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THREE-TERM ARITHMETIC PROGRESSIONS AND SUMSETS

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Abstract Suppose that G is an abelian group and that $A \subset G$ is finite and contains no non-trivial three-term arithmetic progressions. We show that $|A+A| \gg_\varepsilon |A|(\log|A|)^{1/3-\varepsilon}$.

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1. Introduction

In [**6**] Freĭman proved the following qualitative theorem.

Theorem 1.1 (Freĭman). *Suppose that $A \subset \mathbb{Z}$ is finite and contains no non-trivial* three-term arithmetic progressions. Then (by slight abuse of notation) $|A+A|/|A| \to \infty$ as $|A| \to \infty$.*

The best known quantitative version of this theorem is achieved by inserting Bourgain's most recent bound for Roth's theorem (see [**3**]) into a result of Ruzsa's (see [**20**]).

Theorem 1.2 (Bourgain–Ruzsa). *Suppose that $A \subset \mathbb{Z}$ is finite and contains no non-trivial three-term arithmetic progressions. Then*

$$|A+A| \gg |A|\left(\frac{\log|A|}{(\log\log|A|)^3}\right)^{1/6}.$$

This theorem is interesting in its own right but has also been applied (independently) by Schoen in [**23**] and Hegyvári *et al.* in [**17**] to give a witty proof of the following result regarding restricted sumsets.

If A, B are subsets of an abelian group then we write

$$A \mathbin{\widehat{+}} B := \{a+b : a \in A,\ b \in B \text{ and } a \neq b\},$$

and call this the *restricted sum* of A and B.

* A trivial three-term arithmetic progression is one in which all three elements are the same.

Theorem 1.3 (Schoen–Hegyvári–Hennecart–Plagne). *Suppose that A and B are two finite non-empty sets of integers, or residues modulo an integer $m > 1$, and set $n := |A + B|$. Then*

$$\frac{|A \widehat{+} B|}{|A + B|} = 1 + O\left(\frac{(\log\log n)^3}{\log n}\right)^{1/6}.$$

Recently, a lot of work has been done on generalizing additive problems in the integers to other abelian groups (see, for example, [**9**, **13**, **14**, **18**]) and in this paper we not only improve the bounds in Theorems 1.2 and 1.3 but we also extend them to cover arbitrary abelian groups. Specifically, our main result is the following theorem.

Theorem 1.4. *Suppose that G is an abelian group and that $A \subset G$ is finite and contains no non-trivial three-term arithmetic progressions. Then*

$$|A + A| \gg |A|\left(\frac{\log|A|}{(\log\log|A|)^3}\right)^{1/3}.$$

This translates easily to an improvement of Theorem 1.3.

Theorem 1.5. *Suppose that A and B are two finite non-empty subsets of an abelian group G and set $n := |A + B|$. Then*

$$\frac{|A \widehat{+} B|}{|A + B|} = 1 + O\left(\frac{(\log\log n)^3}{\log n}\right)^{1/3}.$$

There are three main aspects to our arguments. Firstly, to effect a complete passage to general abelian groups we have to work slightly harder when the sets in question have elements which differ by an element of order 2. To deal with this we use a generalization of the Bohr set technology of [**2**], as developed in [**10**].

Secondly, we use an energy increment argument in the style of Heath-Brown [**16**] and Szemerédi [**25**] to prove a local version of Roth's theorem that is particularly efficient (essentially because of limitations in the modelling results of Green and Ruzsa [**9**]) in our situation; this type of argument was previously deployed in [**21**].

Finally, we use a result which might be called a weak partially polynomial version of the celebrated Freĭman–Ruzsa theorem. This type of result was first proved for finite fields in [**15**]; the more general case we use was proved in [**10**].

The paper now splits into seven further sections. In §§ 3 and 4 we set up the basic machinery of 'local' Fourier analysis, which lets us prove our local version of Roth's theorem in § 5. In § 6 we prove the partially polynomial version of the Freĭman–Ruzsa theorem, before completing the main arguments in § 7.

In the final section, § 8, we discuss improvements for particular groups G and possible further questions.

2. Notation

The book [**19**] serves as a general reference for the Fourier transform, which we use throughout the paper.

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Suppose that G is a finite abelian group. $\hat{G}$ denotes the *dual group* of G, i.e. the group of homomorphisms $\gamma : G \to S^1$, where $S^1 := \{z \in \mathbb{C} : |z| = 1\}$, and we write $M(G)$ for the space of measures on G endowed with the norm $\|\cdot\|$ defined by $\|\mu\| := \int \mathrm{d}|\mu|$.

There is one element of $M(G)$ worthy of particular note: the Haar probability measure μ_G. This measure is used to define the Fourier transform which takes a function $f : G \to \mathbb{C}$ to

$$\hat{f} : \hat{G} \to \mathbb{C}; \gamma \mapsto \int_{x\in G} f(x)\bar{\gamma}(x)\,\mathrm{d}\mu_G(x) = \frac{1}{|G|}\sum_{x\in G} f(x)\bar{\gamma}(x).$$

We use the Haar probability measure, μ_G, on G to define an inner product on functions $f, g : G \to \mathbb{C}$ by

$$\langle f, g\rangle := \int_{x\in G} f(x)\overline{g(x)}\,\mathrm{d}\mu_G(x).$$

Since μ_G is normalized to be a probability measure, Plancherel's theorem states that

$$\langle f, g\rangle = \sum_{\gamma\in\hat{G}} \hat{f}(\gamma)\overline{\hat{g}(\gamma)}.$$

Similarly, we use μ_G to define the convolution of two functions $f, g : G \to \mathbb{C}$:

$$f * g(y) := \int_{x\in G} f(y-x)g(x)\,\mathrm{d}\mu_G(x),$$

and a simple calculation tells us that $\widehat{f * g} = \hat{f}\hat{g}$.

Finally, it will sometimes be necessary to consider the Fourier transform of a particularly complicated expression E. In this case we may write $E^\wedge$ in place of $\hat{E}$.

3. Bourgain systems

In [**2**], Bourgain showed how to extend some of the techniques of Fourier analysis from groups to a wider class of 'approximate groups'; in [**10**] this was taken further when the notion of a *Bourgain system* was introduced. We refer the reader to that paper for a more comprehensive discussion of Bourgain systems and limit ourselves to recalling the key definitions and tools that we shall require.

Suppose that G is a finite abelian group and $d \geqslant 1$ is real. A *Bourgain system* $\mathcal{B}$ of dimension d is a collection $(B_\rho)_{\rho\in(0,2]}$ of subsets of G such that the following axioms are satisfied.

- Nesting. If $\rho' \leqslant \rho$ we have $B_{\rho'} \subseteq B_\rho$.
- Zero. $0 \in B_\rho$ for all $\rho \in (0, 2]$.
- Symmetry. If $x \in B_\rho$ then $-x \in B_\rho$.
- Addition. For all ρ, ρ' such that $\rho + \rho' \leqslant 1$ we have $B_\rho + B_{\rho'} \subseteq B_{\rho+\rho'}$.
- Doubling. If $\rho \leqslant 1$ then there is a set X with $|X| \leqslant 2^d$ and

$$B_{2\rho} \subset \bigcup_{x\in X} x + B_\rho.$$

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We define the *density* of $\mathcal{B} = (B_\rho)_\rho$ to be $\mu_G(B_1)$ and denote it by $\mu_G(\mathcal{B})$. Frequently we shall consider several Bourgain systems $\mathcal{B}, \mathcal{B}', \mathcal{B}'', \dots$; in this case the underlying sets will be denoted $(B_\rho)_\rho, (B'_\rho)_\rho, (B''_\rho)_\rho, \dots$, and we shall write $B, B', B'', \dots$ for the sets $B_1, B'_1, B''_1, \dots$.

Example 3.1 (Bohr sets). There is a natural valuation on S^1 defined by $\|z\| := (2\pi)^{-1}|\arg z|$, where arg is taken as mapping into $(-\pi, \pi]$. If $\Gamma \subset \hat{G}$ and $\delta \in (0, 1]$, then we set

$$B(\Gamma, \delta) := \{x \in G : \|\gamma(x)\| \leqslant \delta \text{ for all } \gamma \in \Gamma\},$$

and call such a set a *Bohr set*.

It turns out that the system $(B(\Gamma, \rho\delta))_\rho$ is a Bourgain system of density at least $\delta^{|\Gamma|}$ and dimension $2|\Gamma|$, as the next lemma shows. By a slight abuse we call this the Bourgain system *induced* by the Bohr set $B(\Gamma, \delta)$.

Lemma 3.2. *Suppose that $B(\Gamma, \delta)$ is a Bohr set. Then*

$$\mu_G(B(\Gamma, \delta)) \geqslant \delta^{|\Gamma|}$$

and there is a set X of size at most $4^{|\Gamma|}$ such that

$$B(\Gamma, 2\delta) \subset \bigcup_{x \in X} x + B(\Gamma, \delta).$$

The proof of this lemma is a simple averaging argument which may be found, for example, in [**26**, Lemma 4.20].

