 \\ \frac{1}{\mu}|\zeta|^\mu - |\zeta|^s - \frac{1}{\mu} & \text{if } |\zeta| \geq 1, \end{cases}$$

with $p^* > r > \mu > \vartheta > p \geq s > 1$, satisfies hypotheses $H(j)_3$.

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