Returning to Bourgain systems in general, we say that a Bourgain system $\mathcal{B}'$ is a *subsystem* of $\mathcal{B}''$ if $B'_\rho \subset B''_\rho$ for all ρ. We shall be very interested in subsystems and consequently the following dilation and intersection lemmas will be important. The first lemma is immediate.

Lemma 3.3. *Suppose that $\mathcal{B}$ is a Bourgain system of dimension d and $\lambda \in (0, 1]$ is a parameter. Then $\lambda\mathcal{B} := (B_{\lambda\rho})_\rho$ is a Bourgain system of dimension d and density at least $(\lambda/2)^d \mu_G(\mathcal{B})$.*

Lemma 3.4. *Suppose that $\mathcal{B}^{(1)}, \dots, \mathcal{B}^{(k)}$ are, respectively, Bourgain systems of dimensions $d_1, \dots, d_k$. Then*

$$\bigcap_{i=1}^k \mathcal{B}^{(i)} := \left(\bigcap_{i=1}^k B_\rho^{(i)}\right)_\rho$$

is a Bourgain system of dimension at most $2(d_1 + \cdots + d_k)$ and density at least $4^{-(d_1+\cdots+d_{k-1})} 2^{-d_k} \prod_{i=1}^k \mu_G(\mathcal{B}^{(i)})$.

Proof. The conclusion is trivial apart from the doubling and density estimates. For each i with $1 \leqslant i \leqslant k$ there is a set T_i with $|T_i| \leqslant 4^{d_i}$ such that $B^{(i)}_{2\rho} \subset T_i + B^{(i)}_{\rho/2}$. Define a set T as follows: for each $(t_1, \dots, t_k) \in T_1 \times \cdots \times T_k$ place one element of $\bigcap_{i=1}^k (t_i + B^{(i)}_{\rho/2})$ in T if and only if that set is non-empty.

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Now, if $t_0 \in \bigcap_{i=1}^k (t_i + B^{(i)}_{\rho/2})$ then the map $t \mapsto t - t_0$ maps $\bigcap_{i=1}^k (t_i + B^{(i)}_{\rho/2})$ into $\bigcap_{i=1}^k B^{(i)}_\rho$, whence

$$\bigcap_{i=1}^k B^{(i)}_{2\rho} \subset T + \bigcap_{i=1}^k B^{(i)}_\rho,$$

and the intersection has dimension at most $2(d_1 + \cdots + d_k)$.

The density estimate proceeds similarly. For each i with $1 \leqslant i \leqslant k-1$ let T_i be a maximal subset of G such that the sets $(t + B^{(i)}_{1/4})_{t \in T_i}$ are disjoint. It follows that $|T| \leqslant 4^{d_1} \mu_G(\mathcal{B}^{(i)})^{-1}$ and

$$G \subset B^{(i)}_{1/4} - B^{(i)}_{1/4} + T_i \subset B^{(i)}_{1/2} + T_i.$$

Thus, there are some $x_1, \ldots, x_{k-1} \in G$ such that

$$\mu_G\Bigl(\bigcap_{i=1}^{k-1} (x_i + B^{(i)}_{1/2}) \cap B^{(k)}_{1/2}\Bigr) \geqslant 4^{-(d_1+\cdots+d_{k-1})} 2^{-d_k} \prod_{i=1}^k \mu_G(\mathcal{B}^{(i)}).$$

Now, for fixed $x_0 \in \bigcap_{i=1}^{k-1} (x_i + B^{(i)}_{1/2}) \cap B^{(k)}_{1/2}$, the map $x \mapsto x - x_0$ is an injection from

$$\bigcap_{i=1}^{k-1} (x_i + B^{(i)}_{1/2}) \cap B^{(k)}_{1/2}$$

into $\bigcap_{i=1}^k B^{(i)}_1$. The result follows. □

Not all Bourgain systems behave as regularly as we would like; we say that a Bourgain system $\mathcal{B}$ of dimension d is *regular* if

$$1 - 2^3 d|\eta| \leqslant \frac{\mu_G(B_1)}{\mu_G(B_{1+\eta})} \leqslant 1 + 2^3 d|\eta|$$

for all η with $d|\eta| \leqslant 2^{-3}$. Typically, however, Bourgain systems are regular, a fact implicit in the proof of the following proposition.

Proposition 3.5. *Suppose that $\mathcal{B}$ is a Bourgain system of dimension d. Then there is a $\lambda \in [\frac{1}{2}, 1)$ such that $\lambda\mathcal{B}$ is regular.*

Proof. Let $f : [0,1] \to \mathbb{R}$ be the function $f(\alpha) := -(1/d) \log_2 \mu_G(B_{2^{-\alpha}})$ and note that f is non-decreasing in α with $f(1) - f(0) \leqslant 1$. We claim that there is an $\alpha \in [\frac{1}{6}, \frac{5}{6}]$ such that $|f(\alpha + x) - f(\alpha)| \leqslant 3|x|$ for all $|x| \leqslant \frac{1}{6}$. If no such α exists, then for every $\alpha \in [\frac{1}{6}, \frac{5}{6}]$ there is an interval $I(\alpha)$ of length at most $\frac{1}{6}$ having one endpoint equal to α and with

$$\int_{I(\alpha)} \mathrm{d}f > \int_{I(\alpha)} 3 \, \mathrm{d}x.$$

These intervals cover $[\frac{1}{6}, \frac{5}{6}]$, which has total length $\frac{2}{3}$. A simple covering lemma allows us to pass to a disjoint subcollection $I_1 \cup \cdots \cup I_n$ of these intervals with total length at

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least $\frac{1}{3}$. However, we now have

$$1 \geqslant \int_0^1 \mathrm{d}f \geqslant \sum_{i=1}^n \int_{I_i} \mathrm{d}f > \sum_{i=1}^n \int_{I_i} 3\,\mathrm{d}x \geqslant 1,$$

which is a contradiction. It follows that there is an α such that $|f(\alpha+x)-f(\alpha)| \leqslant 3|x|$ for all $|x| \leqslant \frac{1}{6}$. Setting $\lambda := 2^{-\alpha}$, it is easy to see that

$$(1+|\eta|)^{-3d} \leqslant \frac{\mu_G(B_\lambda)}{\mu_G(B_{(1+\eta)\lambda})} \leqslant (1+|\eta|)^{3d}$$

whenever $|\eta| \leqslant \frac{1}{6}$. But, if $3d|\eta| \leqslant \frac{1}{2}$, then $(1+|\eta|)^{-3d} \leqslant 1+6d|\eta|$ and $(1+|\eta|)^{-3d} \geqslant 1-6d|\eta|$; it follows that $\lambda\mathcal{B}$ is a regular Bourgain system. □

4. Fourier analysis local to Bourgain systems

Regular Bourgain systems are the 'approximate groups' to which we extend Fourier analysis; there is a natural candidate for 'approximate Haar measure' on $\mathcal{B}$: if $(B_\rho)_\rho$ is a Bourgain system, then we write β_ρ for the normalized counting measure on B_ρ and simply β for β_1. We adopt similar conventions to before for the Bourgain systems $\mathcal{B}', \mathcal{B}'', \ldots$. It is worth noting that the normalized measures introduced here are different from those in [**10**], where positivity of the Fourier transform was also desired.

Lemma 4.1 (approximate Haar measure). *Suppose that $\mathcal{B}$ is a regular Bourgain system of dimension d. If $y \in B_\eta$, then $\|(y+\beta)-\beta\| \leqslant 2^4 d\eta$.*

Proof. Note that $\mathrm{supp}\,((y+\beta)-\beta) \subset B_{1+\eta} \setminus B_{1-\eta}$, whence we obtain

$$\|(y+\beta)-\beta\| \leqslant \frac{\mu_G(B_{1+\eta} \setminus B_{1-\eta})}{\mu_G(B_1)} \leqslant 2^4 d\eta,$$

by regularity. □

The next two lemmas reflect two ways in which we commonly use the property of regularity.

Lemma 4.2. *Suppose that $\mathcal{B}$ is a regular Bourgain system of dimension d. If $f : G \to \mathbb{C}$, then*

$$\|f*\beta - f*\beta(x)\|_{L^\infty(x+\beta_\eta)} \leqslant 2^4 \|f\|_{L^\infty(\mu_G)} d\eta.$$

Proof. Note that

$$\begin{aligned} |f*\beta(x+y) - f*\beta(x)| &= |f*((-y+\beta)-\beta)(x)| \\ &\leqslant \|f\|_{L^\infty(\mu_G)} \|(-y+\beta)-\beta\|. \end{aligned}$$

The result follows by Lemma 4.1. □

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Lemma 4.3. *Suppose that $\mathcal{B}$ is a regular Bourgain system of dimension d and $\kappa > 0$ is a parameter. Then*

$$\{\gamma : |\hat{\beta}(\gamma)| \geqslant \kappa\} \subset \{\gamma : |1 - \gamma(x)| \leqslant 2^4 d\kappa^{-1}\eta \text{ for all } x \in B_\eta\}.$$

Proof. If $\gamma \in \{\gamma : |\hat{\beta}(\gamma)| \geqslant \kappa\}$ and $y \in B_\eta$, then

$$\begin{aligned}\kappa|1 - \gamma(y)| &\leqslant |\hat{\beta}(\gamma)|\,|1 - \overline{\gamma(y)}| \\ &= \left|\int \gamma(x)\,\mathrm{d}((y+\beta) - \beta)(x)\right| \leqslant 2^4 d\eta\end{aligned}$$

by Lemma 4.1. The lemma follows. □

The final result of the section is a version of Bessel's inequality local to Bourgain systems. Such a result was essentially proved in [**11**, Corollary 8.6], and serves to replace some of the many applications of Parseval's theorem in the local setting.

Proposition 4.4 (local Bessel inequality). *Suppose that $\mathcal{B}$ is a regular Bourgain system of dimension d. Suppose that $f : G \to \mathbb{C}$ and $\epsilon \in (0, 1]$ is a parameter. Write $L_f := \|f\|_{L^1(\beta)}^{-1}\|f\|_{L^2(\beta)}$. Then there is a Bourgain system $\tilde{\mathcal{B}}'$ of dimension $2^2\epsilon^{-2}L_f^2$ such that $\mathcal{B}' := \tilde{\mathcal{B}}' \cap \mathcal{B}$ has*

$$\mu_G(\mathcal{B}') \geqslant 4^{-(d+2\epsilon^{-2}L_f^2)}\mu_G(\mathcal{B})$$

and

$$\{\gamma : |\widehat{f\,\mathrm{d}\beta}(\gamma)| \geqslant \epsilon\|f\|_{L^1(\beta)}\} \subset \{\gamma : |1 - \gamma(x)| \leqslant 2^7(1+d)\epsilon^{-2}L_f^2\eta \text{ for all } x \in B'_\eta\}.$$

To prove this we require an almost-orthogonality lemma due to Cotlar [**5**].

Lemma 4.5 (Cotlar's almost-orthogonality lemma). *Suppose that v and (w_j) are elements of an inner product space. Then*

$$\sum_j |\langle v, w_j\rangle|^2 \leqslant \langle v, v\rangle \max_j \sum_i |\langle w_i, w_j\rangle|.$$

Proof of Proposition 4.4. Let

$$S := \{\gamma \in \hat{G} : |\hat{\beta}(\gamma)| \geqslant \tfrac{1}{2}\epsilon^2 L_f^{-2}\}$$

and

$$\Delta := \{\gamma : |\widehat{f\,\mathrm{d}\beta}(\gamma)| \geqslant \epsilon\|f\|_{L^1(\beta)}\}.$$

Pick $\Lambda \subset \Delta$ maximal such that all the sets $(\lambda + S)_{\lambda\in\Lambda}$ are disjoint. Now if $\gamma \in \Delta$, then there is a $\lambda \in \Lambda$ such that $\lambda + S \cap \gamma + S \neq \emptyset$ by maximality. It follows that $\gamma \in \lambda + S - S$, i.e. $\Delta \subset \Lambda + S - S$.

By Cotlar's lemma (Lemma 4.5) we have

$$\begin{aligned}\sum_{\lambda\in\Lambda} |\widehat{f\,\mathrm{d}\beta}(\lambda)|^2 &\leqslant \|f\|_{L^2(\beta)}^2 \max_{\lambda\in\Lambda} \sum_{\lambda'\in\Lambda} |\hat{\beta}(\lambda - \lambda')| \\ &\leqslant \|f\|_{L^2(\beta)}^2(1 + \tfrac{1}{2}|\Lambda|\epsilon^2 L_f^{-2}),\end{aligned}$$

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since $\lambda, \lambda' \in \Lambda$ and $\lambda - \lambda' \in S$ implies that $\lambda = \lambda'$. Since $\Lambda \subset \Delta$, we conclude that

$$|\Lambda|\epsilon^2 \|f\|_{L^1(\beta)}^2 \leqslant \sum_{\lambda \in \Lambda} |\widehat{f\,\mathrm{d}\beta}(\lambda)|^2.$$

Combining all this we obtain $|\Lambda| \leqslant 2\epsilon^{-2}L_f^2$.

Let $\tilde{\mathcal{B}}'$ be the Bourgain system induced by the Bohr set $B(\Lambda, 1)$ so $\mu_G(\tilde{\mathcal{B}}') = 1$ and $\dim \tilde{\mathcal{B}}' \leqslant 2|\Lambda| \leqslant 2^2\epsilon^{-2}L_f^2$. Recalling that

$$|1 - \gamma(x)| = \sqrt{2(1 - \cos(4\pi\|\gamma(x)\|))} \leqslant 4\pi\|\gamma(x)\|,$$

we certainly have

$$\Lambda \subset \{\gamma : |1 - \gamma(x)| \leqslant 2^6(1 + d)\epsilon^{-2}L_f^2\eta \text{ for all } x \in \tilde{B}'_\eta\}.$$

By Lemma 4.3 S is contained in

$$\{\gamma : |1 - \gamma(x)| \leqslant 2^5 d\epsilon^{-2}L_f^2\eta \text{ for all } x \in B_\eta\},$$

and so, by the triangle inequality,

$$S - S \subset \{\gamma : |1 - \gamma(x)| \leqslant 2^6 d\epsilon^{-2}L_f^2\eta \text{ for all } x \in B_\eta\}.$$

It follows that

$$\Delta \subset \Lambda + S - S \subset \{\gamma : |1 - \gamma(x)| \leqslant 2^7(1 + d)\epsilon^{-2}L_f^2\eta \text{ for all } x \in B_\eta \cap \tilde{B}'_\eta\}.$$

The result follows by Lemma 3.4 on letting $\mathcal{B}' := \tilde{\mathcal{B}}' \cap \mathcal{B}$. □

5. A variant of the Bourgain–Roth theorem

If G is a finite group and $A \subset G$, then we can count the number of three-term arithmetic progressions in A using the following trilinear form:

$$\Lambda(f, g, h) := \int f(x - y)g(x)h(x + y)\,\mathrm{d}\mu_G(x)\,\mathrm{d}\mu_G(y). \tag{5.1}$$

This form has a well-known Fourier expression gained by substituting the inversion formulae for f, g and h into (5.1):

$$\Lambda(f, g, h) = \sum_{\gamma \in \hat{G}} \hat{f}(\gamma)\hat{g}(-2\gamma)\hat{h}(\gamma).$$

In this section we shall prove the following result.

Theorem 5.1. *Suppose that $\mathcal{B}$ is a regular Bourgain system of dimension d. Suppose that $A \subset G$ has $\alpha := \|1_A * \beta\|_{L^\infty(\mu_G)}$ (that is, the relative density of A on the translate of B on which it is largest) positive, and $A - A$ contains no elements of order 2. Then*

$$\Lambda(1_A, 1_A, 1_A) \geqslant \left(\frac{\alpha}{2(1 + d)}\right)^{2^{24}d\log\alpha^{-1} + 2^{52}\alpha^{-3}(\log\alpha^{-1})^2} \mu_G(\mathcal{B})^2.$$

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We prove Theorem 5.1 by iterating the following lemma.

Lemma 5.2 (iteration lemma). *Suppose that $\mathcal{B}$ is a regular Bourgain system of dimension d. Suppose that $A \subset G$ has $\alpha := \|1_A * \beta\|_{L^\infty(\mu_G)} > 0$ and $A - A$ contains no elements of order 2. Then at least one of the following is true.*

(i) (Lots of three-term progressions.)

$$\Lambda(1_A, 1_A, 1_A) \geqslant \frac{\alpha^3}{2^5}\left(\frac{\alpha^3}{2^{44}(1+d)^3}\right)^d \mu_G(\mathcal{B})^2.$$

(ii) (Density increment I.) *There is a regular dilate $\mathcal{B}''$ of $\mathcal{B}$ with*

$$\mu_G(\mathcal{B}'') \geqslant \left(\frac{\alpha^2}{2^{25}(1+d)^2}\right)^d \mu_G(\mathcal{B})$$

such that $\|1_A * \beta''\|_{L^\infty(\mu_G)} \geqslant \alpha(1 + 2^{-12})$.

(iii) (Density increment II.) *There is a regular dilate $\mathcal{B}'''$ of $(\{2x : x \in B_\rho\})_\rho$ with*

$$\mu_G(\mathcal{B}''') \geqslant \frac{\alpha}{2^2}\left(\frac{\alpha^3}{2^{36}(1+d)^3}\right)^d \mu_G(\mathcal{B})$$

such that $\|1_A * \beta'''\|_{L^\infty(\mu_G)} \geqslant \alpha(1 + 2^{-8})$.

(iv) (Density increment III.) *There is a Bourgain system $\tilde{\mathcal{B}}^{(\mathrm{iv})}$ of dimension at most $2^{13}\alpha^{-3}$ and a dilate $\mathcal{B}'''$ of $(\{2x : x \in B_\rho\})_\rho$ such that their intersection, $\mathcal{B}^{(\mathrm{iv})}$, is regular with*

$$\mu_G(\mathcal{B}^{(\mathrm{iv})}) \geqslant \frac{\alpha}{2^2}\left(\frac{\alpha^3}{2^{22}(1+d)}\right)^{2^{13}\alpha^{-3}}\left(\frac{\alpha^5}{2^{48}(1+d)^3}\right)^d \mu_G(\mathcal{B})$$

such that $\|1_A * \beta^{(\mathrm{iv})}\|_{L^\infty(\mu_G)} \geqslant \alpha(1 + 2^{-8})$.

Cases (ii)–(iv) are the outcomes of different parts of the proof; we separate them for ease of understanding.

The proof of the lemma requires the following technical result, which converts energy on non-trivial Fourier modes into a density increment.

Lemma 5.3 (ℓ^2-density increment lemma). *Suppose that $\mathcal{B}$ is a regular Bourgain system of dimension d. Suppose that $A \subset G$ has $\alpha := 1_A * \beta(0_G) > 0$ and $c > 0$ is a parameter. Write $\eta := c\alpha/2^{10}(1+d)$ and suppose that $\mathcal{B}'$ is a subsystem of $\eta\mathcal{B}$ and that there is a set of characters*

$$\Lambda := \{\gamma : |1 - \gamma(x)| \leqslant \tfrac{1}{2} \text{ for all } x \in B'\}$$

such that

$$\sum_{\lambda \in \Lambda} |((1_A - \alpha)1_B)^\wedge(\lambda)|^2 \geqslant c\alpha^2 \mu_G(B).$$

Then $\|1_A * \beta'\|_{L^\infty(\mu_G)} \geqslant \alpha(1 + c/2^3)$.

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Proof. Write $f := 1_A - \alpha$. The triangle inequality shows that if $\lambda \in \Lambda$, then

$$|\widehat{\beta'}(\lambda)| \geqslant \int \mathrm{d}\beta' - \int |1-\lambda|\,\mathrm{d}\beta' \geqslant \tfrac{1}{2},$$

whereupon (from the hypothesis of the lemma)

$$c\alpha^2\mu_G(B)/2^2 \leqslant \sum_{\gamma\in\hat{G}} |\widehat{f1_B}(\gamma)\widehat{\beta'}(\gamma)|^2.$$

Plancherel's theorem (and dividing by $\mu_G(B)$) then gives

$$\langle (f1_B)*\beta', (f\,\mathrm{d}\beta)*\beta'\rangle \geqslant \frac{c\alpha^2}{2^2}.$$

We expand this inner product as follows:

$$\begin{aligned}\langle (f1_B)*\beta', (f\,\mathrm{d}\beta)*\beta'\rangle = {}& \langle (1_A1_B)*\beta', (1_A\,\mathrm{d}\beta)*\beta'\rangle\\ &- \alpha\langle 1_B*\beta', (1_A\,\mathrm{d}\beta)*\beta'\rangle\\ &- \alpha\langle (1_A1_B)*\beta', \beta*\beta'\rangle\\ &+ \alpha^2\langle 1_B*\beta', \beta*\beta'\rangle. \qquad (5.2)\end{aligned}$$

We estimate the last three terms: by Lemma 4.1 we have

$$\begin{aligned}\|\beta*\beta'*\beta'-\beta\| &\leqslant \int \|(y+\beta)-\beta\| d(\beta'*\beta')(y)\\ &\leqslant \sup_{y\in\operatorname{supp}\beta'*\beta'} \|(y+\beta)-\beta\|\\ &\leqslant \sup_{y\in B_2'} \|(y+\beta)-\beta\|\\ &\leqslant \sup_{y\in B_{2\eta}} \|(y+\beta)-\beta\|\\ &\leqslant \frac{c\alpha}{2^5}. \qquad (5.3)\end{aligned}$$

Now

$$\langle 1_B*\beta', (1_A\,\mathrm{d}\beta)*\beta'\rangle = \langle \beta*\beta'*\beta', (1_A1_B)\rangle$$

and

$$|\langle \beta*\beta'*\beta', 1_A1_B\rangle - \langle \beta, 1_A1_B\rangle| \leqslant \frac{c\alpha}{2^5}$$

by (5.3); $\langle \beta, 1_A1_B\rangle = \alpha$, so

$$|\langle 1_B*\beta', (1_A\,\mathrm{d}\beta)*\beta'\rangle - \alpha| \leqslant \frac{c\alpha}{2^5}.$$

By symmetry,

$$|\langle (1_A1_B)*\beta', \beta*\beta'\rangle - \alpha| \leqslant \frac{c\alpha}{2^5}$$

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and, similarly,

$$|\langle 1_B * \beta', \beta * \beta' \rangle - 1| \leqslant \frac{c\alpha}{2^5}.$$

Inserting these last three estimates into (5.2) we get

$$\langle (f1_B) * \beta', (f\,\mathrm{d}\beta) * \beta' \rangle \leqslant \langle (1_A 1_B) * \beta', (1_A\,\mathrm{d}\beta) * \beta' \rangle - \alpha^2 + \frac{c\alpha^2}{2^3}.$$

We conclude that

$$\alpha^2 \left(1 + \frac{c}{2^3}\right) \leqslant \langle (1_A 1_B) * \beta', (1_A\,\mathrm{d}\beta) * \beta' \rangle.$$

Finally,

$$\begin{aligned}\langle (1_A 1_B) * \beta', (1_A\,\mathrm{d}\beta) * \beta' \rangle &\leqslant \|(1_A 1_B) * \beta'\|_{L^\infty(\mu_G)} \|(1_A\,\mathrm{d}\beta) * \beta'\| \\ &\leqslant \|(1_A 1_B) * \beta'\|_{L^\infty(\mu_G)} \|1_A\|_{L^1(\beta)} \|\beta'\| \\ &= \|(1_A 1_B) * \beta'\|_{L^\infty(\mu_G)} \alpha \\ &\leqslant \|1_A * \beta'\|_{L^\infty(\mu_G)} \alpha;\end{aligned}$$

we get the result on dividing by α. □

Proof of Lemma 5.2. Suppose that we are not in case (ii) of the lemma, so we may certainly assume that, for all regular dilates $\mathcal{B}''$ of $\mathcal{B}$ with

$$\mu_G(\mathcal{B}'') \geqslant \left(\frac{\alpha^2}{2^{25}(1+d)^2}\right)^d \mu_G(\mathcal{B}),$$

we have

$$\|1_A * \beta''\|_{L^\infty(\mu_G)} \leqslant \alpha(1 + 2^{-12}). \tag{5.4}$$

Apply Proposition 3.5 to pick λ' so that $\mathcal{B}' := \lambda'\mathcal{B}$ is regular and

$$\frac{\alpha}{2^{16}(1+d)} \leqslant \lambda' < \frac{\alpha}{2^{15}(1+d)}.$$

Apply Proposition 3.5 to pick λ'' so that $\mathcal{B}'' := \lambda''\mathcal{B}'$ is regular and

$$\frac{\alpha}{2^8(1+d)} \leqslant \lambda'' < \frac{\alpha}{2^7(1+d)}.$$

Suppose that $\lambda \in [\lambda''\lambda', \lambda']$. A trivial instance of Young's inequality tells us that

$$\begin{aligned}\|1_A * \beta * \beta_\lambda - 1_A * \beta\|_{L^\infty(\mu_G)} &\leqslant \|1_A\|_{L^\infty(\mu_G)} \|\beta * \beta_\lambda - \beta\| \\ &\leqslant \int \|(y+\beta) - \beta\|\,\mathrm{d}\beta_\lambda(y) \\ &\leqslant \sup_{y \in B_\lambda} \|(y+\beta) - \beta\| \\ &\leqslant 2^4 d\lambda \\ &\leqslant \frac{\alpha}{2^{11}}\end{aligned}$$

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by Lemma 4.1 and the fact that $\lambda \leqslant \lambda'$. Let $x' \in G$ be such that $1_A * \beta(x') = \alpha$. It follows from the previous calculation that

$$|(1_A * \beta_\lambda - \alpha) * \beta(x')| \leqslant \frac{\alpha}{2^{11}}.$$

Moreover, by assumption (5.4) (applicable by Lemma 3.3 and the fact that $\lambda \geqslant \lambda''\lambda'$) we have

$$1_A * \beta_\lambda - \alpha \leqslant \frac{\alpha}{2^{12}}.$$

For functions $g : G \to \mathbb{C}$ we write $g_+ := \frac{1}{2}(|g| + g)$ and $g_- := \frac{1}{2}(|g| - g) = g_+ - g$. Now, combining our last two expressions yields

$$\begin{aligned} |1_A * \beta_\lambda - \alpha| * \beta(x') &= (1_A * \beta_\lambda - \alpha)_+ * \beta(x') + (1_A * \beta_\lambda - \alpha)_- * \beta(x') \\ &= 2(1_A * \beta_\lambda - \alpha)_+ * \beta(x') - (1_A * \beta_\lambda - \alpha) * \beta(x') \\ &\leqslant \frac{\alpha}{2^{10}}. \end{aligned}$$

Applying this expression with $\lambda = \lambda'$ and $\lambda = \lambda''\lambda'$, we get

$$\begin{aligned} \frac{\alpha}{2^9} &\geqslant (|1_A * \beta' - \alpha| + |1_A * \beta'' - \alpha|) * \beta(x') \\ &\geqslant \inf_{x \in G} (|1_A * \beta'(x) - \alpha| + |1_A * \beta''(x) - \alpha|). \end{aligned}$$

By translating A we may assume that the infimum on the right-hand side is attained at $x = 0_G$; we write

$$\alpha' := 1_A * \beta'(0_G), \quad \alpha'' := 1_A * \beta''(0_G), \quad f' := 1_A - \alpha' \quad \text{and} \quad f'' := 1_A - \alpha''$$

and note that

$$|\alpha'' - \alpha| \leqslant \frac{\alpha}{2^9} \quad \text{and} \quad |\alpha' - \alpha| \leqslant \frac{\alpha}{2^9}.$$

Now by trilinearity of Λ we have

$$\begin{aligned} \Lambda(1_A 1_{B'}, 1_A 1_{B''}, 1_A 1_{B'}) = \Lambda(1_A 1_{B'}, 1_A 1_{B''}, \alpha' 1_{B'}) + \Lambda(\alpha' 1_{B''}, 1_A 1_{B''}, f' 1_{B'}) \\ + \Lambda(f' 1_{B'}, \alpha'' 1_{B''}, f' 1_{B'}) + \Lambda(f' 1_{B'}, f'' 1_{B''}, f' 1_{B'}). \quad (5.5) \end{aligned}$$

We can easily estimate the first two terms on the right-hand side using the following fact.

Claim 5.4. *Suppose that $g : G \to \mathbb{C}$ has $\|g\|_{L^\infty(\mu_G)} \leqslant 1$. Then*

$$|\Lambda(g 1_{B'}, 1_A 1_{B''}, 1_{B'}) - \alpha'' g * \beta'(0_G) \mu_G(B'') \mu_G(B')| \leqslant \frac{\alpha'' \alpha' \mu_G(B'') \mu_G(B')}{2^2}.$$

Proof. Recall that $\Lambda(g 1_{B'}, 1_A 1_{B''}, 1_{B'})$ equals

$$\int g(x - y) 1_{B'}(x - y) 1_A(x) 1_{B''}(x) 1_{B'}(x + y) \, \mathrm{d}\mu_G(x) \, \mathrm{d}\mu_G(y)$$

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by definition. By the change of variables $u = x - y$ and symmetry of B' we conclude that this expression is in turn equal to

$$\int g(u)1_{B'}(u)1_A(x)1_{B''}(x)1_{B'}(u-2x)\,\mathrm{d}\mu_G(x)\,\mathrm{d}\mu_G(u).$$

Now the difference between this term and

$$\int g(u)1_{B'}(u)1_A(x)1_{B''}(x)1_{B'}(u)\,\mathrm{d}\mu_G(u)\,\mathrm{d}\mu_G(x)\ (=\alpha''g*\beta'(0_G)\mu_G(B'')\mu_G(B'))$$

is at most

$$\|g\|_{L^\infty(\mu_G)}\int 1_{B'}(u)1_A(x)1_{B''}(x)|1_{B'}(u-2x)-1_{B'}(u)|\,\mathrm{d}\mu_G(x)\,\mathrm{d}\mu_G(u)$$

in absolute value. But if $x \in B''$, then $2x \in B_1''+B_1'' \subset B_2'' = B'_{2\lambda''}$, whence if $u \in B'_{1-2\lambda''}$, then $1_{B'}(u) = 1_{B'}(u-2x)$. It follows that this error term is at most

$$\alpha''\mu_G(B'')\mu_G(B'_1 \setminus B'_{1-2\lambda''}) \leqslant 2^4\alpha''d\lambda''\mu_G(B'')\mu_G(B')$$

by regularity of $\mathcal{B}'$. The claim follows in view of the earlier choice of λ'' and the fact that $\alpha' \geqslant \frac{1}{2}\alpha$. □

It follows by applying this claim with $g = 1_A$ that

$$|\Lambda(1_A1_{B'}, 1_A1_{B''}, \alpha'1_{B'}) - \alpha''\alpha'^2\mu_G(B'')\mu_G(B')| \leqslant \frac{\alpha''\alpha'^2\mu_G(B'')\mu_G(B')}{2^2}. \tag{5.6}$$

Moreover, since $f' * \beta'(0_G) = 0$ the claim applied with $g = f'$ gives

$$|\Lambda(\alpha'1_{B'}, 1_A1_{B''}, f'1_{B'})| \leqslant \frac{\alpha''\alpha'^2\mu_G(B'')\mu_G(B')}{2^2}. \tag{5.7}$$

In view of (5.6), (5.7) and the decomposition (5.5) we conclude (by the triangle inequality) that

(i)
$$|\Lambda(1_A1_{B'}, 1_A1_{B''}, 1_A1_{B'})| \geqslant \frac{\alpha''\alpha'^2\mu_G(B'')\mu_G(B')}{2^2},$$
and we are in case (i) of the lemma, or

(ii)
$$|\Lambda(f'1_{B'}, \alpha''1_{B''}, f'1_{B'})| \geqslant \frac{\alpha''\alpha'^2\mu_G(B'')\mu_G(B')}{2^3},$$
and it turns out that we are in case (iii) of the lemma, or

(iii)
$$|\Lambda(f'1_{B'}, f''1_{B''}, f'1_{B'})| \geqslant \frac{\alpha''\alpha'^2\mu_G(B'')\mu_G(B')}{2^3},$$
and it turns out that we are in case (iv) of the lemma.

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The first conclusion is immediate. The second and third are verified (respectively) in the following two claims.

Claim 5.5. *If*

$$|\Lambda(f'1_{B'}, \alpha''1_{B''}, f'1_{B'})| \geqslant \frac{\alpha''\alpha'^2\mu_G(B'')\mu_G(B')}{2^3},$$

then we are in case (iii) of the lemma.

Proof. In view of the Fourier expression for Λ we get

$$\frac{\alpha'^2\mu_G(B'')\mu_G(B')}{2^3} \leqslant \sum_{\gamma\in\hat{G}} |\widehat{1_{B''}}(2\gamma)|\,|\widehat{f'1_{B'}}(\gamma)|^2. \tag{5.8}$$

It turns out that the characters for which $|\widehat{1_{B''}}(2\gamma)|$ is large support a lot of the mass of the sum on the right: let $\epsilon = \alpha'/2^4$ and set

$$\Lambda := \{\gamma \in \hat{G} : |\widehat{1_{B''}}(2\gamma)| \geqslant \epsilon\mu_G(B'')\}.$$

Then

$$\begin{aligned}\sum_{\gamma\notin\Lambda} |\widehat{1_{B''}}(2\gamma)|\,|\widehat{f'1_{B'}}(\gamma)|^2 &\leqslant \epsilon\mu_G(B'')\sum_{\gamma\in\hat{G}} |\widehat{f'1_{B'}}(\gamma)|^2\\ &= \epsilon\mu_G(B'')\mu_G(B')\|f'\|^2_{L^2(\beta')},\end{aligned}$$

by the triangle inequality and Parseval's theorem. Now $\|f'\|^2_{L^2(\beta')} = \alpha' - \alpha'^2$, so it follows that this last expression is at most $\epsilon\alpha'\mu_G(B'')\mu_G(B')$ and hence by the triangle inequality and (5.8) we have

$$\frac{\alpha'^2\mu_G(B')}{2^4} \leqslant \sum_{\gamma\in\Lambda} |\widehat{f'1_{B'}}(\gamma)|^2.$$

Note that $(\{2x : x \in B''_\rho\})_\rho$ is a Bourgain system of dimension d. Apply Proposition 3.5 to pick λ''' so that $\mathcal{B}''' := \lambda'''(\{2x : x \in B''_\rho\})_\rho$ is regular and

$$\frac{\alpha}{2^{11}(1+d)} \leqslant \lambda''' < \frac{\alpha}{2^{10}(1+d)};$$

since $\mathcal{B}''$ is a dilate of $\mathcal{B}$, $\mathcal{B}'''$ is a dilate of $(\{2x : x \in B_\rho\})_\rho$. By Lemma 4.3 we have that

$$\Lambda \subset \{\gamma : |1-(2\gamma)(x)| \leqslant \tfrac{1}{2} \text{ for all } x \in B''_{\lambda'''}\} = \{\gamma : |1-\gamma(x)| \leqslant \tfrac{1}{2} \text{ for all } x \in B'''\}.$$

Now $\mathcal{B}'''$ is a subsystem of $(\alpha'/2^{14}(1+d))\mathcal{B}'$ so we apply Lemma 5.3 with $c = 2^{-4}$ to see that

$$\|1_A * \beta'''\|_{L^\infty(\mu_G)} \geqslant \alpha'(1+2^{-7}) \geqslant \alpha(1+2^{-8}).$$

It remains only to verify the bound on the density of $\mathcal{B}'''$. Note that

$$\|1_A * \beta''_{\lambda'''} * \beta' - 1_A * \beta'\|_{L^\infty(\mu_G)} \leqslant 2^4 d\lambda'''\lambda'' \leqslant \tfrac{1}{2}\alpha'$$

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by Lemma 4.1. Whence we obtain

$$1_A * \beta''_{\lambda'''} * \beta'(x') \geqslant 1_A * \beta'(x') - \frac{\alpha'}{2} \geqslant \frac{\alpha}{2^2}.$$

By averaging, it follows that there is some $x'' \in G$ such that $1_A * \beta''_{\lambda'''}(x'') \geqslant \alpha/2^2$. Since $A - A$ contains no elements of order 2 we have that $x \mapsto 2x$ is injective when restricted to A; we conclude that

$$\begin{aligned} \mu_G(2B''_{\lambda'''}) &= \mu_G(2(x'' + B''_{\lambda'''})) \\ &\geqslant \mu_G(2(A \cap (x'' + B''_{\lambda'''}))), \\ \mu_G(A \cap (x'' + B''_{\lambda'''})) &\geqslant \frac{\alpha}{2^2}\mu_G(B''_{\lambda'''}) \\ &\geqslant \frac{\alpha}{2^2}\left(\frac{\lambda'\lambda''\lambda'''}{2}\right)^d \mu_G(\mathcal{B}), \end{aligned}$$

by Lemma 3.3. The claim follows. □

Claim 5.6. *If*

$$|\Lambda(f'1_{B'}, f''1_{B''}, f'1_{B'})| \geqslant \frac{\alpha''\alpha'^2\mu_G(B'')\mu_G(B')}{2^3},$$

then we are in case (iv) of the lemma.

Proof. In view of the Fourier expression for Λ we have

$$\frac{\alpha''\alpha'^2\mu_G(B'')\mu_G(B')}{2^3} \leqslant \sum_{\gamma\in\hat{G}} |\widehat{f''1_{B''}}(2\gamma)|\,|\widehat{f1_{B'}}(\gamma)|^2. \tag{5.9}$$

As in the previous claim we may ignore the characters supporting small values of $\widehat{f''1_{B''}}(\gamma)$: let $\epsilon = \alpha''\alpha'/2^4$ and set

$$\Lambda := \{\gamma \in \hat{G} : |\widehat{f''1_{B''}}(2\gamma)| \geqslant \epsilon\mu_G(B'')\}.$$

Then

$$\begin{aligned} \sum_{\gamma\notin\Lambda} |\widehat{f''1_{B''}}(2\gamma)|\,|\widehat{f'1_{B'}}(\gamma)|^2 &\leqslant \epsilon\mu_G(B'') \sum_{\gamma\in\hat{G}} |\widehat{f'1_{B'}}(\gamma)|^2 \\ &= \epsilon\mu_G(B'')\mu_G(B')\|f'\|^2_{L^2(\beta')}, \end{aligned}$$

by the triangle inequality and Parseval's theorem. Now $\|f'\|^2_{L^2(\beta')} = \alpha' - \alpha'^2$, so it follows that the latter expression is at most $\alpha''\alpha'^2\mu_G(B'')\mu_G(B')/2^4$, and hence by the triangle inequality and (5.9) we have

$$\sum_{\gamma\in\Lambda} |\widehat{f''1_{B''}}(2\gamma)|\,|\widehat{f'1_{B'}}(\gamma)|^2 \geqslant \frac{\alpha''\alpha'^2\mu_G(B'')\mu_G(B')}{2^4}.$$

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Since

$$\|f''1_{B''}\|_{L^1(\mu_G)} = 2(\alpha'' - \alpha''^2)\mu_G(B''),$$

we have $|\widehat{f''1_{B''}}(2\gamma)| \leqslant 2\alpha''\mu_G(B'')$ and so

$$\sum_{\gamma\in\Lambda} |\widehat{f'1_{B'}}(\gamma)|^2 \geqslant \frac{\alpha'^2\mu_G(B')}{2^5}.$$

We apply Proposition 4.4 to obtain a system $\tilde{\mathcal{B}}'''$ with

$$\dim \tilde{\mathcal{B}}''' \leqslant 2^{10}\alpha''^{-1}\alpha'^{-2} \leqslant 2^{13}\alpha^{-3},$$

such that $\tilde{\mathcal{B}}''' \cap \mathcal{B}''$ has

$$\mu_G(\tilde{\mathcal{B}}''' \cap \mathcal{B}'') \geqslant 4^{-d-2^{12}\alpha^{-3}}\mu_G(\mathcal{B}'')$$

and

$$\Lambda \subset \{\gamma : |1-(2\gamma)(x)| \leqslant 2^{18}\alpha^{-3}d\eta \text{ for all } x \in \tilde{B}'''_\eta \cap B''_\eta\}.$$

Apply Proposition 3.5 to pick $\lambda^{(\mathrm{iv})}$ so that

$$\mathcal{B}^{(\mathrm{iv})} := \lambda^{(\mathrm{iv})}((\{2x : x \in \tilde{B}'''_\rho\})_\rho \cap (\{2x : x \in B''_\rho\})_\rho)$$

is regular and

$$\frac{\alpha^3}{2^{20}(1+d)} \leqslant \lambda^{(\mathrm{iv})} < \frac{\alpha^3}{2^{19}(1+d)}.$$

Set $\tilde{\mathcal{B}}^{(\mathrm{iv})} := \lambda^{(\mathrm{iv})}(\{2x : x \in \tilde{B}'''_\rho\})_\rho$ and $\mathcal{B}''' := \lambda^{(\mathrm{iv})}\mathcal{B}''$. Now

$$\begin{aligned}\Lambda &\subset \{\gamma : |1-(2\gamma)(x)| \leqslant \tfrac{1}{2} \text{ for all } x \in \tilde{B}'''_{\lambda^{(\mathrm{iv})}} \cap B''_{\lambda^{(\mathrm{iv})}}\}\\ &= \{\gamma : |1-\gamma(x)| \leqslant \tfrac{1}{2} \text{ for all } x \in B^{(\mathrm{iv})}\}.\end{aligned}$$

$\mathcal{B}^{(\mathrm{iv})}$ is a subsystem of $(\alpha'/2^{14}(1+d))\mathcal{B}'$ so we may apply Lemma 5.3 with $c = 2^{-4}$ to see that

$$\|1_A * \beta^{(\mathrm{iv})}\|_{L^\infty(\mu_G)} \geqslant \alpha'(1+2^{-7}) \geqslant \alpha(1+2^{-8}).$$

It remains only to verify the bound on the density of $\mathcal{B}^{(\mathrm{iv})}$. Note that

$$\|1_A * \beta'''_{\lambda^{(\mathrm{iv})}} * \beta' - 1_A * \beta'\|_{L^\infty(\mu_G)} \leqslant 2^4 d\lambda^{(\mathrm{iv})}\lambda'' \leqslant \tfrac{1}{2}\alpha'$$

by Lemma 4.1. Whence,

$$1_A * \beta'''_{\lambda^{(\mathrm{iv})}} * \beta'(x') \geqslant 1_A * \beta'(x') - \frac{\alpha'}{2} \geqslant \frac{\alpha}{2^2}.$$

By averaging it follows that there is some $x'' \in G$ such that $1_A * \beta'''_{\lambda^{(\mathrm{iv})}}(x'') \geqslant \alpha/2^2$. Since $A - A$ contains no elements of order 2 we have that $x \mapsto 2x$ is injective when restricted

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to A; we conclude that

$$\begin{aligned}\mu_G(2B'''_{\lambda^{(\mathrm{iv})}}) &= \mu_G(2(x''+B'''_{\lambda^{(\mathrm{iv})}}))\\ &\geqslant \mu_G(2(A\cap(x''+B'''_{\lambda^{(\mathrm{iv})}})))\\ &= \mu_G(A\cap(x''+B'''_{\lambda^{(\mathrm{iv})}}))\\ &\geqslant \frac{\alpha}{2^2}\mu_G(B'''_{\lambda^{(\mathrm{iv})}})\\ &\geqslant \frac{\alpha}{2^2}\left(\frac{\lambda^{(\mathrm{iv})}}{2}\right)^{d+2^{13}\alpha^{-3}} 4^{-d-2^{12}\alpha^{-3}}\left(\frac{\lambda'\lambda''}{2}\right)^d\mu_G(\mathcal{B}),\end{aligned}$$

by Lemma 3.3. The claim follows. □

The lemma is proved. □

Proof of Theorem 5.1. We construct two sequences of Bourgain systems $\tilde{\mathcal{B}}_k$ and $\mathcal{B}'_k$; we write $\tilde{d}_k$ for the dimension of $\tilde{\mathcal{B}}_k$, $\mathcal{B}_{k+1}$ for the intersected system $\tilde{\mathcal{B}}_k\cap\mathcal{B}'_k$, d_k for the dimension of $\mathcal{B}_k$, δ_k for the density of $\mathcal{B}_k$, β_k for the measure on $\mathcal{B}_k$ and $\alpha_k := \|1_{A_k}*\beta_k\|_{L^\infty(\mu_G)}$.

For $k\leqslant 2^{14}\log\alpha^{-1}$ we shall show inductively that these sequences satisfy the following conditions:

(i) $\tilde{d}_k\leqslant 2^{13}\alpha^{-3}$;

(ii) $\mathcal{B}'_k$ is a dilate of either $\mathcal{B}_{k-1}$ or $(\{2x : x\in(B_{k-1})_\rho\})_\rho$;

(iii) $\mathcal{B}_k$ is a regular Bourgain system;

(iv) $d_k\leqslant 2d+2^{14}\alpha^{-3}k$;

(v) $\delta_k\geqslant(\alpha/(2(1+d)))^{(2^8d+2^{36}\alpha^{-3}\log\alpha^{-1})k}\mu_G(\mathcal{B})$;

(vi) $\alpha_k\geqslant(1+2^{-12})^k\alpha$.

We initialize the set-up with $\mathcal{B}_0=\mathcal{B}$ (or, if preferred, $\tilde{\mathcal{B}}_{-1}$ as the trivial system and $\mathcal{B}'_{-1}=\mathcal{B}$) so that the properties are trivially satisfied. At stage $k\leqslant 2^{13}\log\alpha^{-1}$ apply Lemma 5.2 to $\mathcal{B}_k$. It follows that either

$$\Lambda(1_A,1_A,1_A)\geqslant\frac{\alpha_k^3}{2^5}\left(\frac{\alpha_k^3}{2^{44}(1+d_k)^3}\right)^{d_k}\mu_G(\mathcal{B}_k)^2 \tag{5.10}$$

or there is a (possibly trivial) Bourgain system $\tilde{\mathcal{B}}_k$ with dimension $\tilde{d}_k\leqslant 2^{13}\alpha_k^{-3}\leqslant 2^{13}\alpha^{-3}$ and another $\mathcal{B}'_k$ which is a dilate either of $\mathcal{B}_k$ or of $(\{2x : x\in(B_k)_\rho\})_\rho$ such that $\mathcal{B}_{k+1}=\tilde{\mathcal{B}}_k\cap\mathcal{B}'_k$ is regular,

$$\begin{aligned}\delta_{k+1} &\geqslant \frac{\alpha_k}{2^2}\left(\frac{\alpha_k^3}{2^{22}(1+d_k)}\right)^{2^{13}\alpha_k^{-3}}\left(\frac{\alpha_k^5}{2^{48}(1+d_k)^3}\right)^{d_k}\delta_k\\ &\geqslant\left(\frac{\alpha}{2(1+d)}\right)^{(2^8d+2^{36}\alpha^{-3}\log\alpha^{-1})(k+1)}\mu_G(\mathcal{B})\end{aligned}$$

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and

$$\alpha_{k+1} \geqslant (1+2^{-12})\alpha_k \geqslant (1+2^{-12})^{k+1}\alpha.$$

It remains to check the bound on d_{k+1}, which follows by Lemma 3.4 on noting that $\mathcal{B}_{k+1}$ is the intersection of a system of dimension d and $k+1$ systems of dimension at most $2^{13}\alpha^{-3}$.

In view of the lower bound on α_k and the fact that $\alpha_k \leqslant 1$, it follows that there is some $k \leqslant 2^{13}\log\alpha^{-1}$ such that (5.10) holds; this yields the result. □

6. An argument of Bogolioùboff and Chang

In this section we shall prove the following proposition, which draws on techniques of Bogolioùboff [**1**] as refined by Chang [**4**]. An argument of this type is contained in [**10**].

Proposition 6.1. *Suppose that G is a finite abelian group. Suppose that $A \subset G$ has density $\alpha > 0$ and that $|A+A| \leqslant K|A|$. Then there is a regular Bourgain system $\mathcal{B}$ with*

$$\dim \mathcal{B} \leqslant 2^5 K\log\alpha^{-1} \quad \text{and} \quad \mu_G(\mathcal{B}) \geqslant \left(\frac{1}{2^{14}K^2(1+\log\alpha^{-1})}\right)^{2^4K\log\alpha^{-1}}$$

such that

$$\|1_A * \beta\|_{L^\infty(\mu_G)} \geqslant \frac{1}{2K}.$$

We require Chang's theorem [**9**, Proposition 3.2].

Proposition 6.2 (Chang's theorem). *Suppose that $A \subset G$ is a set of density $\alpha > 0$ and $\epsilon \in (0,1]$ is a parameter. Let $\Lambda := \{\gamma \in \hat{G} : |\widehat{1_A}(\gamma)| \geqslant \epsilon\alpha\}$. Then there is a set of characters Γ with $|\Gamma| \leqslant 2\epsilon^{-2}\log\alpha^{-1}$ such that $\Lambda \subset \langle\Gamma\rangle$, where we recall that*

$$\langle\Gamma\rangle := \left\{\sum_{\lambda\in\Gamma}\sigma_\lambda\lambda : \sigma \in \{-1,0,1\}^\Gamma\right\}.$$

Proof of Proposition 6.1. Let ϵ be a parameter to be chosen later. Apply Chang's theorem (Proposition 6.2) to the set A with parameter $\sqrt{\epsilon/3}$ to obtain a set of characters Γ with $|\Gamma| \leqslant 6\epsilon^{-1}\log\alpha^{-1}$ and $\Lambda := \{\gamma : |\widehat{1_A}(\gamma)| \geqslant \sqrt{\epsilon/3}\alpha\} \subset \langle\Gamma\rangle$.

Write $\mathcal{B}'$ for the Bourgain system induced by $B(\Gamma, \epsilon/2^6(1+|\Gamma|))$ and apply Proposition 3.5 to pick $\eta \in [\frac{1}{2}, 1)$ so that $\mathcal{B} := \eta\mathcal{B}'$ is regular. It follows that $\mathcal{B}$ has dimension at most $2|\Gamma|$ and density at least

$$\left(\frac{1}{4}\right)^{2|\Gamma|} \times \left(\frac{\epsilon}{2^6(1+|\Gamma|)}\right)^{|\Gamma|} \geqslant \left(\frac{\epsilon^2}{2^{12}(1+\log\alpha^{-1})}\right)^{|\Gamma|}.$$

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If $\lambda \in \Lambda$, then $\lambda = \sum_{\gamma\in\Gamma}\sigma_\gamma\gamma$ so

$$\begin{aligned}|1-\lambda(h)| &\leqslant \sum_{\gamma\in\Gamma}|1-\gamma(h)|\\ &= \sum_{\gamma\in\Gamma}\sqrt{2(1-\cos(4\pi\|\gamma(h)\|))}\\ &\leqslant \sum_{\gamma\in\Gamma}4\pi\|\gamma(h)\|\\ &\leqslant 4\pi|\Gamma|\sup_{\gamma\in\Gamma}\|\gamma(h)\|.\end{aligned}$$

So if $\lambda \in \Lambda$, then

$$|1-\hat{\beta}(\lambda)| \leqslant \sup_{h\in B}|1-\lambda(h)| \leqslant \tfrac{1}{3}\epsilon.$$

Hence, $|\langle 1_A * 1_A, 1_A * 1_A\rangle - \langle 1_A * 1_A, 1_A * 1_A * \beta\rangle|$ is at most

$$\begin{aligned}\Big|\sum_{\gamma\in\hat{G}}|\widehat{1_A}(\gamma)|^4(1-\overline{\hat{\beta}(\gamma)})\Big| &\leqslant \sup_{\gamma\in\Lambda}|1-\hat{\beta}(\gamma)|\sum_{\gamma\in\hat{G}}|\widehat{1_A}(\gamma)|^4 + 2\sup_{\gamma\notin\Lambda}|\widehat{1_A}(\gamma)|^2\sum_{\gamma\in\hat{G}}|\widehat{1_A}(\gamma)|^2\\ &\leqslant \tfrac{1}{3}\epsilon\alpha^2\sum_{\gamma\in\hat{G}}|\widehat{1_A}(\gamma)|^2 + 2(\tfrac{1}{3}\epsilon)\alpha^2\sum_{\gamma\in\hat{G}}|\widehat{1_A}(\gamma)|^2\\ &\leqslant \epsilon\alpha^3.\end{aligned}$$

Moreover,

$$\langle 1_A * 1_A, 1_A * 1_A\rangle \geqslant \mu_G(\operatorname{supp} 1_A * 1_A)^{-1}\Big(\int 1_A * 1_A \,\mathrm{d}\mu_G\Big)^2 \geqslant \frac{\alpha^3}{K}$$

by the Cauchy–Schwarz inequality and the fact that $|A+A| \leqslant K|A|$. It follows from the triangle inequality that if we take $\epsilon = 1/2K$, then

$$\begin{aligned}\frac{\alpha^3}{2K} &\leqslant |\langle 1_A * 1_A, 1_A * 1_A * \beta_{\Gamma,\delta}\rangle|\\ &= |\langle 1_A * 1_A * 1_{-A}, 1_A * \beta_{\Gamma,\delta}\rangle|\\ &\leqslant \|1_A * \beta_{\Gamma,\delta}\|_{L^\infty(\mu_G)}\alpha^3.\end{aligned}$$

Dividing by α^3, the result is proved. □

7. The main arguments

In this section we prove the following theorem, which is the real heart of the paper.

Theorem 7.1. *Suppose that G is an abelian group and that $A \subset G$ is finite with $|A+A| \leqslant K|A|$. If $A-A$ contains no elements of order 2, then A contains at least $\exp(-CK^3\log^3(1+K))|A|^2$ three-term arithmetic progressions for some absolute positive constant C.*

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Recall that if G and G' are two abelian groups with subsets A and A', respectively, then $\phi : A \to A'$ is a *Freĭman homomorphism* if

$$a_1 + a_2 = a_3 + a_4 \implies \phi(a_1) + \phi(a_2) = \phi(a_3) + \phi(a_4).$$

If ϕ has an inverse which is also a homomorphism, then we say that ϕ is a *Freĭman isomorphism*. For us the key property of Freĭman isomorphisms is that if A and A' are Freĭman isomorphic, then the three-term arithmetic progressions in A and A' are in one-to-one correspondence. It follows that each set has the same number of these.

To leverage the work of § 5 we need A to be a large proportion of G. This cannot be guaranteed, but the following proposition will allow us to move A to a setting where this is true.

Proposition 7.2 (Green and Ruzsa [9, Proposition 1.2]). *Suppose that G is an abelian group and $A \subset G$ is finite with $|A + A| \leqslant K|A|$. Then there is an abelian group G' with $|G'| \leqslant (20K)^{10K^2}|A|$ such that A is Freĭman isomorphic to a subset of G'.*

Proof of Theorem 7.1. We apply Proposition 7.2 to obtain a subset A' of a group G' with density at least $(20K)^{-10K^2}$ such that A' Freĭman isomorphic to A. Since A' is Freĭman isomorphic we have $|A| = |A'|$, $|A' + A'| \leqslant K|A'|$ and $A' - A'$ contains no elements of order 2. We apply Proposition 6.1 to get a regular Bourgain system $\mathcal{B}$ with

$$\dim \mathcal{B} \leqslant 2^9 K^3 \log(1 + K) \quad \text{and} \quad \mu_{G'}(\mathcal{B}) \geqslant (2K)^{-2^{13}K^3 \log(1+K)}$$

such that $\|1_{A'} * \beta\|_{L^\infty(\mu_{G'})} \geqslant 1/2K$. We now apply Theorem 5.1 to obtain the result. □

The proof of Theorem 1.4 is now rather straightforward.

Proof of Theorem 1.4. Write $K := |A + A|/|A|$ and suppose that $a, a' \in A$ have $a - a'$ of order 2. Then $a + a = 2a'$ is a non-trivial three-term progression in A which contradicts the hypothesis. It follows that we may apply Theorem 7.1 to conclude that A contains at least $\exp(-CK^3 \log^3(1 + K))|A|^2$ progressions; however, we know this to be at most $|A|$, whence

$$\exp(CK^3 \log^3(1 + K)) \geqslant |A|.$$

The result follows on rearranging. □

Proving Theorem 1.5 simply requires us to apply Theorem 1.4 in more or less the same manner as Schoen applies Theorem 1.2.

Proof of Theorem 1.5. Write

$$S := \{a \in A \cap B : \nexists a' \in A,\ b' \in B \text{ with } a' \neq b' \text{ such that } a' + b' = 2a\},$$

and note that crucially we have

$$(A + B) \setminus (A \mathbin{\widehat{+}} B) = 2S, \tag{7.1}$$

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and, moreover, that S contains no three-term progressions (a, b, c) with $a + b = 2c$ and $a \neq b$.

Let S' be a subset of S such that for all $s \in 2S$ there is exactly one $s' \in S'$ such that $2s' = s$. It is easy to see that $|S'| = |2S|$.

We claim that S' contains no non-trivial three-term progressions. Suppose that $a, b, c \in S'$ have $a+b = 2c$. Since $S' \subset S$ we conclude that $a = b$, but in this case we have $2a = 2c$, which, by choice of S', implies that $a = c$. The claim follows.

Consequently, we may apply Theorem 1.4 to conclude that

$$|S' + S'| \gg |S'|\left(\frac{\log |S'|}{(\log\log |S'|)^3}\right)^{1/3}.$$

Recalling that $n = |A + B|$, we can rearrange this expression to give

$$|S'| \ll |S' + S'|\left(\frac{(\log\log |S' + S'|)^3}{\log |S' + S'|}\right)^{1/3} \ll |A + B|\left(\frac{(\log\log n)^3}{\log n}\right)^{1/3},$$

since the middle expression is an increasing function of $|S' + S'|$ and $S' + S' \subset A + B$. The result follows from (7.1) and the fact that $|S'| = |2S|$. □

8. Concluding remarks

The extension of Theorem 1.2 to the groups $\mathbb{Z}^r$ and $\mathbb{Z}/N\mathbb{Z}$ (with the same bound) is implicit in [**2**, **20**, **24**]. Moreover, since there are particularly good versions of the modelling proposition (Proposition 7.2) for these groups, it seems very likely that our Proposition 6.1 could be used in conjunction with a more traditional ℓ^∞-density increment argument [**2**] to prove the following.

Theorem 8.1. *Suppose that G is $\mathbb{Z}^r$ or $\mathbb{Z}/N\mathbb{Z}$ and that $A \subset G$ is finite with $|A+A| \leqslant K|A|$. Then A contains at least $\exp(-CK^{2+o(1)})|A|^2$ three-term arithmetic progressions for some absolute $C > 0$.*

Indeed, it appears that with the methods of [**22**] one could replace $K^{2+o(1)}$ by $K^2 \log(1 + K)$, thereby directly generalizing Bourgain's version of Roth's theorem from [**2**].

We have not considered how the ideas in [**3**] might come into play to give an even stronger result; the following is a natural question.

Problem 8.2. Find a direct generalization of the result of [**3**] to sets with small sumset. That is, show that if $A \subset \mathbb{Z}/N\mathbb{Z}$ is finite with $|A + A| \leqslant K|A|$, then A contains at least $\exp(-CK^{3/2}\log^3(1 + K))|A|^2$ three-term arithmetic progressions for some absolute $C > 0$.

Among other things Theorem 1.4 immediately improves a result of Stanchescu [**24**], who, answering a further question of Freĭman [**6**], used Theorem 1.2 to bound from below the size of $|A + A|/|A|$ when $A \subset \mathbb{Z}^2$ is finite and contains no three collinear points. This is an intriguing question because one appears to have so much extra information to play

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with: not only does A not contain any three-term progressions but it also avoids any triples (a, b, c) with $\lambda a + \mu b = (\lambda + \mu)c$ for any positive integers λ and μ.

Problem 8.3. Find a constant an absolute constant $c > \frac{2}{3}$ such that if $A \subset \mathbb{Z}^2$ is finite and contains no three collinear points, then $|A + A| \gg |A| \log^c |A|$.

Moves to generalize additive problems to arbitrary abelian groups have also spawned the observation (see, for example, [**8**, **12**]) that some arguments can be modelled very cleanly (and often more effectively) in certain well-behaved abelian groups. It would be surprising if one could not prove the following theorem using the methods outlined above.

Theorem 8.4. *Suppose that G is a vector space over $\mathbb{F}_3$ and $A \subset G$ is finite with $|A + A| \leqslant K|A|$. Then A contains at least $\exp(-CK)|A|^2$ three-term arithmetic progressions for some absolute constant $C > 0$.*

In a different direction it may be that the following problem captures the essence of Roth's theorem in a natural general setting.

Problem 8.5. Suppose that $A \subset \mathbb{Z}$ has at least $\delta|A|^3$ additive quadruples. Find a good absolute constant $c > 0$ such that we can conclude that A contains at least $\exp(-C\delta^{-c})|A|^2$ three-term arithmetic progressions.

It is immediate from the quantitative Balog–Szemerédi–Gowers theorem (see [**7**]) that there is some $c > 0$; the problem is to find a good value.

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